Cohomology and Geometry for Frobenius kernels of Algebraic Groups

by

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(Under the Direction of Daniel K. Nakano)

Abstract

This dissertation contains many results related to cohomology of Frobenius kernels for algebraic structures. It is divided into four themes. First, we compute the cohomology ring for the first Frobenius kernel of the unipotent radical of a simple algebraic group. New results on computing cohomology of the Frobenius kernels for parabolic subgroups are also obtained in this part. Next, we provide generalizations of all former results to the cohomology of small quantum groups at a root of unity. Third, we focus on cohomology computations for SL_2. Some observations about low degree cohomology for B_r and G_r are given beforehand. Finally, we look at the geometrical aspect of the cohomology for Frobenius kernels, namely nilpotent commuting varieties of r-tuple, and prove various properties in the case of low rank Lie algebras.

Index words: Lie algebra cohomology, Frobenius kernel, Quantum group, Commuting variety

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Contents

Chapter 1

Introduction

1.1 Background and History

1.1.1

In the investigation of cohomology for general algebraic structures like groups and Lie algebras, it is important to understand the cohomology of unipotent and nilpotent algebraic structures. For example, it is well-known that the cohomology of a finite group in characteristic $p > 0$ embeds in the cohomology of any one of its p-Sylow subgroups. On the other hand, the cohomology of general p-groups is not well-understood, which makes the task of computing cohomology for finite groups challenging. In particular, it is not even known how to compute the cohomology of the subgroup of the finite general linear group $GL_n(\mathbb{F}_p)$ consisting of upper triangular unipotent matrices.

For the case of a Lie algebra over a field k of characteristic $p > 0$, far more is known. For example, let $\mathfrak{g}$ be the Lie algebra of a simple algebraic group G over k, let $\mathfrak{p}_J \subset \mathfrak{g}$ be a standard parabolic subalgebra of $\mathfrak{g}$ corresponding to a subset of simple roots J, and let $\mathfrak{u}_J \subset \mathfrak{p}_J$ be the nilradical of $\mathfrak{p}_J$. If $p \geq h - 1$ (where h is the Coxeter number of $\mathfrak{g}$), then an analog of Kostant's famous cohomology formula applies, and one can compute the ordinary Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{u}_J, L(\lambda))$ with coefficients in a simple G-module $L(\lambda)$ having highest weight in the bottom p-alcove (cf. [FP1],[PT],[UGA1]). Moreover, in the case when $L(\lambda)$ is the trivial module k and $J = \emptyset$, then the ring structure of $\mathrm{H}^\bullet(\mathfrak{u}, k)$ is also known.

In Chapter 3, 4, 5 and 6, we demonstrate how to compute cohomology for the infinitesimal Frobenius kernels of unipotent and parabolic algebraic group schemes and of their quantum analogs (i.e., small quantum groups). Our calculations include several new ideas and are highly dependent on the aforementioned calculations for ordinary Lie algebra cohomology and on analogous calculations for quantized enveloping algebras at an ℓ-th root of unity (cf. [UGA2]). To compute cohomology for the Frobenius kernels of parabolic algebraic group schemes, we also rely on deep geometric results established in [KLT] concerning the vanishing of line bundle cohomology for the flag variety.

1.1.2

Given a reduced group G, we can identify $\mathrm{H}^i(G, M)$ with the inverse limit $\varprojlim \mathrm{H}^i(G_r, M)$ under some mild assumptions on G and M [Jan2, I.9]. This has motivated studying the cohomology of Frobenius kernels. Representations and cohomology for the first Frobenius kernels were investigated in the works of Friedlander-Parshall, Andersen-Jantzen and Kumar-Lauritzen-Thomsen (cf. [AJ], [FP1], [FP2], [KLT]). Cohomology for higher Frobenius kernels remains an open problem for a long time. Some progress was recently made by Bendel-Nakano-Pillen on computing the first and second degree cohomology of the r-th Frobenius kernels (cf. [BNP1], [BNP2]). Using a different spectral sequence, we compute some low degree cohomology spaces for G_r in Chapter 7. The results highly depend on the value of prime p. However, in the case when $G = SL_2$, the computation turns out to be very nice for arbitrary $r > 0$ and $p > 2$. We give a complete answer on cohomology of the r-th Frobenius kernels for this special case in Chapter 8. It should be noticed that this work has not been found anywhere in the literature.

1.1.3

We turn our concerns, from Chapter 9, to geometric aspects of the cohomology ring for Frobenius kernels. The first contribution to this trend is from Friedlander and Parshall. They proved that the cohomology ring $\mathrm{H}^\bullet(G_1, k)$ is isomorphic to the coordinate algebra of the nullcone $\mathcal{N} \subset \mathfrak{g}$ when $p \geq 3(h-1)$ [FP2] (hence the variety Spec $\mathrm{H}^\bullet(G_1, k)$ is homeomorphic to $\mathcal{N}$). Andersen and Jantzen improved this result for $p \geq h$ [AJ]. In 1997, Bendel-Friedlander-Suslin [BFS1] showed that there is a homeomorphism from Spec $\mathrm{H}^{\mathrm{ev}}(G_r, k)$ to the commuting variety of r nilpotent elements in $\mathfrak{g}$ (the so-called nilpotent commuting variety of r-tuples in Lie algebra $\mathfrak{g}$).

