Math 122B

Exam III

April 15th 2014

The University of Arizona

Name: KEY

Answers without adequate justification will not receive full credit, including multiple choice. Include units with your answer when appropriate, and box all answers unless an answer line is provided. By signing below I am agreeing to abide by the University of Arizona academic integrity policies and that all work done on this test is my own.

Signature: ______________________

Tips for Success:

- Look through the entire test before starting to prioritize questions.
- If you get stuck on a question, move on and come back to it later.
- Do a quick reality check after each question: does my answer make sense? Did I include units? Did I show all my work?
- Read over the entire test at the end to make sure you didn't miss anything.
- For each question: take a deep breath, think slowly and deliberately at first, then work quickly once you see what to do.

Requests:

- Show all your steps.
- Please box answers when possible.

1. Let $f(x)$ be a **continuous** function with domain $\mathbb{R}$. Below is a sign chart for $f(x)$ and its first two derivatives. 'UND' means undefined, '+' means *strictly* bigger than 0, '−' means *strictly* less than 0, and 0 means 0.

$f''(x)$	$-$	0	$-$	UND	$-$	0	$+$
$f'(x)$	$+$	0	$-$	UND	$+$	0	$+$
$f(x)$	$-$	$-$	$-$	$-$	$-$	0	$+$
x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$

(a) Classify the critical points of $f(x)$ using the sign chart.

$X=-1$: f': $+$ 0 $-$ at $x=-1$; / \ Local Max

$X=0$: f': $-$ UND $+$ at $x=0$; \ / Local min

$X=1$: f': $+$ 0 $+$ at $x=1$; / / Neither

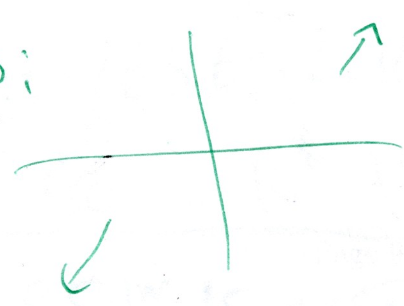
(b) According to the chart, does $f(x)$ have any inflection points? If so, where and why?

Yes - $X=1$ is an inflection PT since $f''(x)$ changes from $-$ to $+$.

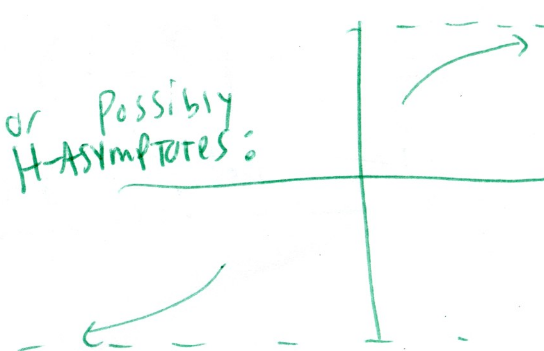
$f''(x)$: $-$ 0 $+$ at $x=1$; $\cap$ $\cup$

(c) According to the sign chart, do you think $f(x)$ will have a global maximum or minimum? Remember, the domain of $f(x)$ is $\mathbb{R}$.

