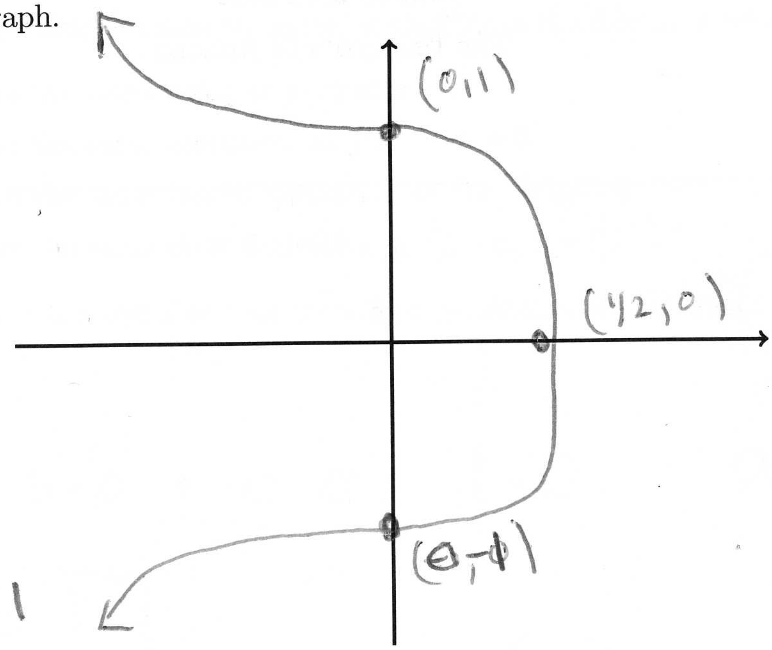
1. Consider the implicit curve defined by $(ax)^3 + y^2 = 1$, where $a > 0$.

   (a) Plot the curve when $a = 2$. Hint: solve for y to obtain *two* explicit functions describing the top and bottom half. You may use your calculator, but be sure to label your graph.



$$8x^3 + y^2 = 1$$

$$\Rightarrow y^2 = 1 - 8x^3$$

$$\Rightarrow y = \pm\sqrt{1-8x^3}$$

   (b) Find $\frac{dy}{dx}$.

(Assuming $a=2$ is O.K.)

$$\frac{d}{dx}\left[8x^3 + y^2\right] = \frac{d}{dx}[1]$$

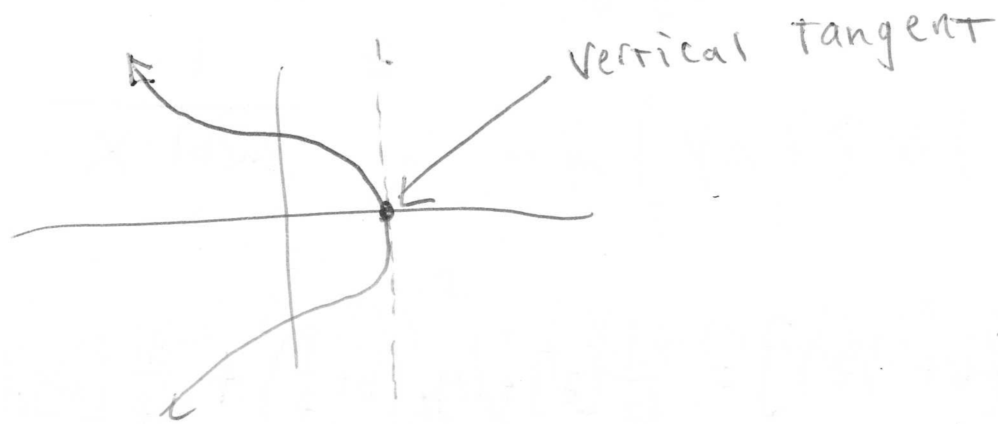
$$\Rightarrow 8\frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^2\right] = 0$$

$$\Rightarrow 24x^2 + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cancel{24}^{12}x^2}{\cancel{2}y} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-12x^2}{y}}$$

(c) Where is $\frac{dy}{dx}$ undefined? Use the graph from part (a) to explain *why* this derivative is not defined.

$\frac{dy}{dx}$ is undefined when $y = 0$.



(d) Find the equation of the tangent line to the curve when $y = 2$.

