Math 122B

Exam II

March 28th 2014

The University of Arizona

Name: ______________________________

Answers without adequate justification will not receive full credit, including multiple choice. Include units with your answer when appropriate, and box all answers unless an answer line is provided. By signing below I am agreeing to abide by the University of Arizona academic integrity policies and that all work done on this test is my own.

Signature: ______________________________

Tips for Success:

- Look through the entire test before starting to prioritize questions.
- If you get stuck on a question, move on and come back to it later.
- Do a quick reality check after each question: does my answer make sense? Did I include units? Did I show all my work?
- Read over the entire test at the end to make sure you didn't miss anything.
- For each question: take a deep breath, think slowly and deliberately at first, then work quickly once you see what to do.

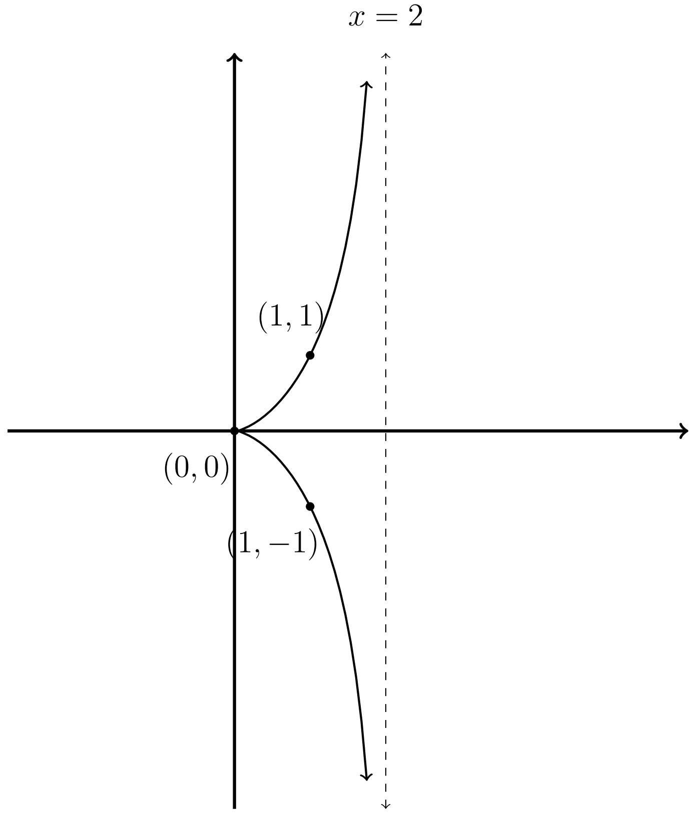
Requests:

- Show all your steps.
- Please box answers when possible.

1. The curve defined by $x(x^2 + y^2) = 2ay^2$ is called the *cissoid of Diocles* after the ancient Greek mathematician Diocles (240 BC-180 BC).

For the remainder of the problem we will assume $a = 1$.

(a) Plot the curve. Hint: solve for y to obtain two explicit functions for the top half and bottom half. You may use your calculator, but you must label your graph appropriately.



To plot the curve, we solve for y, taking care to include $\pm$ when we take the square root:

$$\begin{aligned} x(x^2 + y^2) &= 2y^2 \\ \Leftrightarrow x^3 + xy^2 &= 2y^2 \\ \Leftrightarrow x^3 &= y^2(2 - x) \\ \Leftrightarrow y^2 &= \frac{x^3}{2 - x} \\ \Leftrightarrow y &= \pm\sqrt{\frac{x^3}{2 - x}} \end{aligned}$$

We can then plot the two explicit functions $y = \sqrt{\frac{x^3}{2-x}}$ and $y = -\sqrt{\frac{x^3}{2-x}}$ using the calculator. Notice that I label three key points $(0, 0), (1, 1)$ and $(1, -1)$, as well as the vertical[1] asymptote at $x = 2$.

[1] or 'verticle', according to many of you ;)

(b) Find $\frac{dy}{dx}$.