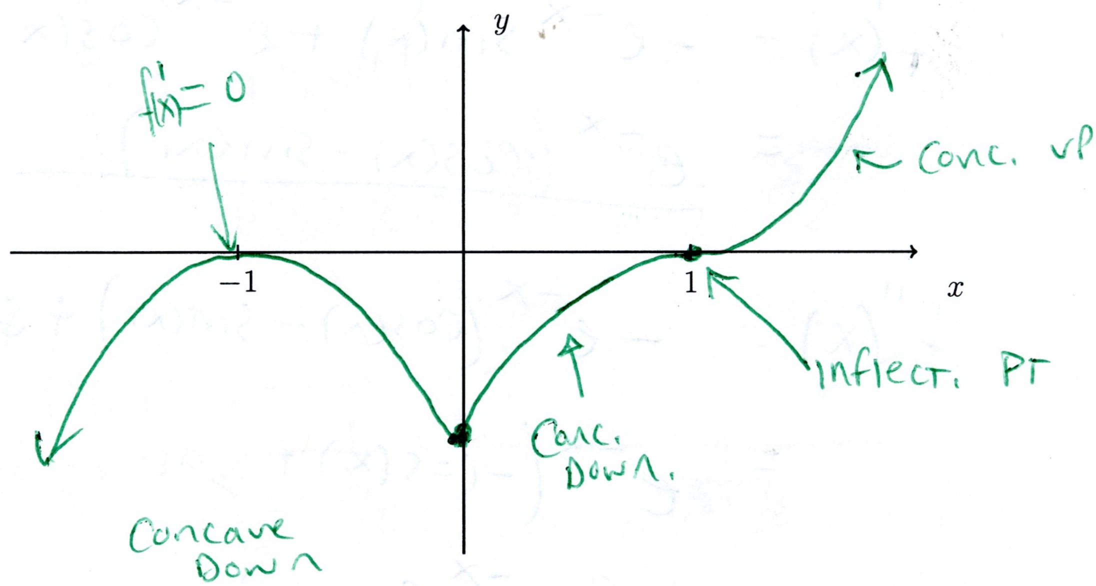
NO - Domain is $\mathbb{R}$ & $f'(x)$ is positive for $X<-1$ & $X>1$, so "end behavior" is like This:



or Possibly H-Asymptotes:



(d) Draw a reasonable sketch of $y = f(x)$ given the information above. Hint: $f(x)$ will have two pieces, $x < 0$ and $x > 0$, with a corner at $x = 0$. Exact values are not necessary unless they are specified in the sign chart.



2. Find values of the constants $a > 0$ and $b > 0$ so that $g(t) = ate^{-bt}$ has a local maximum at $(t, y) = (2, 3)$.

$$g'(T) = ae^{-bT} - abTe^{-bT}$$
$$= ae^{-bT}(1 - bT)$$

So $g'(T) = 0$ if $1 = bT \Rightarrow T = \frac{1}{b}$

So, want $\frac{1}{b} = 2$ or $\boxed{b = 1/2} \Rightarrow g(T) = aTe^{-T/2}$

Then, want $g(2) = 3$ so

$g(2) = a \cdot 2e^{-1} = 3 \Rightarrow \boxed{a = \frac{3e}{2}}$

Sign Chart for $g'(t)$: (To check if Lmax) (Remember $a>0$ & $b>0$)

$g'(T)$: + 0 −, T: $1/b$

3. Let $f(x) = e^{-x}\sin(x)$ for $0 \le x \le 2\pi$.

(a) Find and simplify $f'(x)$ and $f''(x)$.

$$f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x)$$
$$= \underline{e^{-x}(\cos(x) - \sin(x))}$$

$$f''(x) = -e^{-x}(\cos(x) - \sin(x)) + e^{-x}(-\sin(x) - \cos(x$$
$$= e^{-x}(-\cos(x) + \cancel{\sin(x)} - \cancel{\sin(x)} - \cos(x$$
$$= \underline{-2e^{-x}\cos(x)}$$

(b) Find the critical points of f, and use the first or second derivative test to classify them. You must show work - provide a sign chart or values of f''.

$f'(x) = 0$ if $\cos(x) = \sin(x)$. On $[0, 2\pi$

This happens when $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$

$f''\left(\frac{\pi}{4}\right) = -2e^{-\pi/4}\cos\left(\frac{\pi}{4}\right) < 0$ Conc. Down,

$\left(\text{since } \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} > 0\right)$ $\boxed{\text{Local max}}$ $\boxed{@ x = \frac{\pi}{4}}$

$f''\left(\frac{5\pi}{4}\right) = -2e^{-\frac{5\pi}{4}}\cos\left(\frac{5\pi}{4}\right) > 0$ Conc up,

$\left(\text{since } \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} < 0\right)$ $\boxed{\text{Local min}}$ $\boxed{@ x = \frac{5\pi}{4}}$

(c) Find the inflection points of $f(x)$.