For a fixed Lie algebra $\mathfrak{g}$, we denote by $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. Then the nilpotent commuting variety $C_r(\mathcal{N})$ over $\mathfrak{g}$ is defined as the collection of all r-tuples of pairwise commuting nilpotent elements in $\mathfrak{g}$. In particular, for a positive integer r, we can write

$$C_r(\mathcal{N}) = \{(v_1, \ldots, v_r) \in \mathcal{N}^r \mid [v_i, v_j] = 0 \text{ with } 1 \leq i \leq j \leq r\}.$$

Without the nilpotency condition, the irreducibility of this variety was studied by Gerstenhaber, Guralnick-Sethuraman, Kirillov-Neretin (cf. [G], [GS], [KN]) in case $\mathfrak{g} = \mathfrak{gl}_n$. Nevertheless, the study about $C_r(\mathcal{N})$ for arbitrary r and Lie algebra $\mathfrak{g}$ remains mysterious even in simple cases. Using techniques in geometry and commutative algebra, we are able to show that $C_r(\mathcal{N})$ is irreducible, Cohen-Macaulay and normal where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{sl}_2$.

When $r = 2$, these varieties are called ordinary commuting varieties. Notice that great deal is known for these commuting varieties. In particular, the variety $C_2(\mathfrak{g})$ over a semisimple Lie algebra $\mathfrak{g}$ was first proved to be irreducible by Richardson in 1979 for characteristic 0. For positive characteristic, Levy showed in his paper [L] that this variety is irreducible under certain mild assumptions on G. In 2003, Premet completely proved the irreducibility of $C_2(\mathcal{N})$ over arbitrary reductive Lie algebra in characteristic 0 or p a good prime for the root system of G, (see Section 2.1 for the definition of a good prime). Explicitly, he showed that the nilpotent commuting variety $C_2(\mathcal{N})$ equals to the union of irreducible components $\mathcal{C}(e_i) = \overline{G \cdot (e_i, \mathrm{Lie}(Z_{[G,G]}(e_i)))}$ where e_i's are representatives of *distinguished nilpotent orbits* in $\mathfrak{g}$ (which contain elements whose centralizers are in the nilpotent cone). In case of simple Lie algebras, the classification of distinguished nilpotent orbits (Bala-Carter theorem, [C, 9.5]) gives us that $C_2(\mathcal{N})$ is irreducible only if $\mathfrak{g}$ is of type A. Then the surjective projection from $C_r(\mathcal{N})$ to $C_2(\mathcal{N})$ implies immediately that $C_r(\mathcal{N})$ is reducible except $\mathfrak{g} = \mathfrak{sl}_n$ for $r > 2$. Hence an interesting question now is whether $C_r(\mathcal{N})$ is irreducible with $r > 2$ in type A.

Up to now, the properties of being Cohen-Macaulay and normal for (ordinary) commuting varieties are in general mysterious [K]. There is a long-standing conjecture saying that the commuting variety $C_2(\mathfrak{g})$ is always normal (cf. [Po], [Pr]). Artin and Hochster claimed that $C_2(\mathfrak{gl}_n)$ is a Cohen-Macaulay integral domain (cf. [K], [MS]). This is verified up to $n = 4$ by the computer program Macaulay [Hr]. There is not much hope for nilpotent commuting varieties since their defining ideals are not radical, thus creating a great difficulty for computer calculations. Premet indicated to me that all components of $C_2(\mathcal{N})$ share the origin 0, so if it is reducible then it can never be normal. Another motivation of this problem has arisen from understanding the punctual Hilbert schemes of smooth algebraic curves (see [BI], [Ba]).

1.2 Main results

1.2.1 Cohomology of infinitesimal unipotent algebraic groups and parabolic subgroups

Let U_J be the unipotent radical of the parabolic subgroup $P_J = L_J \ltimes U_J$ (J a subset of simple roots). Write $(U_J)_1$ for the first Frobenius kernel of U_J. In Chapter 3, we prove for

$\lambda \in C_{\mathbb{Z}}$ that there exists an isomorphism of graded L_J-modules

$$\mathrm{H}^\bullet((U_J)_1, L(\lambda)) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, L(\lambda)).$$

Here $S^\bullet(\mathfrak{u}_J^*)$ is the symmetric algebra on $\mathfrak{u}_J^* := \mathrm{Hom}_k(\mathfrak{u}_J, k)$, generated in degree two. Our result strengthens an observation previously made by Friedlander and Parshall [FP1, Remark 2.7(b)]. In Chapter 4, we prove for $p > 2(h-1)$ that there exists a graded algebra isomorphism

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k),$$

where the algebra structure on the right-hand side is the ordinary tensor product of algebras. This computation answers the 25-year-old problem concerning the ring structure of $\mathrm{H}^\bullet(U_1, k)$. As T-modules, this identification was obtained by Friedlander and Parshall in 1986, provided $p > h$ [FP1, Corollary 2.6]. To obtain the identification of graded algebras, however, the lower bound on $p > 2(h-1)$ cannot in general be improved, as seen from Example 4.2.1.[1] In Chapter 4 we also obtain for $p > 3(h-1)$ the ring structure of the cohomology ring $\mathrm{H}^\bullet((U_J)_1, k)$ for arbitrary $J \subseteq \Pi$.