When $y = 2$, we have

$$8x^3 + y^2 = 1 \Rightarrow 8x^3 + (2)^2 = 1$$
$$\Rightarrow 8x^3 = -3$$
$$\Rightarrow x^3 = -\frac{3}{8}$$
$$\Rightarrow x = \left(\frac{-3}{8}\right)^{1/3}$$

This is a real # since we're taking an odd root

So, $\frac{dy}{dx} = \frac{-12(-3/8)^{2/3}}{2}$

$= -6\left(\frac{9}{16}\right)^{1/3}$

So,

$$y = y_0 + m(x - x_0)$$

$$\boxed{y = 2 + \underbrace{\left(-6\left(\tfrac{9}{16}\right)^{1/3}\right)}_{m}\left(x - \left(\tfrac{-3}{8}\right)^{1/3}\right)}$$

2. Compute the following derivatives and simplify your answer.

(a) $\frac{d}{dz}[z\log_2(b+1/z)]$

"Change of base"

Recall That $\frac{d}{dx}[\log_b(x)] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(b)}\right]$

$= \frac{1}{\ln(b)\cdot x}$

So,

$\frac{d}{dz}[z\log_2(b+1/z)] = \frac{d}{dz}[z]\cdot\log_2(b+\frac{1}{z}) + \frac{d}{dz}[\log_2(b+\frac{1}{z})]\cdot z$

Chain rule!

$= \log_2(b+\frac{1}{z}) + \frac{1\cdot\boxed{-z^{-2}}}{\ln(2)(b+\frac{1}{z})}\cdot z$

$= \log_2(b+\frac{1}{z}) - \frac{\cancel{z}}{z^{\cancel{2}}\ln(2)(b+\frac{1}{z})}$

$= \log_2(b+\frac{1}{z}) - \frac{1}{z\ln(2)(b+1/z)}$

(b) $\frac{d}{da}\left[\frac{R}{Ya+\sinh(1+a^2)}\right] = \frac{d}{da}\left[R\left(\underbrace{\boxed{Ya+\sinh(1+a^2)}}_{\star}\right)^{-1}\right]$

$$= R\frac{d}{da}\left[\left(Ya+\sinh(1+a^2)\right)^{-1}\right]$$

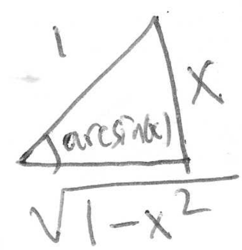
$$= -R\left(Ya+\sinh(1+a^2)\right)^{-2}\cdot\frac{d}{da}\left[\star\right]$$

$$= -R\left(Ya+\sinh(1+a^2)\right)^{-2}\cdot\left(Y+\cosh(1+a^2)\cdot\underbrace{2a}_{\text{Chain rule}}\right)$$

$$= \boxed{\frac{-R\left(Y+\cosh(1+a^2)\cdot 2a\right)}{\left[Ya+\sinh(1+a^2)\right]^2}}$$

(c) $A'(\theta)$ if $A(\theta) = \arcsin(\theta^{2/3})$.

Recall $\frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\frac{d}{dy}\left[\sin(y)\right]\Big|_{y=\arcsin(x)}} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$



So,

$$A'(\theta) = \frac{1}{\sqrt{1-(\theta^{2/3})^2}}\cdot\underbrace{\frac{d}{d\theta}\left[\theta^{2/3}\right]}_{\text{Chain rule}}$$

$$= \frac{\frac{2}{3}\theta^{-1/3}}{\sqrt{1-\theta^{4/3}}}$$

$$= \boxed{\frac{2}{3\theta^{1/3}\sqrt{1-\theta^{4/3}}}}$$

(d) $(y^{-1})'(t)$ if $y(t) = t^{1/3}$ $\boxed{y^{-1}(t) = t^3}$

Method 1:

$$(y^{-1})'(t) = \frac{1}{y'(y^{-1}(t))} = \frac{1}{\frac{1}{3}(t^3)^{-2/3}}$$

$$= \frac{3}{t^{-2}} = 3t^2$$

Method 2:

$y^{-1}(t) = t^3$ so $(y^{-1})'(t) = 3t^2$

(much easier!)

(e) y' if $y = x^{x^2}$. Hint: take a logarithm of both sides of $y = x^{x^2}$.

$$\ln(y) = \ln(x^{x^2}) = x^2 \ln(x)$$

So $$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[x^2 \ln(x)]$$

$$\Rightarrow \frac{y'}{y} = \frac{d}{dx}[x^2] \cdot \ln(x) + \frac{d}{dx}[\ln(x)] \cdot x^2$$

$$= 2x \ln(x) + \frac{1}{x} \cdot x^2$$

$$\Rightarrow \frac{y'}{y} = 2x \ln(x) + x \quad (y = x^{x^2})$$

$$\Rightarrow \boxed{y' = x^{x^2}(2x \ln(x) + x)}.$$

3. Use the table below to evaluate the following:

x	0	1	2	3	4	5
$f(x)$	-1	2	1	5	-2	2.5
$f'(x)$	1.5	4.1	-4.2	7.5	3	-1

(a) $(f^{-1})'(2)$

$(f(1) = 2 \text{ so } f^{-1}(2) = 1)$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)}$$

$$= \boxed{\frac{1}{4.1}}$$

(b) $Q'(3)$ if $Q(x) = \ln((f(x))^2)$.