$f''(x) = -2e^{-x}\cos(x)$ so f'' is defined everywhere on $[0, 2\pi]$.

$f''(x) = 0$ if $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Sign chart:

$f''(x)$	$-$	0	$+$	0	$-$
x		$\pi/2$		$3\pi/2$	

Both $\frac{\pi}{2}$ & $\frac{3\pi}{2}$ are inflect. PTS.

(d) Find the global maximum and minimum of $f(x)$. You must show work by computing function values. $\boxed{\text{Crit PTS: } \pi/4 \text{ \& } 5\pi/4}$

$f\left(\frac{\pi}{4}\right) = e^{-\pi/4}\sin(\pi/4) \approx 0.322$

GLOBAL Max @ $x = \pi/4$

$f\left(\frac{5\pi}{4}\right) = e^{-5\pi/4}\sin\left(\frac{5\pi}{4}\right) \approx -0.322$

Global min @ $x = \frac{5\pi}{4}$

EndPTS:

$f(0) = e^0 \sin(0) = 1 \cdot 0 = 0$

$f(2\pi) = e^{-2\pi}\sin(2\pi) = 0$

4. An object at a distance p from a thin glass lens produces an image at a distance q from the lens according to the *thin lens formula*:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

The constant f is called the *focal length* of the lens. Suppose an object is moving at 2mm per second towards a lens of focal length 10mm. How fast and in what direction (towards or away from the lens) is the *image* moving at the moment when the object is 50mm from the lens? Hint: we are looking for $\frac{dq}{dt}$.

Leave eqn as-is to compute $\frac{dq}{dt}$!

$$\frac{d}{dt}\left[\frac{1}{p} + \frac{1}{q}\right] = \underbrace{\frac{d}{dt}\left[\frac{1}{f}\right]}_{0 \text{ since } f \text{ is constant}}$$

$$\frac{d}{dt}\left[p^{-1} + q^{-1}\right] = 0$$

$$-p^{-2}\frac{dp}{dt} - q^{-2}\frac{dq}{dt} = 0$$

So $$-\frac{1}{q^2}\frac{dq}{dt} = \frac{dp}{dt}\cdot\frac{1}{p^2}$$

$$\Rightarrow \frac{dq}{dt} = -\frac{dp}{dt}\frac{q^2}{p^2}$$

If $f = 10$ mm & $p = 50$ mm, then

$$\frac{1}{50} + \frac{1}{q} = \frac{1}{10}$$

$$\Rightarrow \frac{1}{q} = \frac{5}{50} - \frac{1}{50}$$

$$\Rightarrow 1/q = 4/50$$

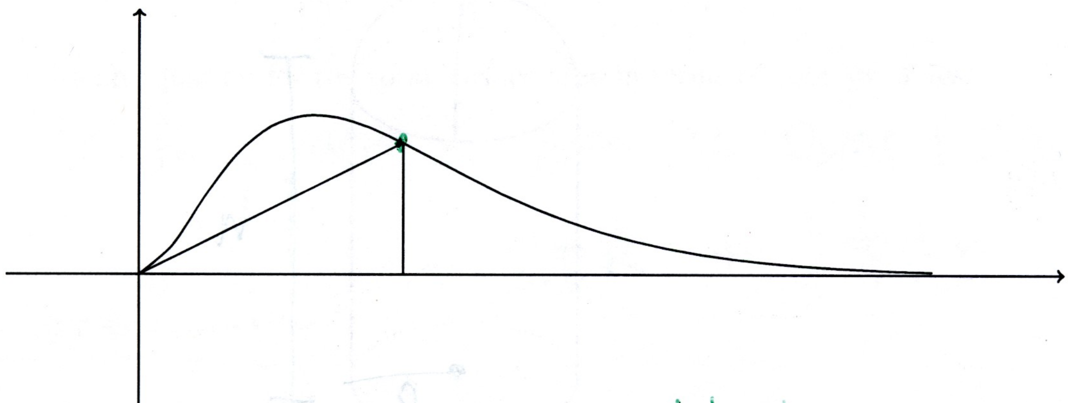
$$\Rightarrow q = \frac{50}{4} = \frac{25}{2}$$

