Math 122B

Exam I PRACTICE

February 2014

The University of Arizona

Name: ____

Answers without adequate justification will not receive full credit, including multiple choice. Include units with your answer when appropriate, and box all answers unless an answer line is provided. By signing below I am agreeing to abide by the University of Arizona academic integrity policies and that all work done on this test is my own.

Signature: ____

Tips for Success:

- Look through the entire test before starting to prioritize questions.
- If you get stuck on a question, move on and come back to it later.
- Do a quick reality check after each question: does my answer make sense? Did I include units? Did I show all my work?
- Read over the entire test at the end to make sure you didn't miss anything.
- For each question: take a deep breath, think slowly and deliberately at first, then work quickly once you see what to do.

Requests:

- Show all your steps.
- Please box answers when possible.

1. Compute the derivative of the following functions using differentiation rules:

(a) $f(x) = x^{\pi/2} + 3\sqrt{x} - \frac{2}{x^{1/3}}$

Solution: We will use the following differentiation rules:

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x), \quad \frac{d}{dx}[cf(x)] = cf'(x), \quad \frac{d}{dx}[x^a] = ax^{a-1}$$

So,

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^{\pi/2} + 3\sqrt{x} - \frac{2}{x^{1/3}}] \\ &= \frac{d}{dx}[x^{\pi/2}] + 3\frac{d}{dx}[x^{1/2}] - 2\frac{d}{dx}[x^{-1/3}] \\ &= \frac{\pi}{2}x^{-\pi/2} + 3\frac{1}{2}x^{-1/2} - 2\frac{-1}{3}x^{-4/3} \\ &= \frac{\pi}{2x^{\pi/2}} + \frac{3}{2\sqrt{x}} + \frac{2}{3x^{4/3}} \end{aligned}$$

thus

$$\boxed{f'(x) = \frac{\pi}{2x^{\pi/2}} + \frac{3}{2\sqrt{x}} + \frac{2}{3x^{4/3}}}$$

(b) $g(y) = ay^3 + ya^3$

Solution: Using the same differentiation rules as above, and remembering that a is a constant, we obtain

$$g'(y) = \frac{d}{dy}[ay^3 + ya^3] = a\frac{d}{dy}y^3 + a^3\frac{d}{dy}[y] = 3ay^2 + a^3$$

hence

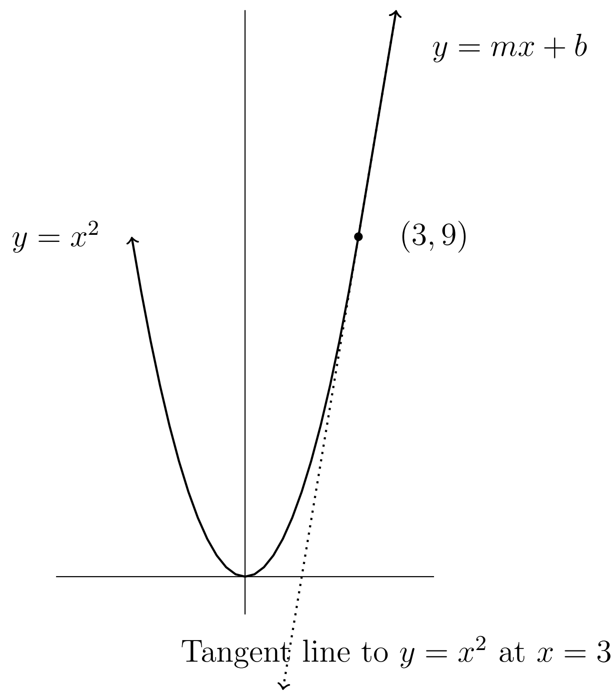
$$\boxed{g'(y) = 3ay^2 + a^3}$$

2. Let $g(x)$ be defined as follows:

$$g(x) = \begin{cases} x^2 & x \leq 3 \\ mx + b & x > 3 \end{cases}$$

Find values of m and b so that $g(x)$ is continuous and differentiable everywhere.