In Chapter 5, we apply our calculations to obtain new results on the cohomology of $(P_J)_1$ with coefficients in a simple G-module having highest weight in the closure of bottom p-alcove. In particular, we compute that if $\lambda = w \cdot 0 + p\sigma$, i.e., λ is weakly p-linked to 0, then there exists a P_J-module isomorphism

$$\mathrm{H}^j((P_J)_1, L(\lambda))^{(-1)} \cong \begin{cases} \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] & \text{if } j \equiv \ell(w) \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

One of the primary ingredients is the calculation for $p > h$ by Kumar, Lauritzen and Thom-

[1] In her thesis, Crane [Cra] claims for all primes that if the underlying root system is of type A_n, then the cohomology ring $\mathrm{H}^\bullet(U_1, k)$ is an integral extension of a polynomial algebra. For $p < h$ this is easily seen to be false using well-established results on the spectrum of the cohomology ring and support varieties. In this thesis we verify Crane's claim for $p > 2(h-1)$.

sen [KLT] of the cohomology of G_1 with coefficients in an induced module, which employs the existence of Frobenius splittings on the cotangent bundle of the flag variety. Our calculations for $(P_J)_1$ significantly extend the earlier calculations of Friedlander and Parshall [FP1, Corollary 2.6 and Remark 2.7(b)].

1.2.2 Cohomolgy for Quantum groups

Chapter 6 contains quantum analogs of our results for infinitesimal Frobenius kernels. These results pertain to computing the cohomology of the small quantum groups $u_\zeta(\mathfrak{p}_J)$ and $u_\zeta(\mathfrak{u}_J)$ with parameter specialized to a primitive ℓ-th root of unity $\zeta \in \mathbb{C}$. Different techniques are required here than for algebraic groups due to the lack of quantum analogs for various tools available in the study of algebraic groups. In particular, we show that if ℓ satisfies the following conditions:

(i) ℓ is odd,

(ii) $\ell > h$,

(iii) ℓ is coprime to $n+1$ if Φ has type A_n, and ℓ is coprime to 3 if Φ has type E_6 or G_2,

and suppose $\lambda \in C_{\mathbb{Z}}$ and $w \in W$, then there exists an isomorphism of left U_ζ^0-modules and of left $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$-modules

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda)),$$

with $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acting via the cup product on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$, and via left multiplication on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda))$. Quantum version of U_1-cohomology ring can be translated as follows: Assume in addition that $\ell > 2(h-1)$. Then there exists a graded ring isomorphism

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), \mathbb{C}) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C}).$$

Let $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$, and let $J \subseteq \Pi$. We also prove the quantum analog of parabolic cohomology that

- $\mathrm{H}^\bullet(u_\zeta(\mathfrak{p}_J), L^\zeta(\lambda)) = 0$ unless λ is weakly ℓ-linked to zero.

- Suppose $\lambda = w \cdot 0 + \ell\sigma$. Then there exists a P_J-module isomorphism

$$\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), L^\zeta(\lambda)) \cong \begin{cases} \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] & \text{if } j \equiv \ell(w) \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

1.2.3 Low Degree Cohomology and Computations for SL_2

At first, we keep assuming that G is a simple, simply-connected algebraic group. Following from the spectral sequence for Frobenius kernels in [Jan2, Proposition 9.14], a new spectral sequence is introduced to compute B_r-cohomology in Chapter 7, Theorem 7.1.1. Let c be the largest coefficient in the expression of all positive roots in terms of simple roots. Let p be a very good prime (which will be defined in Section 2.1). Then for each $n \leq \frac{p}{c}$, we can prove that there are B-module isomorphisms

$$\mathrm{H}^n(B_r, k) \cong \mathrm{H}^n(B_1, k)^{(r-1)} \cong \begin{cases} S^{\frac{n}{2}}(\mathfrak{u}^*)^{(r)} & \text{if } n \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mathrm{H}^n(G_r, k)^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^n(B_r, k)^{(-r)}) \cong \begin{cases} \mathrm{ind}_B^G S^{\frac{n}{2}}(\mathfrak{u}^*) & \text{if } n \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\mathrm{H}^n(G_r, k) \cong \mathrm{H}^n(G_1, k)^{(r-1)}$ as a G-module. Apply our results with $n = 1$ and $p \geq 3$, we immediately get $\mathrm{H}^1(B_r, k) = 0$ for G of type B, C, D. Some argument is needed for the other types. This result agrees with the ones in [BNP1] and [Jan2, II.12]. Moreover, the result allows us to understand higher degree cohomology for Frobenius kernels with big p. It can be observed that the cohomology for G_r tends to be like the one for G_1 as p increases. We also notice that this property was studied by Friedlander-Parshall in [FP1, Theorem 1.8]. Although their result is more precise, it only applies for type A.

