Math 122B

Exam I

February 27th 2014

The University of Arizona

Name: KEY

Answers without adequate justification will not receive full credit, including multiple choice. Include units with your answer when appropriate, and box all answers unless an answer line is provided. By signing below I am agreeing to abide by the University of Arizona academic integrity policies and that all work done on this test is my own.

Signature: C.F. GAUSS

Tips for Success:

- Look through the entire test before starting to prioritize questions.
- If you get stuck on a question, move on and come back to it later.
- Do a quick reality check after each question: does my answer make sense? Did I include units? Did I show all my work?
- Read over the entire test at the end to make sure you didn't miss anything.
- For each question: take a deep breath, think slowly and deliberately at first, then work quickly once you see what to do.

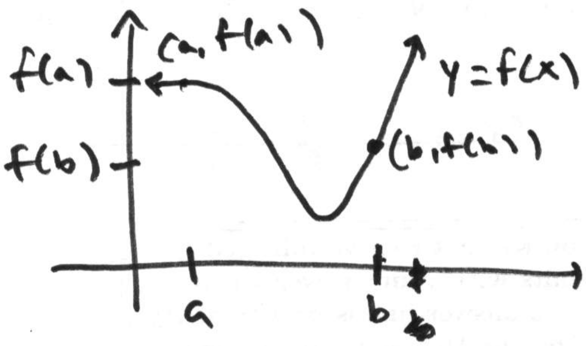
Requests:

- Show all your steps.
- Please box answers when possible.

1. For the following questions, give a *brief* explanation and/or sketch to justify your answer.

   (a) If $f'(a) < f'(b)$, then $f(a) < f(b)$.

   TRUE FALSE

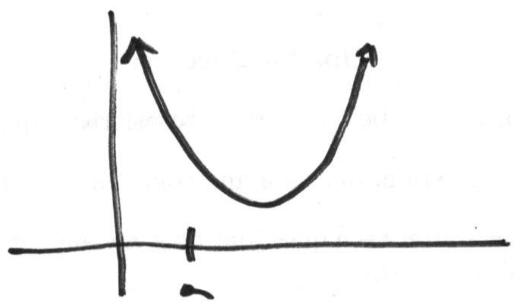


$f'(a) < f'(b)$ but $f(b) < f(a)$

   (b) If $f'(a) < 0$, $f''(a) < 0$.

   TRUE FALSE

decr. conc. down



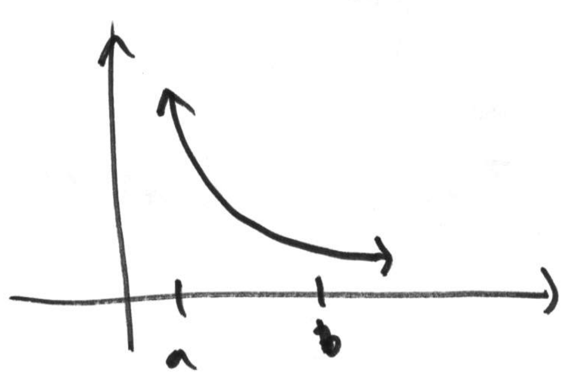
$f'(a) < 0$ but $f''(a) > 0$.

   (c) If $f''(x) > 0$ on the interval (a, b), then $f(x)$ is increasing on the interval (a, b).

   TRUE FALSE

conc. up

$f' > 0$



$f''(x) > 0$ on (a,b) but f is <u>decreasing</u> on (a,b)

2. Let $f(x) = \sqrt{ax}$, where $a > 0$ is constant.

(a) *Using the limit definition*, compute $f'(x)$. Your answer will involve a; no points will be given for just using the power rule (but it's a good way to check your answer!).

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{a(x+h)} - \sqrt{ax}}{h} \cdot \frac{\sqrt{a(x+h)} + \sqrt{ax}}{\sqrt{a(x+h)} + \sqrt{ax}}$$

$$= \lim_{h\to 0} \frac{a(x+h) - ax}{h\left(\sqrt{a(x+h)} + \sqrt{ax}\right)}$$

$$= \lim_{h\to 0} \frac{a\cancel{h}}{\cancel{h}\left(\sqrt{a(x+h)} + \sqrt{ax}\right)}$$

$$= \lim_{h\to 0} \frac{a}{\sqrt{a(x+h)} + \sqrt{ax}} = \frac{a}{2\sqrt{ax}} = \frac{\sqrt{a}}{2\sqrt{x}}$$

Pwr rule: $f = \sqrt{a}\sqrt{x}$ so $f' = \sqrt{a} \cdot \frac{1}{2\sqrt{x}} =$ ✓

(b) Suppose $a = 2$. *Estimate* $f'(3)$. Use your calculator, but include a table of values.