Solution As both 'parts' of this piecewise function are continuous and differentiable, we only need to worry about the jump point $x = 3$. Basically, we want the picture to look like this:



1. To have differentiability, we need the slope of the line $y = mx + b$ to be the same as the slope of the tangent line to $y = x^2$ at $x = 3$. Using the power rule, we know that $\frac{d}{dx}[x^2]\Big|_{x=3} = 2 \cdot 3 = 6$. Thus, the slope of our line must be 6.
2. To get continuity, we require that

$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} g(x)$$

Evaluating these limits (using $y = 6x + b$ from above), we find that

$$\lim_{x \to 3^-} g(x) = 9$$
$$\lim_{x \to 3^+} g(x) = 3 \cdot 6 + b$$

hence we must have $18 + b = 9$, or $b = 9$. So, finally, we have

$$\boxed{m = 6, \quad b = 9}$$

3. Let $f(x) = \dfrac{a}{\sqrt{bx}}$, where a and b are nonzero constant real numbers. *Using the limit definition*, compute $f'(2)$. Leave answer exact, involving a and b.

Solution: This is straightforward but *very* messy. Write slowly, clearly and concisely and you won't make any mistakes!

$$
\begin{aligned}
f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{\frac{a}{\sqrt{b(2+h)}}-\frac{a}{\sqrt{2b}}}{h}\\
&= \lim_{h\to 0}\frac{\frac{a\sqrt{2b}-a\sqrt{b(2+h)}}{\sqrt{2b}\sqrt{b(2+h)}}}{h}\\
&= \lim_{h\to 0}\frac{a\sqrt{2b}-a\sqrt{b(2+h)}}{h\sqrt{2b}\sqrt{b(2+h)}}\\
&= \lim_{h\to 0}\frac{a\sqrt{2b}-a\sqrt{b(2+h)}}{h\sqrt{2b}\sqrt{b(2+h)}}\left(\frac{a\sqrt{2b}+a\sqrt{b(2+h)}}{a\sqrt{2b}+a\sqrt{b(2+h)}}\right)\\
&= \lim_{h\to 0}\frac{2a^2b-a^2(b(2+h))}{h\sqrt{2b}\sqrt{b(2+h)}\left(a\sqrt{2b}+a\sqrt{b(2+h)}\right)}\\
&= \lim_{h\to 0}\frac{-a^2bh}{h\sqrt{2b}\sqrt{b(2+h)}\left(a\sqrt{2b}+a\sqrt{b(2+h)}\right)}\\
&= \lim_{h\to 0}\frac{-a^2b}{\sqrt{2b}\sqrt{b(2+h)}\left(a\sqrt{2b}+a\sqrt{b(2+h)}\right)}\\
&= \frac{-a^2b}{\sqrt{2b}\sqrt{2b}\left(a\sqrt{2b}+a\sqrt{2b}\right)}\\
&= \frac{-a}{4\sqrt{2b}}
\end{aligned}
$$

4. One of the consequences of Einstein's theory of *general relativity* is that the mass m of an object depends on its velocity v according to the following function:

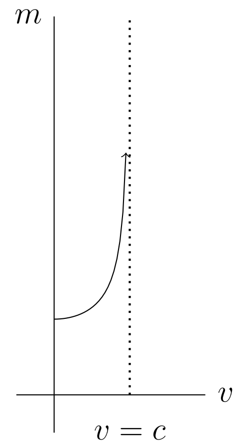
$$m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \text{where } c \text{ is the speed of light}$$

(a) Evaluate and interpret $m(0)$ and $m(c/2)$ in the context of the problem.

Solution: $m(0) = m_0$, which is the mass of the object when its velocity is 0. Meanwhile $m(c/2) = \frac{m_0}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}}m_0 \approx 1.1547m_0$, so travelling at half the speed of light the mass of the object is about 15.47% *more* than at rest.

(b) Evaluate and interpret $\lim_{v\to c} m(v)$ in the context of the problem.

Solution: It helps to draw a graph:



As we can see, the mass goes to $+\infty$ as $v \to c$. So technically speaking, the limit does not exist, though it is more useful to say

$$\lim_{v\to c} m(v) = \infty$$

and thus 'as the velocity of the object goes to the speed of light, the mass goes to infinity'. Pretty weird idea!

For the rest of the problem, suppose $m_0 = 10$ and $c = 1$ Apologies for the typo in this problem! I was supposed to say $c = 1$.

(c) *Approximate* $m'(c/2)$ using your calculator. Start by writing down the difference quotient:

Solution:

$$\Delta_m(h) = \frac{m(c/2+h) - m(c/2)}{h} = \frac{1}{h}\left[\frac{10}{\sqrt{1-(1/2+h)^2}} - \frac{10}{\sqrt{1-(1/2)^2}}\right]$$

Then, fill in the table below with 4 approximations (round to 3 decimal places):

Solution: Make sure you include positive and negative h values!