Next we narrow our concerns down to the case when $G = SL_2$. Corollary 8.1.2, Proposition 8.2.1 and Theorem 8.4.1 describe the cohomology for U_r, B_r and G_r as a module. An algorithm is introduced in Remark 8.2.4 to find the character multiplicity of weight β in $\mathrm{H}^\bullet(B_r, \lambda)$; hence of $\mathrm{H}^0(\beta)$ in $\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))$. The reduced part of the rings $\mathrm{H}^\bullet(B_r, k)$ and $\mathrm{H}^\bullet(G_r, k)$ are also computed in Theorems 8.3.2 and 8.5.3. Finally, we show that there is a homeomorphism on spectra of the reduced G_r-cohomology ring and $k[G \times^B \mathfrak{u}^r]$ (cf. Proposition 8.5.4).

1.2.4 Commuting varieties

In Chapter 10 and 11 we study commuting varieties of r-tuples. We tackle the irreducibility, normality and Cohen-Macaulayness for simple cases. The main results are Proposition 10.1.2, Corollary 10.1.3, Theorem 10.3.5, Proposition 10.4.1, and Theorem 11.2.3. In particular, by the isomorphism (10.1), we reduce to showing these properties for the variety $C_r(\mathfrak{sl}_2)$. Then applying results in determinantal rings, we prove that both varieties $C_r(\mathfrak{gl}_2)$ and $C_r(\mathfrak{sl}_2)$ are irreducible, Cohen-Macaulay, and normal for each $r \geq 1$.

Next we study the structure of the variety $C_r(\mathcal{N}(\mathfrak{sl}_2))$ by intersecting it with a hypersurface. In order to obtain the Cohen-Macaulayness of $C_r(\mathcal{N}(\mathfrak{sl}_2))$, we use a deep concept in commutative algebra, namely *Principal Radical System* [BV], to show a certain family of ideals are radical. Then we show that the moment map $m : G \times^B \mathfrak{u}^r \to C_r(\mathcal{N}(\mathfrak{sl}_2))$ admits rational singularities (cf. Proposition 10.4.1). Therefore, we can compute the characters of the coordinate algebra for this variety as a G-module (cf. Theorem 10.5). Combining this with Theorem 8.5.4, we obtain a homeomorphism between the spectrum of the reduced G_r-cohomology ring and the variety $C_r(\mathcal{N}(\mathfrak{sl}_2))$.

Chapter 11 involves new results about the structure of commuting varieties over various sets of 3 by 3 matrices. We are able to show that $C_r(\mathcal{N}(\mathfrak{sl}_3))$ is irreducible and satisfies the condition $(R1)$ of Serre's criterion. In particular, we classify singularities of $C_r(\mathcal{N}(\mathfrak{sl}_3))$ and show that they are in codimension greater than or equal to 2 (cf. Theorem 11.2.3).

Chapter 2

Notation

2.1 Root systems and combinatorics

Let k be an algebraically closed field of characteristic p. Let G be a simple, simply-connected algebraic group over k, defined and split over the prime field $\mathbb{F}_p$. Fix a maximal torus $T \subset G$, also split over $\mathbb{F}_p$, and let Φ be the root system of T in G. Fix a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots in Φ, and let Φ^+ be the corresponding set of positive roots.

A prime p is called *bad* for the root system Φ if there exists a closed subsystem Φ' of Φ such that $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ has p-torsion. If p is not bad, then it is called a *good prime* for Φ. Alternatively, p is good if and only if the decomposition of the maximal root in Φ as an integral linear combination of simple roots does not have p as a coefficient. For the reader's convenience, we list all good primes for irreducible root systems as follows:

- Φ of type A_n, all p;
- Φ of type B_n, C_n, D_n, $p \geq 3$;
- Φ of type $E_6, E_7, F_4, G_2, p \geq 5$;
- Φ of type $E_8, p \geq 7$.

Now a positive integer m (> 1) is called good for Φ provided that m is not divisible by any bad prime for Φ. Otherwise, m is bad for Φ. Additionally, a good integer m is called *very good* for Φ provided that if Φ is of type A_n, then m is coprime to $n+1$.

Let W be the Weyl group of Φ; it is generated by the set of simple reflections $\{s_\alpha : \alpha \in \Pi\}$. Write $\ell : W \to \mathbb{N}$ for the standard length function on W, and let $w_0 \in W$ be the longest element. Let $(\cdot,\cdot)$ be the standard W-invariant inner product on the Euclidean space $\mathbb{E} := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Given $\alpha \in \Phi$, let $\alpha^\vee := 2\alpha/(\alpha,\alpha)$ be the corresponding coroot. Set α_0 to be the highest short root of Φ, and ρ to be one-half the sum of all positive roots in Φ. Then the Coxeter number of Φ is $h = (\rho, \alpha_0^\vee) + 1$.