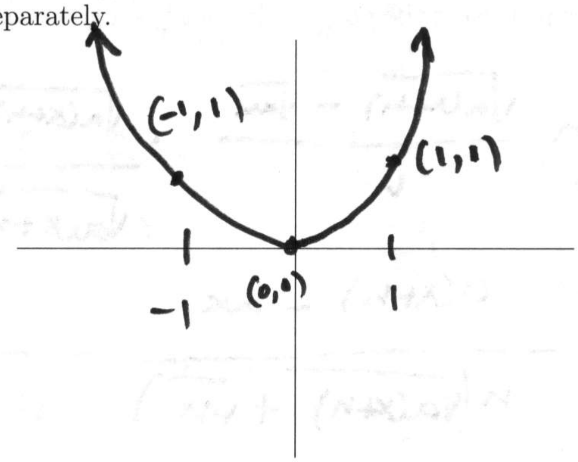
$f(x) = \sqrt{2x}$ $\qquad \dfrac{f(3+h) - f(3)}{h}$ for small h, $+$ & $-$

h	-0.01	0.01	-0.001	0.001
$\frac{f(3+h)-f(3)}{h}$	.4085	0.4079	0.4083	0.4082

So $f'(3) \approx 0.408$

3. Let $F(t) = |t^3|$.

(a) Sketch a graph of $y = F(t)$ on the axes below, and label 3 points. Hint: consider $t < 0$ and $t > 0$ separately.



(b) Is $F(t)$ differentiable everywhere? If not, pick a point where you think $F'(t)$ does not exist and *explain why it doesn't exist by **estimating** limits.* You may use your calculator, but you must explain your work and include values.

Yes - F(t) is differentiable everywhere.

Explanation: F(t) is a piecewise function: F(t) = t^3 for t >= 0 and F(t) = -t^3 for t<0. Thus F is certainly differentiable for t > 0 and t < 0, since it's just a polynomial for those t values. What about t=0 (the "transition" point)?

To convince someone that F'(0) exists, we must estimate or compute the limit

(F(0+h) - F(0))/h = |h^3| / h

This is a piecewise function which is equal to h^2 for h >= 0 and -h^2 for h<0. So,

lim_{h->0} (F(0+h) - F(0))/h = lim_{h->0} |h^3|/h = 0,

since the limit from the left is zero and the limit from the right is zero.

One could also provide a table of pairs (h, |h^3|/h) to convince someone that the limit is 0.

4. While riding a roller coaster, you record your height (in feet) above the ground at various times t (in seconds). Here are a few of your values:

t	30	31	32	33	34
H	300	210	90	10	50

(a) Compute your average (vertical) velocity from $t = 30$ to $t = 34$.

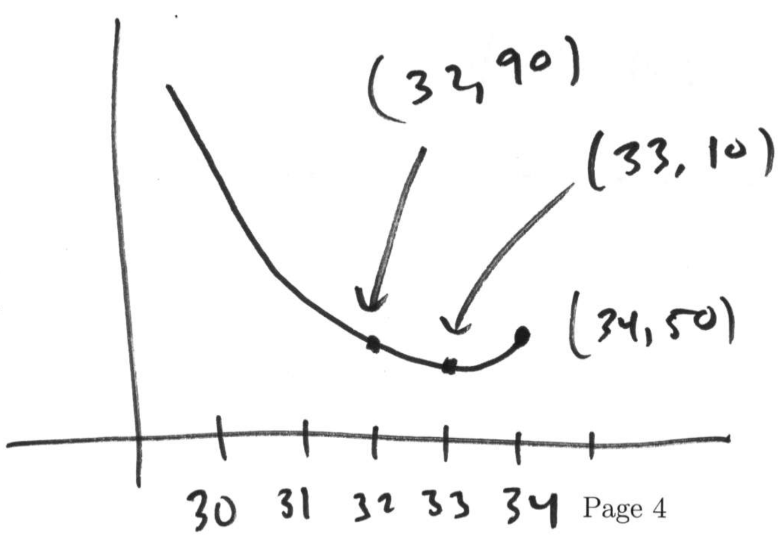
$$\frac{H(34) - H(30)}{4} = \frac{50 - 300}{4} = \frac{-250}{4} = \boxed{-62.5 \text{ ft/s}}$$

(b) Approximate and interpret $H'(32)$.

$$H'(32) \approx \frac{H(33) - H(31)}{2} = \frac{10 - 210}{2} = -100 \text{ ft/s}$$

At 32 seconds, we are dropping @ a rate of 100 feet per second.

(c) What do you think the *sign* of $H''(33)$ is? Explain using a sketch.