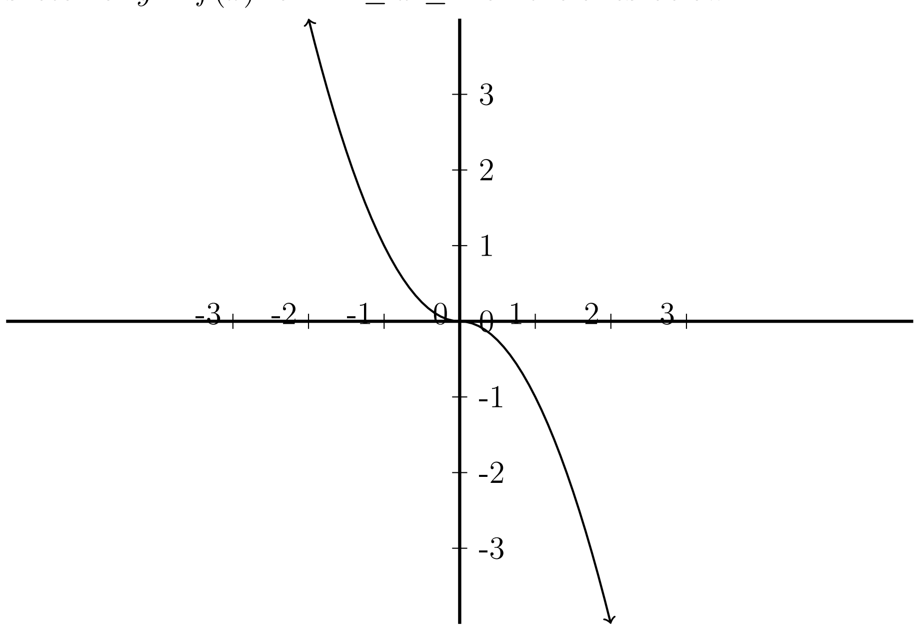
h	-0.001	0.001	-0.0001	0.0001
$\Delta_m(h)$	7.687	7.7134	7.697	7.699

So I estimate that $\boxed{m'(1/2) \approx 7.7}$

5. Consider the following function:

$$f(x) = \begin{cases} -x^2 & x \le 0 \\ x^2 & x > 0 \end{cases}$$

(a) Draw a sketch of $y = f(x)$ for $-2 \le x \le 2$ on the axes below:



(b) Show that $f(x)$ is differentiable at 0 by using the limit definition of $f'(0)$.

Solution: We must show that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists, meaning that we take the right hand and left hand limits and show they are the same. So, first consider the left hand limit. As $h < 0$, $f(0+h) = -(0+h)^2 = -h^2$ and thus we have

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^2}{h} = \lim_{h\to 0^-} -h = 0$$

Now, if we take the limit from the right, we have $h > 0$ and so $f(0+h) = (0+h)^2 = h^2$ and so

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

Since the left hand limit equals the right hand limit (they are both zero), the limit $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists and so f is differentiable at 0.

(c) What is $f'(x)$? (Hint: like f, f' is a piecewise function).

Solution: Each 'piece' of the piecewise function f is simply a quadratic, which we know how to take the derivative of. And, since we know that f is differentiable at 0, and $f'(0) = 0$, we can claim that

$$f'(x) = \begin{cases} -2x & x < 0 \\ 0 & x = 0 \\ 2x & x > 0 \end{cases}$$

If you look closely, this is exactly the function $y = |2x|$!

(d) Does $f''(0)$ exist?

Solution: No - as we have discussed in class, the derivative of the absolute value function doesn't exist at 0 because there is a sharp corner there.

6. For the following questions, give a *brief* explanation and/or sketch to justify for your answer.

   (a) If a function is continuous at $x = a$, it is differentiable at a.

   TRUE FALSE

   Solution: False - take $f(x) = |x|$ and $x = 0$. Then, f is continuous at 0 but *not* differentiable at 0.

   (b) If a function is *not* differentiable at $x = a$, then x is *not* continuous at $x = a$.

   TRUE FALSE

   Solution: False - take the same example as part (a).

   (c) If $f''(a) = 0$, then $f(x)$ changes concavity at $x = a$.