Note: $Q(x) = 2\ln(f(x))$ by $\ln(a^b) = b\ln(a)$

$$Q'(x) = \frac{d}{dx}\left[2\ln(f(x))\right] = \frac{2f'(x)}{f(x)}$$

So

$$Q'(2) = \frac{2f'(2)}{f(2)} = \frac{2}{-4.2} = \boxed{\frac{-1}{2.1}}$$

4. Let $f(x) = \exp(-\cosh(x))$ (note: $\exp(y) = e^y$).

(a) Find the local linearization (aka tangent line approximation) for $f(x)$ at $x = 1$.

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$$f'(x) = \frac{d}{dx}\left[e^{-\cosh(x)}\right]$$

$$= e^{-\cosh(x)} \cdot \frac{d}{dx}\left[-\cosh(x)\right]$$

$$= e^{-\cosh(x)}(-\sinh(x))$$

$$= -\sinh(x)\, e^{-\cosh(x)}$$

So

$$f'(1) = -\sinh(1)\, e^{-\cosh(1)}$$

$$= -\left(\frac{e^1 - e^{-1}}{2}\right) e^{-\left(\frac{e^1 + e^{-1}}{2}\right)}$$

$$y = f(1) + f'(1)(x-1)$$

$$\Rightarrow \boxed{y = \exp(-\cosh(1)) - \sinh(1)\exp(-\cosh(1)) \cdot (x-1)}$$

(b) Is the local linearization an *over* estimate or *under* estimate near $x = 1$? Explain using calculus.

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left[-\sinh(x)\exp(-\cosh(x))\right]$$

$$= -\left(\frac{d}{dx}[\sinh(x)]\exp(-\cosh(x)) + \frac{d}{dx}[\exp(-\cosh(x))]\sinh(x)\right)$$

$$= -\left(\cosh(x)\exp(-\cosh(x)) + -\sinh^2(x)\exp(-\cosh(x))\right)$$

$$= -\cosh(x)\exp(-\cosh(x)) + \sinh^2(x)\exp(-\cosh(x))$$

So $f''(1) \approx -0.035 < 0$ so concave down
(calculator)

$\boxed{\text{OVER est}}$

5. Answer the following true/false questions; no explanation is required.

(a) $\frac{d}{dx}[f(x)g(h(x))] = f'(x)g(h(x)) + f(x)g'(h(x))$

TRUE FALSE

Missing chain rule for $\frac{d}{dx}[g(h(x))]$.

(b) Suppose $f(x)$ is differentiable on $(0, 1)$. Then, there exists a c with $0 < c < 1$ such that $f'(c) = f(1) - f(0)$

TRUE FALSE

Not necessarily - We need $f(x)$ to be continuous on $[0,1]$.

(c) If $x = y^{2/3}$, then $\frac{dy}{dx} = \frac{3}{2}y^{1/3}$.

TRUE FALSE

$\frac{d}{dx}[x] = \frac{d}{dx}[y^{2/3}]$ so $1 = \frac{2}{3}y^{-1/3}y'$

$\Rightarrow y' = \frac{3}{2}y^{1/3}$ ✓

6. Suppose we want to approximate $f(x) = e^x$ with a *cubic* function near $x = 0$, i.e.

$$e^x \approx a + bx + cx^2 + dx^3$$

We would like this cubic function to 'agree' with $f(x)$ in the following ways:

1. It should have the same *value* as $f(x)$ at $x = 0$
2. It should have the same *derivative* as $f(x)$ at $x = 0$.
3. It should have the same *second* derivative as $f(x)$ at $x = 0$
4. It should have the same *third* derivative as $f(x)$ at $x = 0$.

Find the values of a, b, c and d so that these four conditions are satisfied.

1. $f(0) = e^0 = a + b\cdot 0 + c\cdot 0 + d\cdot 0 = a$

So $\boxed{a = 1}$

2. $f'(0) = e^0 = \frac{d}{dx}\left[a + bx + cx^2 + dx^3\right]\Big|_{x=0}$

$= b + 2c(0)^2 + 3d(0)^2 = b$

So $\boxed{b = 1}$

3. $f''(0) = e^0 = (2c + 6d(0)) = 2c$

So $2c = 1$ or $\boxed{c = \frac{1}{2}}$

$f'''(0) = e^0 = 6d$ so $\boxed{d = \frac{1}{6}}$