Let X be the weight lattice of Φ, defined by the $\mathbb{Z}$-span of the fundamental weights $\{\omega_1, \ldots, \omega_n\}$, and let $X^+ \subset X$ be the set of dominant weights. The dot action of W on X is defined for $w \in W$ and $\lambda \in X$ by $w \cdot \lambda = w(\lambda+\rho) - \rho$. The bottom p-alcove and its closure are defined, respectively, by

$$C_{\mathbb{Z}} := \left\{\lambda \in X : 0 < (\lambda+\rho, \beta^\vee) < p \ \text{ for all } \beta \in \Phi^+\right\},$$
$$\overline{C}_{\mathbb{Z}} := \left\{\lambda \in X : 0 \le (\lambda+\rho, \beta^\vee) \le p \ \text{ for all } \beta \in \Phi^+\right\}.$$

Given $J \subseteq \Pi$, let $\Phi_J = \mathbb{Z}J \cap \Phi$ be the subroot system of Φ generated by J, and let $W_J \subseteq W$ be the standard parabolic subgroup generated by the set of simple reflections $\{s_\beta : \beta \in J\}$. Set $\Phi_J^+ = \Phi_J \cap \Phi^+$. The set of J-dominant weights is defined by

$$X_J^+ = \left\{\mu \in X : \forall \beta \in \Phi_J^+, (\mu, \beta^\vee) \in \mathbb{N}\right\},$$

and the set of J-restricted dominant weights is defined by

$$(X_J)_1 = \left\{\mu \in X_J^+ : \forall \beta \in J, (\mu, \beta^\vee) < p\right\}.$$

Write ${}^J W$ for the set of the minimal length right coset representatives of W_J in W. Then

${}^{J}W$ is also given by ${}^{J}W = \{w \in W | w^{-1}(\Phi_J^+) \subseteq \Phi^+\}$.

2.2 Algebraic groups and Frobenius kernels

Let $B \subseteq G$ be the Borel subgroup of G containing T and corresponding to Φ^+, and let $U \subseteq B$ be the unipotent radical of B. Write $U^- \subseteq B^-$ for the opposite subgroups. Given $J \subseteq \Pi$, let P_J be the standard parabolic subgroup of G containing B and corresponding to J, let U_J be the unipotent radical of P_J, and let L_J be the Levi factor of P_J. Then $P_J = L_J \ltimes U_J$. Set $\mathfrak{g} = \text{Lie}(G)$, the Lie algebra of G, $\mathfrak{b} = \text{Lie}(B)$, $\mathfrak{u} = \text{Lie}(U)$, $\mathfrak{p}_J = \text{Lie}(P_J)$, $\mathfrak{u}_J = \text{Lie}(U_J)$, and $\mathfrak{l}_J = \text{Lie}(L_J)$. Denote by $S^\bullet(\mathfrak{u}_J^*)$ the symmetric algebra over $\mathfrak{u}_J^*$, generated in degree two. Sometimes we use S^m (resp. $S^\bullet$) for abbreviating $S^m(\mathfrak{u}^*)$ (resp. $S^\bullet(\mathfrak{u}^*)$). Similarly, exterior algebra $\Lambda^\bullet(\mathfrak{u}_J^*)$ over $\mathfrak{u}_J^*$ will be abbreviated $\Lambda^\bullet$.

For given positive integer r, the map $f \mapsto f^{p^r}$ on the algebra $k[G]$ is an endormorphism. It induces the r-th Frobenius morphism $F_r : G \to G$. For instance, if G is a linear algebraic group, then F_r sends (a_{ij}) to $\left(a_{ij}^{p^r}\right)$. Given a closed F-stable subgroup H of G, we write H_r for the scheme-theoretic kernel of the morphism $F_r|_H : H \to H$. Given a rational H-module M, write $M^{(r)}$ for the module obtained by twisting the structure map for M by F_r. Alternately, as all of the algebraic groups mentioned in the previous paragraph are defined and split over $\mathbb{F}_p$, we have F induces an isomorphism $H/H_r \cong H$. If M is an H/H_r-module, write $M^{(-r)}$ for the space M considered as an H-module via the isomorphism $H/H_r \cong H$. Write M^H for the module of all fixed points under the H-action.

For $\lambda \in X^+$, let $L(\lambda)$ be the simple G-module of highest weight λ; it is the G-socle of the induced module $H^0(\lambda) = \text{ind}_{B^-}^G \lambda$. Similarly, given $J \subseteq \Pi$ and $\lambda \in X_J^+$, write $L_J(\lambda)$ for the simple L_J-module of highest weight λ.

2.3 Cohomology

2.3.1 Definition

Let A be either an augmented k-algebra or an algebraic group defined over k. Let $\mathcal{C}$ be the category of A-modules. In either case $\mathcal{C}$ has enough injective objects, so we can construct right derived functors $\mathrm{Ext}_A^n(k,-)$ of the left exact functor $\mathrm{Hom}_A(k,-)$ from an injective resolution. In particular, for any A-module M in $\mathcal{C}$, the n-th cohomology of A with coefficients in M is defined as

$$\mathrm{H}^n(A,M) = \mathrm{Ext}_A^n(k,M).$$

We further denote

$$\mathrm{H}^\bullet(A,M) = \bigoplus_{n=0}^{\infty} \mathrm{H}^n(A,M).$$

This is a graded ring with multiplication described below in Subsection 2.3.3. Sometimes, we use $\mathrm{H}^{ev}(A,M)$ or $\mathrm{H}^{odd}(A,M)$ to indicate the direct sum which is taken over all even or odd n.