Solution We apply the 'derivative with respect to x' operator to both sides of the (original) equation, and use implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}\left[x^3 + xy^2\right] &= \frac{d}{dx}\left[2y^2\right] \\
\frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[xy^2\right] &= 2\frac{d}{dx}\left[y^2\right] \\
3x^2 + \frac{d}{dx}\left[x\right] y^2 + \frac{d}{dx}\left[y^2\right] x &= 2 \cdot 2y\frac{dy}{dx} \\
3x^2 + y^2 + 2y\frac{dy}{dx}x &= 4y\frac{dy}{dx} \\
3x^2 + y^2 + 2xy\frac{dy}{dx} &= 4y\frac{dy}{dx} \qquad (\star)
\end{aligned}$$

Then, we solve for $\frac{dy}{dx}$ by isolating it on one side of the equation, factoring it out, and dividing:

$$\begin{aligned}
(\star) \Rightarrow 3x^2 + y^2 &= 4y\frac{dy}{dx} - 2xy\frac{dy}{dx} \\
\Rightarrow 3x^2 + y^2 &= \frac{dy}{dx}(4y - 2xy) \\
\Rightarrow \frac{3x^2 + y^2}{4y - 2xy} &= \frac{dy}{dx}
\end{aligned}$$

Thus, after factoring the denominator, we have

$$\boxed{\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2 - x)}}$$

Notes: Some common mistakes include:

- Leaving 'lonely' $\frac{d}{dx}$ symbols around. $\frac{d}{dx}[\cdot]$ is an *operator* - it must be applied to something in order to be happy. Writing $\frac{d}{dx}$ by itself is similar to writing sin by itself - meaningless!
- Don't take $\frac{d}{dx}[\cdot]$ of an *entire* equation - it doesn't make sense to take the derivative of an *equation*, only of a function (explicitly or implicitly defined). This is why we take $\frac{d}{dx}[\cdot]$ of *both sides* of the equation.

(c) Where is $\frac{dy}{dx}$ undefined? Use the graph from part (a) to explain *why* this derivative is not defined.

Solution Note that in the expression for $\frac{dy}{dx}$ derived above, the denominator is zero if either $y = 0$ or if $x = 2$. When $y = 0$, we see that the original graph has a 'cusp' (a kind of corner). When $x = 2$, the curve is actually not defined - $x = 2$ is an asymptote. In fact, the curve itself is not defined for $x \leq 0$ or $x \geq 2$, so indeed the derivative is not defined there. The correct answer should be that $\frac{dy}{dx}$ is not defined on the interval $\boxed{(-\infty, 0] \cup [2, \infty)}$. If you gave me $y = 0$ and $x = 2$, I believe you received 5.5/6 points.

2. Compute the following derivatives and simplify your answer. **I should have probably specified that a, b, A and B are all *constants*. I think everyone realized this.**

Solution

(a) We use the product rule and chain rule (predominantly):

$$\begin{aligned}\frac{d}{dr}\left[ar\ln(b+r^2)\right] &= \frac{d}{dr}[ar]\ln(b+r^2) + ar\cdot\frac{d}{dr}[\ln(b+r^2)]\\ &= a\ln(b+r^2) + ar\frac{1}{b+r^2}\frac{d}{dr}[b+r^2]\\ &= a\ln(b+r^2) + \frac{ar}{b+r^2}2r\\ &= a\ln(b+r^2) + \frac{2ar^2}{b+r^2}\end{aligned}$$

Note: The 'logarithmic differentiation rule' is very useful here - maybe a good idea to remember this one:

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

It's just the chain rule, but it comes up a lot.

(b) There are essentially two ways to do this. We can either use the quotient rule, with top = A and bottom = $Bz + \exp(\sin(2z))$, or the chain rule with outside = Ay^{-1} and inside = $Bz + \exp(\sin(2z))$. I prefer the chain rule:

$$\begin{aligned}
\frac{d}{dz}\left[\frac{A}{Bz + e^{\sin(2z)}}\right] &= A\frac{d}{dz}\left[(Bz + \exp(sin(2z)))^{-1}\right] \\
&= -A\,(Bz + \exp(\sin(2z)))^{-2}\,\frac{d}{dz}\,[Bz + \exp(\sin(2z))] \\
&= -\frac{A}{(Bz + \exp(\sin(2z)))^2}\left(B + \frac{d}{dz}[\exp(\sin(2z))]\right) \\
&= -\frac{A}{(Bz + \exp(\sin(2z)))^2}\big(B + \exp(\sin(2z)) \cdot 2\cos(2z)\big) \\
&= -\frac{A\,(B + 2\cos(2z)\exp(\sin(2z)))}{(Bz + \exp(\sin(2z))^2}
\end{aligned}$$