   TRUE FALSE

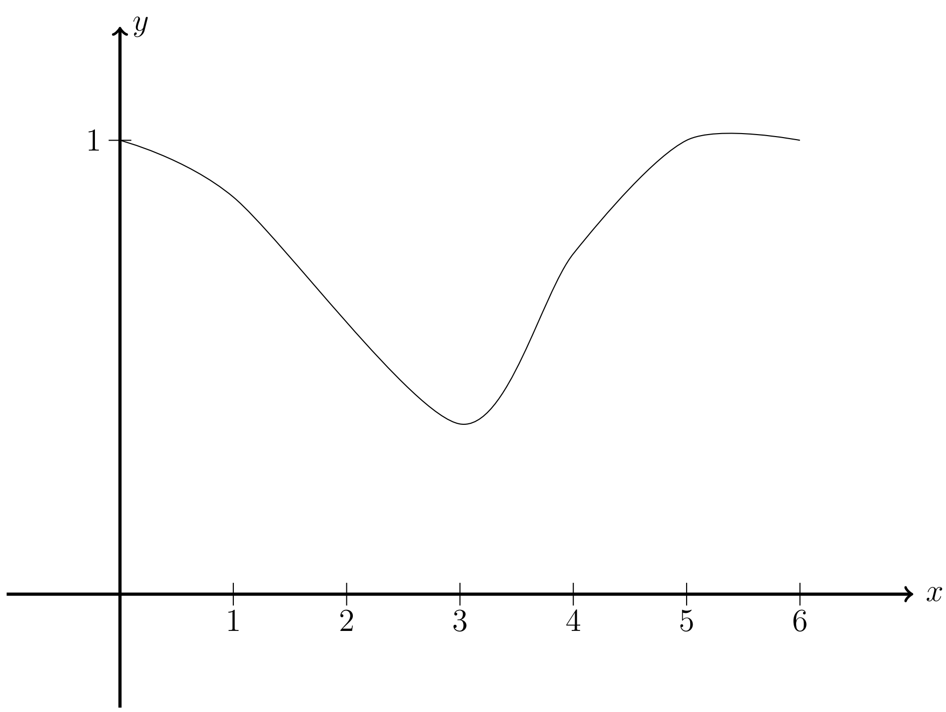
   Solution: False - this *can* be the case - i.e. a can be an inflection point, but take for example $f(x) = x^4$. $f''(x) = 12x^2$, and so $f''(0) = 0$ but $f(x)$ is always concave up (the second derivative is always positive!)

7. On the axes below, draw the graph of a function $y = f(x)$ with the following properties:

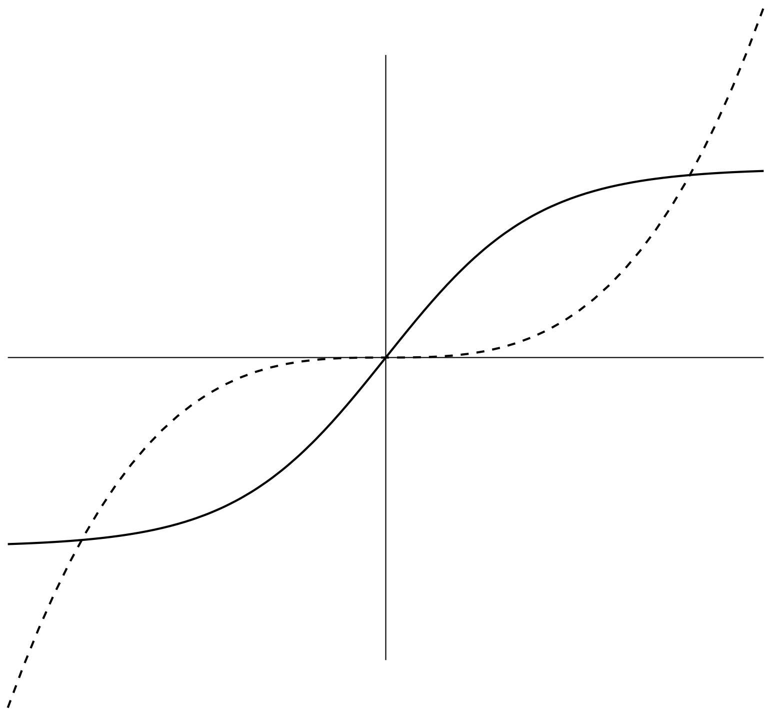
- $f(0) = f(6) = 1$
- $f'(0) = f'(5) = 0$
- $f'(x) < 0$ on $(0, 3)$
- $f'(3) = 0$
- $f'' < 0$ on $(0, 2)$ and $(4, 6)$

Solution: Below is one possibility:

x	0	1	2	3	4	5	6
y	1	0.8	1/2	1/4	0.6	1	1



8. Using the graph of $y = f''(x)$ shown below, sketch a possible graph of $y = f(x)$ on the same axes. You may assume that $f(0) = 0$.



Solution: The function must be concave down for negative x and concave up for positive x (this is really all I was looking for). There are many possibilities.