Let $\mathfrak{g}$ be a Lie algebra over k and $\mathbb{U}(\mathfrak{g})$ be its universal enveloping algebra. Suppose M is a $\mathfrak{g}$-module. Then for each n, the cohomology of the Lie algebra $\mathfrak{g}$ is defined as

$$\mathrm{H}^n(\mathfrak{g},M) = \mathrm{H}^n(\mathbb{U}(\mathfrak{g}),M) = \mathrm{Ext}_{\mathbb{U}(\mathfrak{g})}^n(k,M).$$

2.3.2 Extensions

Given M and N be A-modules, an n-extension E of N by M is an exact sequence

$$E : 0 \to N \to C_{n-1} \to C_{n-2} \to \ldots \to C_0 \to M \to 0$$

of A-modules. By [ML, III.5, VII.3], the set of all equivalence classes of n-extensions of N by M forms an abelian group under the Baer sum. This group is denoted by $\mathcal{E}xt_A^n(M,N)$.

It is well-known that for each $n \geq 0$

$$\mathcal{E}xt_A^n(M,N) \cong \operatorname{Ext}_A^n(M,N)$$

as a vector space.

2.3.3 Cohomological products

If there exists a comultiplication map $\Delta : A \to A \otimes A$, then we can define the *cup product* on the ring $\mathrm{H}^\bullet(A,k)$ satisfying $[\alpha] \cup [\beta] = (-1)^{ij}([\beta] \cup [\alpha])$ with $[\alpha] \in \mathrm{H}^i(A,k), [\beta] \in \mathrm{H}^j(A,k)$ which is called *graded commutativity*. In fact, this multiplication is a special case of Yoneda product which will be defined in more general context below.

Let M, N, P be A-modules. Yoneda product is a bilinear map

$$\circ : \operatorname{Ext}_A^\bullet(N,P) \otimes \operatorname{Ext}_A^\bullet(M,N) \to \operatorname{Ext}_A^\bullet(M,P)$$

constructed as follows: for any $[\alpha] \in \operatorname{Ext}_A^i(N,P)$ and $[\beta] \in \operatorname{Ext}_A^j(M,N)$, we represent α, β by the i- and j-extension, respectively,

$$E_\alpha : 0 \to P \to C_{i-1} \to C_{i-2} \to \ldots \to C_0 \to N \to 0,$$
$$E_\beta : 0 \to N \to C'_{j-1} \to C'_{j-2} \to \ldots \to C'_0 \to M \to 0.$$

Now splicing E_α and E_β together, we obtain the new extension

$$E_{\alpha\beta} : 0 \to P \to C_{i-1} \to \ldots \to C_0 \to C'_{j-1} \to \ldots \to C'_0 \to M \to 0.$$

It is easy to see that $[E_{\alpha\beta}] \in \mathcal{E}xt_A^{i+j}(M,P) = \operatorname{Ext}_A^{i+j}(M,P)$. Finally, we set $[\alpha] \circ [\beta] = [E_{\alpha\beta}]$.

2.4 Nilpotent orbits

For the chapters on commuting varieties, we will follow the same conventions as in [Hum2] and [Jan3]. Let V be a G-variety, and let v be a point in V. We then denote by $\mathcal{O}_v$ the G-orbit of v (i.e., $\mathcal{O}_v = G\cdot v$). For example, consider the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ as a G-variety with the adjoint action, we have the following well-known orbits: $\mathcal{O}_{\text{reg}} = G\cdot v_{\text{reg}}, \mathcal{O}_{\text{subreg}} = G\cdot v_{\text{subreg}}$ and $\mathcal{O}_{\min} = G\cdot v_{\min}$. Denote by $z(v)$ and $Z(v)$ respectively the centralizers of v in $\mathfrak{g}$ and G. It is well-known that $\dim z(v) = \dim Z(v)$ and $\dim \mathcal{O}_v = \dim G - \dim z(v)$. It is also useful to keep in mind that every orbit is a smooth variety. Sometimes, we use $\mathcal{O}_V$ for the structure sheaf of a variety V (see [CM], [Hum2] for more details).

2.5 Basic algebraic geometry conventions

We use R_{red} to denote the reduced ring $R/\operatorname{Nilrad}(R)$ where $\operatorname{Nilrad}(R)$ is the nilradical ideal of R which consists of all nilpotent elements of R. Let Spec R be the spectrum of all prime ideals of R. Define a topology on Spec R in which every closed set is defined as

$$V(I) = \{P \in \operatorname{Spec} R \mid I \subset P\}$$

for some ideal I. This is known as the Zariski topology on Spec R.