Notes Another useful differentiation rule to remember is the general exponential rule:

$$\frac{d}{dx}\left[b^{f(x)}\right] = \ln(b)b^{f(x)}f'(x)$$

Again - this is just the chain rule - but it comes up frequently enough that committing it to memory is useful.

(c) $Z'(t)$ if $Z(t) = \arcsin(\ln(t))$. This is just an application of the chain rule = symbolically, we will have

$$Z'(t) = \arcsin'(\ln(t))\frac{d}{dt}[\ln(t)] = \frac{1}{t}\arcsin'(\ln(t))$$

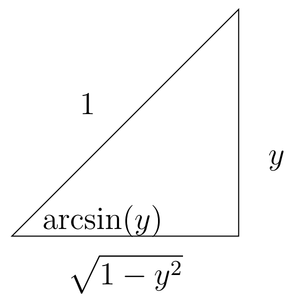
Now, if you remember what the derivative of $f(y) = \arcsin(y)$ is, great - I never remember. What I do remember is:

$$\frac{d}{dy}[f^{-1}(y)] = \frac{1}{f'(f^{-1}(y)}$$

With $\arcsin(y)$, we would have $f(y) = \sin(y)$ and so

$$\arcsin'(y) = \frac{1}{\sin'(\arcsin(y))} = \frac{1}{\cos(\arcsin(y))}$$

Then, remember the 'triangle trick':



So, $\cos(\arcsin(y)) = \sqrt{1-y^2}$, and so

$$\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$$

So finally, we have

$$\boxed{Z'(t) = \frac{1}{t\sqrt{1-(\ln(t))^2}}}$$

3. Consider the following statement, called *Rolle's theorem*:

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there is a number c such that $a < c < b$ and $f'(c) = 0$.

Explain why this is true using a theorem discussed in class.

Solution: This is a consequence of the *mean value theorem*. To see why, note first (very important!) that $f(x)$ satisfies the conditions of the MVT since it is continuous on $[a, b]$ and differentiable on (a, b). Then, the MVT says that there exists a c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Well, we are assuming that $f(b) = f(a)$, and so the right hand side of the above equation is indeed zero. Thus we know there exists a c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

Notes

- We are implicitly assuming $a \neq b$. If $a = b$, we have that f is differentiable on (a, a) which is completely meaningless!
- If you really want to dig into it, it is actually easier to prove Rolle's theorem and then derive the Mean Value Theorem from it! So really, Rolle's theorem is more 'fundamental' in a way. You would do this in say, a 300 or 400 level mathematical analysis course.
- I gave nearly full points for just mentioning the MVT; I took off a small amount for not mentioning that f satisfies the conditions of the MVT (if it didn't, we couldn't use MVT!)

4. Use the table below to evaluate the following:

x	0	1	2	3	4	5
$f(x)$	-2	-1	3	6	-3	2
$f'(x)$	1	4	-4	7	5	0

(a) $(f^{-1})'(3)$

Solution: We use the inverse function derivative rule:

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

So, we have

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)} = -\frac{1}{4}$$

(b) $h'(3)$ if $h(x) = \exp(Af(x))$. *Note:* $\exp(y) = e^y$

Solution: This is a simple application of the chain rule - but be careful! I saw a lot of solutions where $h'(x) = A\exp(Af(x))$. Never forget the chain rule!

$$h'(x) = \frac{d}{dx}\left[e^{Af(x)}\right] = e^{Af(x)}\frac{d}{dx}\left[Af(x)\right] = e^{Af(x)} \cdot Af'(x)$$

So,

$$\boxed{h'(3) = Af'(3)e^{Af(3)} = 7Ae^{6A}}$$

(c) $R'(4)$ if $R(x) = f(\sinh(x-4)+x)$. *Note: hyperbolic sine, not regular sine*