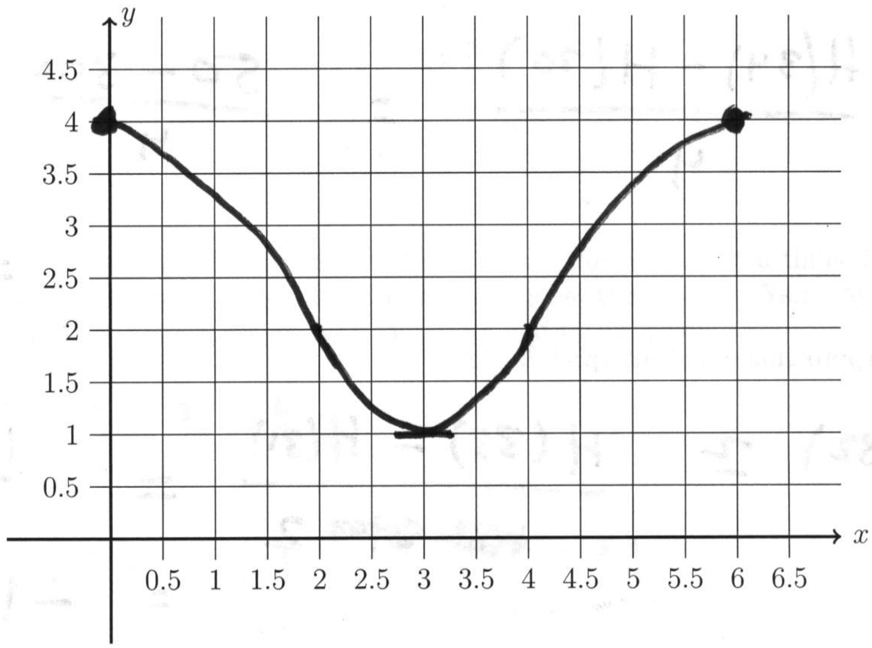


Positive

Concave up.

5. On the axes below, draw the graph of a function $y = f(x)$ with the following properties:

- $f(0) = f(6) = 4$
- $f'(x) < 0$ on $[0, 3)$
- $f'(3) = 0$
- $f'' < 0$ on $[0, 2)$ and $(4, 6]$
- $f'' > 0$ on $(2, 4)$.



6. Find the equation of the tangent line to $y = x^{-1/2} \cdot 2^x$ at $x = 1$.

$f(x) = x^{-1/2} \quad g(x) = 2^x \quad f'(x) = -\frac{1}{2}x^{-3/2} \quad g'(x) = \ln(2)2^x$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2} \cdot 2^x + \ln(2)2^x x^{-1/2} \quad \text{(Prod. rule)}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{2} \cdot 1^{-3/2} \cdot 2^1 + \ln(2) \cdot 2^1 \cdot 1^{-1/2}$$

$$= -\frac{1}{2} \cdot 2 + \ln(2) \cdot 2 \cdot 1$$

$$= -1 + 2\ln(2)$$

$$(x, y) = (1, 1^{-1/2} \cdot 2^1) = (1, 2)$$

So

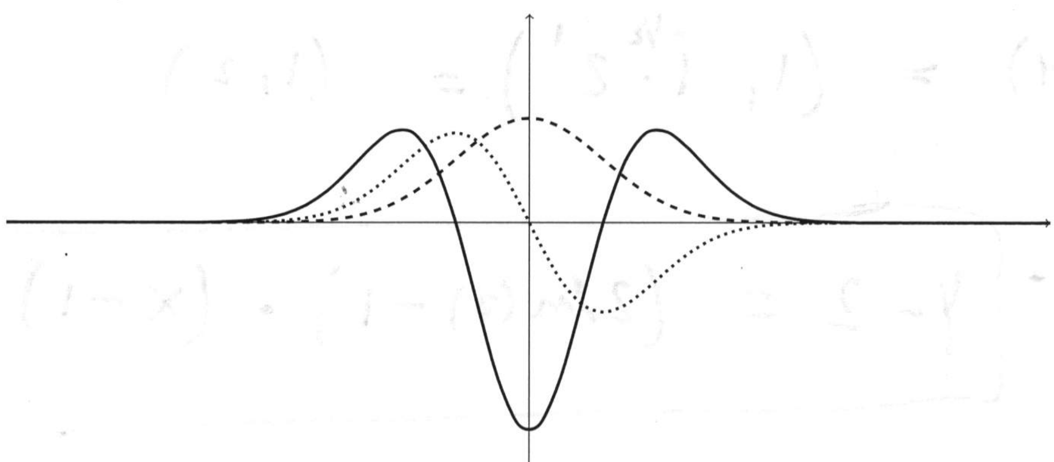
$$\boxed{y - 2 = (2\ln(2) - 1) \cdot (x - 1)}$$

7. Suppose that $g(3) = A$, $h(3) = B$, $g'(3) = 1/2$ and $h'(3) = -1$. Write the equation of the tangent line to the graph $y = 2g(x) - h(x)$ at $x = 3$.

$$y'(3) = 2g'(3) - h'(3)$$
$$= 2 \cdot \tfrac{1}{2} - (-1)$$
$$= 2$$
$$(x, y) = (3, 2 \cdot A - B)$$

so $\boxed{y - (2A - B) = 2(x - 3)}$

8. The graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ are shown below. Identify each graph below using 'solid', 'dashed' or 'dotted'.



$f(x)$: Dashed $f'(x)$: Dotted $f''(x)$: Solid