Given a G-variety V, it can be seen that B acts freely on $G\times V$ by setting $b\cdot(g,v) = (gb^{-1}, bv)$ for all $b\in B, g\in G$ and $v\in V$. The notation $G\times^B V$ stands for the fiber bundle associated to the projection $\pi : G\times^B V \to G/B$ with fiber V. Topologically, $G\times^B V$ is a quotient space of $G\times V$ in which the equivalence relation is given as

$$(g,v)\sim(g',v') \Leftrightarrow (g',v') = b\cdot(g,v) \quad \text{for some} \quad b\in B.$$

In other words, each equivalence class of $G\times^B V$ represents a B-orbit in $G\times V$. The map $m : G\times^B V \to G\cdot V$ defined by mapping $[g,v]$ to $g\cdot v$ for all $g\in G, v\in V$ is called *the moment* *morphism*. It is obviously surjective. Let X be an affine variety. Then we always write $k[X]$ for the coordinate ring of X which is the same as the ring of global sections $\mathcal{O}_X(X)$. Although $G \times^B V$ is not affine, we still denote by $k[G \times^B V]$ the ring of global sections on this variety. It is sometimes useful to make the following identification: $k[G \times^B V] \cong k[G \times V]^B$.

Let $\mathfrak{g}$ be the Lie algebra of an algebraic group G. Suppose $V_1, \ldots, V_r$ are subvarieties of $\mathfrak{g}$. Then we define the commuting variety of r-tuples (which is a subvariety of $V_1 \times \cdots \times V_r$) as follows:

$$C(V_1, \ldots, V_r) = \{(v_1, \ldots, v_r) \in V_1 \times \cdots \times V_r \mid [v_i, v_j] = 0,\ 1 \leq i \leq j \leq r\}.$$

In the case when $V_1 = \cdots = V_r = V$, we simply write $C_r(V)$ for this variety.

Let $f : X \to Y$ be a morphism of varieties. Denote by f_* the direct image functor from the category of sheaves over X to the category of sheaves over Y. One can see that this is a left-exact functor. Hence, we have right derived functors of this direct image. We call these functors *higher direct images* and denote them by $R^i f_*$ with $i \geq 0$. In particular, if $Y =$ Spec A is affine and $\mathcal{F}$ is a quasi-coherent sheaf on X, then we have $R^i f_*(\mathcal{F}) \cong \mathcal{L}(\mathrm{H}^i(X, \mathcal{F}))$ where $\mathcal{L}$ is an exact functor mapping an A-module M to its associated sheaf $\mathcal{L}(M)$. Here we follow conventions in [H].

Chapter 3

Cohomology of $(U_J)_1$ with Coefficients in $L(\lambda)$

3.1 Weight combinatorics and the Weyl group

We begin with some elementary observations concerning weights. The first lemma is a restatement of [Hum1, Lemma 13.2A].

Lemma 3.1.1. *Let $w_1, w_2 \in W$, and let $\lambda \in X^+$. If $w_1 \cdot \lambda = w_2 \cdot \lambda$, then $w_1 = w_2$.*

Lemma 3.1.2. *Let $w_1, w_2 \in W$, $\lambda \in C_{\mathbb{Z}}$, and $\sigma \in \mathbb{Z}\Phi$. If $w_1 \cdot \lambda = w_2 \cdot \lambda + p\sigma$, then $\sigma = 0$.*

Proof. Suppose $w_1 \cdot \lambda = w_2 \cdot \lambda + p\sigma$ as in the statement of the lemma. Without loss of generality, we may assume that $\sigma \in \mathbb{Z}\Phi \cap X^+$. Indeed, if σ is not already dominant, choose $y \in W$ such that $y\sigma \in X^+$. Then $y\cdot(w_1\cdot\lambda) = y\cdot(w_2\cdot\lambda)+p(y\sigma)$, i.e., $(yw_1)\cdot\lambda = (yw_2)\cdot\lambda+p(y\sigma)$. So suppose $w_1 \cdot \lambda = w_2 \cdot \lambda + p\sigma$ for some $w_1, w_2 \in W$, $\lambda \in C_{\mathbb{Z}}$, and $\sigma \in \mathbb{Z}\Phi \cap X^+$. Observe that

$$\begin{aligned}(w_1 \cdot \lambda, \alpha_0^\vee) &= (w_1(\lambda + \rho), \alpha_0^\vee) - (\rho, \alpha_0^\vee) \\ &= (\lambda + \rho, w_1^{-1}(\alpha_0^\vee)) - (h-1) \\ &\leq (\lambda + \rho, \alpha_0^\vee) - (h-1) \qquad \text{because } w_1^{-1}(\alpha_0^\vee) \leq \alpha_0^\vee \text{ in } \Phi^\vee, \\ &< p - (h-1).\end{aligned}$$

because $\lambda \in C_{\mathbb{Z}}$. Then $p(\sigma, \alpha_0^\vee) + (w_2 \cdot \lambda, \alpha_0^\vee) = (w_1 \cdot \lambda, \alpha_0^\vee) < p - (h-1)$. Now suppose $\sigma \neq 0$. Since $\sigma \in \mathbb{Z}\Phi \cap X^+$, σ can not be a minuscule weight. In particular, if $\sigma \neq 0$, then $(\sigma, \alpha_0^\vee) \geq 2$, in which case $(w_2 \cdot \lambda, \alpha_0^\vee) < -p - (h-1)$. Observe also that

$$\begin{aligned}(w_2 \cdot \lambda, \alpha_0^\vee) &= (\lambda + \rho, w_2^{-1}(\alpha_0^\vee)) - (h-1) \\ &\geq (\lambda + \rho, -\alpha_0^\vee) - (h-1) \qquad \text{because } w_2^{-1}(\alpha_0^\vee) \geq -\alpha_0^\vee \text{ in } \Phi^\vee \\ &> -p - (h-1).\end{aligned}$$