Solution: Again - this is simply an application of the chain rule. For some reason a lot of people seemed to think that f was a constant, when in fact $f(\cdot)$ is the same function as parts (a) and (b). The derivative is

$$R'(x) = f'\left(\sinh(x-4)+x\right)\frac{d}{dx}\left[\sinh(x-4)+x\right] = f'\left(\sinh(x-4)+x\right)\cdot\left(\cosh(x-4)+1\right)$$

Thus

$$R'(4) = f'(\sinh(0)+4)\cdot(\cosh(0)+1) = f'(4)\cdot 2 = 10$$

Note: One tip: if possible, always compute the derivative *function* first before evaluating a derivative. Notice that above I computed $h'(x)$, $R'(x)$ before finding $h'(3)$, $R'(4)$. This will prevent many mistakes!

5. Let $g(x) = x - \ln(\cos(x))$.

(a) Find the local linearization (aka tangent line approximation) for $g(x)$ at $x = \pi/4$.

Solution: Recall that the tangent line approximation to the differentiable function $f(x)$, at $x = a$, is given by the line

$$y = f(a) + f'(a)(x - a)$$

So, we need to compute $g(\pi/4)$ and $g'(\pi/4)$:

$$\begin{aligned} g(\pi/4) &= \frac{\pi}{4} - \ln(\cos(\pi/4)) = \frac{\pi}{4} - \ln\left(\frac{\sqrt{2}}{2}\right) \\ g'(x) &= 1 - \frac{1}{\cos(x)}\frac{d}{dx}[\cos(x)] = 1 - \frac{-\sin(x)}{\cos(x)} = 1 + \tan(x) \\ \Rightarrow g'(\pi/4) &= 1 + \tan(\pi/4) = 2 \end{aligned}$$

Thus the tangent line is given by

$$\begin{aligned} y &= g(\pi/4) + g'(\pi/4)(x - \pi/4) \\ &= \frac{\pi}{4} - \ln\left(\frac{\sqrt{2}}{2}\right) + 2(x - \pi/4) \end{aligned}$$

(It's fine to leave it like this - definitely don't approximate the numbers, leave them exact.

(b) Is the local linearization an *over* estimate or *under* estimate on the interval $(0, \pi/2)$? Explain using calculus.

Solution: The *only* correct way to answer this question *using calculus* is to compute the second derivative of g and look at its sign (positive or negative) on the **entire** interval $(0, \pi/2)$. Any other answers are not correct!

$$g''(x) = \frac{d}{dx}[1 + \tan(x)] = \sec^2(x)$$

So, on $(0, \pi/2)$, $g''(x) > 0$. In fact, $g''(x) > 0$ for any x where $\sec(x)$ is defined, i.e. when $\cos(x) \neq 0$. Thus $g(x)$ is concave up, and so the tangent line approximation is an *under* estimate.

6. Answer the following true/false questions; no explanation is required.

(a) $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(x)}$

TRUE FALSE

False This is the inverse function derivative rule, which you should know is

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

(Sort of) challenge question: when (i.e. for what function $f(x)$) is the stated rule actually true?

(b) Suppose $h(x) = f(g(x))$, $h(1)$ is defined, but h is not differentiable for $x = 1$. Then, either $f'(1)$ or $g'(1)$ must not exist.

TRUE FALSE

False This one was tricky - look at the chain rule:

$$h'(x) = f'(g(x))g'(x)$$

For the expression on the right hand side to be defined at $x = 1$, we need $g'(1)$ and $\boxed{f'(g(1))}$ to be defined. A counter example: let $f(x) = |x|$ and $g(x) = x - 1$. Then, both $f'(1)$ and $g'(1)$ exist, but $h(x) = |x - 1|$ and so $h'(1)$ does *not* exist.

(c) The function $\sinh(x)$ is periodic.

TRUE FALSE

False Look at the graph of $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$; periodic functions must 'repeat' themselves, and $\sinh(x)$ is actually *monotonic* i.e. it *never* repeats itself.