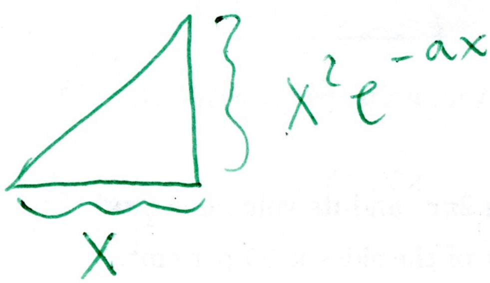
If $\frac{dp}{dt} = 2$ mm/s, $p = 50$mm & $q = \frac{25}{2}$mm,

$$\frac{dq}{dt} = -2\cdot\frac{\left(\frac{25}{2}\right)^2}{50^2} = -2\cdot\frac{\frac{25^2}{4}}{4\cdot 25^2} = \frac{-2}{16} = \boxed{-\frac{1}{8}}$$

So img is moving @ $\frac{1}{8}$ mm/s ~~[illegible]~~ Towards ~~[illegible]~~ the lens.

5. The hypotenuse of a right triangle has one end at the origin $(x, y) = (0, 0)$ and the other end on the graph of $y = x^2 e^{-ax}$, i.e. $(x, y) = (x, x^2 e^{-ax})$ - see picture below. Here $a > 0$ is a constant. Find the maximum possible area of this triangle if $x \geq 0$. Your answer will depend on a.





$\text{Area} = \frac{1}{2} b \cdot h$

$A(x) = \frac{1}{2} x^3 e^{-ax}, \quad x \geq 0$

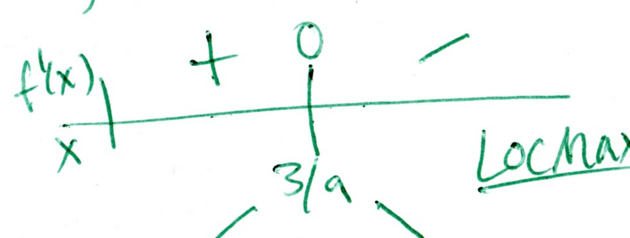
$A'(x) = \frac{3}{2} x^2 e^{-ax} - \frac{a}{2} x^3 e^{-ax}$

$= e^{-ax} \left(\frac{3}{2} x^2 - \frac{a}{2} x^3 \right)$

So $A'(x) = 0$ if $\frac{3}{2} x^2 - \frac{a}{2} x^3 = 0$

$\frac{x^2}{2}(3 - ax) = 0$

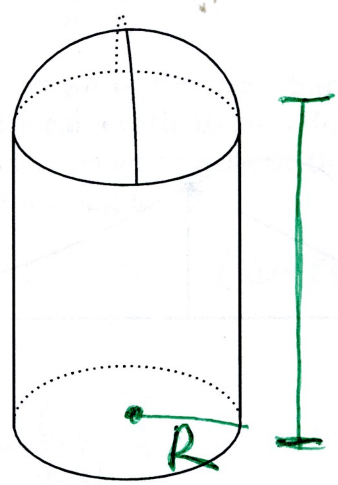
So $x = 0$ or $x = 3/a$



$\boxed{\begin{array}{l} \text{ENDPTS: } A(0) = 0 \\ \lim_{x \to \infty} A(x) = 0 \end{array}}$

So MAX Area is $\boxed{A(3/a) = \frac{1}{2}\left(\frac{3}{a}\right)^3 e^{-3}}$

6. The goal is to design our very own R2D2 robot at **minimal construction cost**. If you haven't seen the original Star Wars, make it a priority. This robot looks roughly like a cylinder with a hemispherical dome on top:



We will use the following information:

- The dome is a perfect hemisphere (half-sphere).
- The surface area of a hemisphere with radius r is $2\pi r^2$ and its volume is $\frac{2}{3}\pi r^3$.
- The cost of the dome is \$10 per cm^2 and the cost of the sides is \$5 per cm^2.
- We will neglect the cost of the bottom.
- To house the necessary electronics, we require that the volume of the robot be $0.1\text{m}^3 = 100,000\text{cm}^3$.

(a) Write down, **in words**, exactly what we are trying to minimize. Break it into relevant terms.

Want to minimize Cost of Robot

= Cost of sides + cost of Dome.

(b) Define variables and label the picture on the previous page.

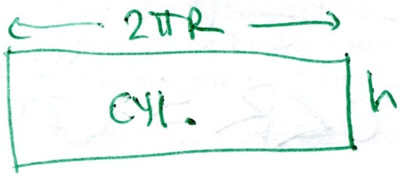
$R \sim$ Radius

$h \sim$ height of cyl. section

(c) Write an equation for the total surface area in terms of your variables.

Tot surface area = Area of Dome + Area of Cyl

$$= 2\pi R^2 + 2\pi R h$$



(d) Write an equation for the total *cost* in terms of your variables. Hint: $\$ = \frac{\$}{\text{cm}^2} \cdot \text{cm}^2$.

Tot Cost = Cost of Dome + Cost of Cyl

$$= 10 \cdot 2\pi R^2 + 5 \cdot 2\pi R h$$

$$= 20\pi R^2 + 10\pi R h$$

(e) Write an equation for the total volume in terms of your variables. Remember, the total volume is *fixed* at $100,000\text{cm}^3$.

Tot Vol = Vol Cyl + Vol Dome

$$V = \pi R^2 h + \frac{2}{3}\pi R^3$$

So

$$100,000 = \pi R^2 h + \frac{2}{3}\pi R^3$$

Solve for h

$$\Rightarrow h = \frac{100\,000 - \frac{2}{3}\pi R^3}{\pi R^2}$$

(f) Use the equations found above to write a *function* of a single variable expressing the total construction cost. Don't forget domain!

$$C(R) = 20\pi R^2 + 10\cancel{\pi} R\left(\frac{100000 - \frac{2}{3}\pi R^3}{\cancel{\pi} R^{\cancel{2}}}\right)$$

$h = 0$ if $R = \left(\frac{300000}{2\pi}\right)$

$$= 20\pi R^2 + 10\left(\frac{100000 - \frac{2}{3}\pi R^3}{R}\right)$$

$$= 20\pi R^2 + \frac{1000000}{R} - \frac{20}{3}\pi R^2$$

$$= \boxed{\frac{40}{3}\pi R^2 + \frac{1000000}{R}} \qquad 0 < R \leq \left(\frac{300000}{2\pi}\right)^{1/3}$$

(g) **Using calculus**, find the robot dimensions (radius and height) that minimize total cost. You may round your answer to three decimal places, but you must use calculus!

$$C'(R) = \frac{80\pi}{3} R - \frac{1000000}{R^2}$$

$$C'(R) = 0 \text{ if } \frac{80\pi}{3} R^3 = 1{,}000{,}000$$

$$\rightarrow R^* = \left(\frac{3{,}000{,}000}{80\pi}\right)^{1/3} \approx 22.85$$

$$C(R^*) = 65{,}634.29$$

End PTS:

$C(0)$ "=" ∞

$$C\left(\left(\frac{300000}{2\pi}\right)^{1/3}\right) = 268682.161 > C(R^*)$$

So R^* is the global min

Dims:

$$\boxed{(R, h) \approx (22.854\text{mm}, 15.235\text{mm})}$$