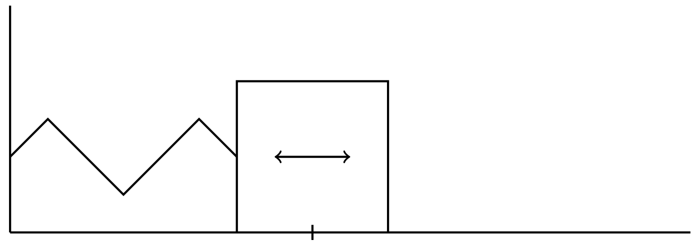
(d) If $z = 3y^{1/3}$, then $\frac{dy}{dz} = y^{2/3}$.

TRUE FALSE

True This is a simple implicit derivative. Take $\frac{d}{dz}[\cdot]$ of both sides:

$$\frac{d}{dz}[z] = \frac{d}{dz}[3y^{1/3}] \Rightarrow 1 = y^{-2/3}\frac{dy}{dz} \Rightarrow \frac{dy}{dz} = y^{2/3}$$

7. A mass is attached to a spring and oscillates back and forth on a table:



The function $x(t)$ given below describes the position of the center of mass of the box relative to $x = 0$, in cm:

$$x(t) = 2\sin\left(\sqrt{2}\,t + \phi\right)$$

Note: it seems like about 30-40% of the class thought that $x(t) = 2\sin(\sqrt{2t}+\phi)$. I understand the confusion - the typesetting made it look that way. I was nice with points on part (b).

(a) What does ϕ need to be in order to have $x(0) = 1$?

Solution: $x(0) = 2\sin(\phi) = 1$ so we need $\sin(\phi) = 1/2$. The *full* set of solutions is thus

$$\phi \in \{\pi/6 + 2k\pi : k \in \mathbb{Z}\} \cup \{5\pi/6 + 2k\pi : k \in \mathbb{Z}\}$$

I was happy with $\phi = \pi/6$, because the resulting functions are all equivalent anyway.

(b) By taking derivatives and simplifying, show that

$$x''(t) + 2x(t) = 0$$

Note: you can answer this question without answering part (a).

Solution: Take the second derivative:

$$x'(t) = 2\sqrt{2}\cos(\sqrt{2}\,t + \phi)$$
$$\Rightarrow x''(t) = -2\sqrt{2}\sqrt{2}\sin(\sqrt{2}\,t + \phi) = -4\sin(\sqrt{2}\,t + \phi)$$

So,

$$x''(t) + 2x(t) = -4\sin(\sqrt{2}\,t + \phi) + 4\sin(\sqrt{2}\,t + \phi) = 0 \quad \checkmark$$

General Notes:

- Always be clear with your notation - your work should follow a logical sequence, not just a mess of scratch work. If you're still not sure what this means, come talk to me and I will explain the difference
- Relating to the above: don't leave 'lonely' derivative operators lying around. For example, say we want $\frac{d}{dx}[x^2]$. The following is meaningless:

  $$\frac{d}{dx} = 2x$$

  Just remind yourself that the derivative always needs something inside its brackets! The correct way to write the above is

  $$\frac{d}{dx}[x^2] = 2x$$

  or alternately, you could say 'let $f(x) = x^2$. Then, $f'(x) = 2x$.'
- Also relating to notation: be very careful to separate a function and its value at a point! For example, I see this kind of thing all the time: say $g(x) = x^2 + x$ and we want $g'(2)$. Then, people write

  $$g'(2) = 2x + x = 2 \cdot 2 + 2 = 6 \qquad (\otimes)$$

  or

  $$g'(x) = 2x + x = 2 \cdot 2 + 2 = 6 \qquad (\otimes)$$

  or

  $$g'(x) = 2 \cdot 2 + 2 = 6 \qquad (\otimes)$$

  The correct way to do this is to find the derivative *function* first, then *evaluate* it on a separate line:

  $$\begin{aligned} g'(x) &= 2x + x \\ \Rightarrow g'(2) &= 2 \cdot 2 + 2 \end{aligned}$$

- If you're not sure exactly how to proceed with a problem, try to break it down into pieces. If you think you'll need the derivative at some point, compute it - chances are you will get some partial credit for writing it down (for instance, the last problem on the test).