This contradicts the inequality $(w_2 \cdot \lambda, \alpha_0^\vee) < -p - (h-1)$, so we must have $\sigma = 0$ as required. □

Remark 3.1.3. The conclusion of Lemma 3.1.2 need not hold if the weight λ is merely in the closure $\overline{C}_{\mathbb{Z}}$ of the bottom p-alcove. For example, suppose $p = 5$ and Φ is of type A_2. Then $\lambda := 2\omega_1 + \omega_2 \in \overline{C}_{\mathbb{Z}}$, but $w_0 \cdot \lambda = 1 \cdot \lambda + 5(-\alpha_1 - \alpha_2)$, where 1 denotes the identity element of W.

Lemma 3.1.4. *Let $w_1, w_2, w_3 \in W$, and suppose $w_1 \cdot 0 + w_2 \cdot 0 = w_3 \cdot 0 + p\sigma$ for some $\sigma \in \mathbb{Z}\Phi$. If $p > 2(h-1)$, then $\sigma = 0$.*

Proof. Suppose $w_1 \cdot 0 + w_2 \cdot 0 = w_3 \cdot 0 + p\sigma$ for some $\sigma \in \mathbb{Z}\Phi$. Then $(w_3^{-1}w_1) \cdot 0 + w_3^{-1}(w_2 \cdot 0) = p(w_3^{-1}\sigma)$. Choose $y \in W$ such that $yw_3^{-1}\sigma \in X^+$. Then

$$y((w_3^{-1}w_1) \cdot 0) + yw_3^{-1}(w_2 \cdot 0) = p(yw_3^{-1}\sigma) \in \mathbb{Z}\Phi \cap X^+.$$

Set $w_1' = w_3^{-1}w_1$, $y' = yw_3^{-1}$, and $\sigma' = yw_3^{-1}\sigma$. Then

$$y(w_1' \cdot 0) + y'(w_2 \cdot 0) = p\sigma' \in \mathbb{Z}\Phi \cap X^+. \tag{3.1}$$

Now, $w_1' \cdot 0$ and $w_2 \cdot 0$ are each sums of distinct roots in Φ. Then the same is true for $y(w_1' \cdot 0)$ and $y'(w_2 \cdot 0)$, so $p\sigma' \leq 2\rho + 2\rho = 4\rho$. Taking the inner product with $\alpha_0^\vee$ preserves the inequality, so we get $p(\sigma', \alpha_0^\vee) \leq 4(h-1)$. Now suppose $\sigma \neq 0$. Then also $\sigma' \neq 0$, hence σ' is a nonzero dominant weight in the root lattice. If $(\sigma', \alpha^\vee) \leq 1$ for all $\alpha \in \Phi^+$, then this would imply that σ' is a minuscule dominant weight [Hum1, Exercise 13.13], a contradiction because the minuscule dominant weights are not in the root lattice. Then $(\sigma', \alpha^\vee) \geq 2$ for some $\alpha \in \Phi^+$. Since $\alpha_0^\vee$ is the unique highest root in the dual root system $\Phi^\vee$, this implies that also $(\sigma', \alpha_0^\vee) \geq 2$. Now $2p \leq p(\sigma', \alpha_0^\vee) \leq 4(h-1)$, hence $p \leq 2(h-1)$. So if $p > 2(h-1)$, then necessarily $\sigma = 0$. □

Example 3.1.5. The conclusion of Lemma 3.1.4 need not hold if $p < 2(h-1)$. Indeed, suppose that $p = 5$, and that Φ is of type B_2, so that $h = 4$. Write $\Pi = \{\alpha, \beta\}$ with α a long root. Then $(s_\beta s_\alpha) \cdot 0 = -\alpha - 3\beta$ and $(s_\alpha s_\beta) \cdot 0 = -2\alpha - \beta$, so that $(s_\beta s_\alpha) \cdot 0 + (s_\beta s_\alpha) \cdot 0 = (s_\alpha s_\beta) \cdot 0 + 5(-\beta)$.

The following lemma generalizes Lemma 3.1.4.

Lemma 3.1.6. *Let $J \subseteq \Pi$. For $i \in \{1, 2, 3\}$, let $w_i \in {}^JW$, and let μ_i be a weight of T in $L_J(w_i \cdot 0)$. Suppose $p > 3(h-1)$, and that $\mu_1 + \mu_2 = \mu_3 + p\sigma$ for some $\sigma \in \mathbb{Z}\Phi$. Then $\sigma = 0$.*

Proof. Suppose $\mu_1 + \mu_2 = \mu_3 + p\sigma$ for some $\sigma \in \mathbb{Z}\Phi$. Choose $w \in W$ such that $\sigma' := w\sigma \in X^+$, and set $\mu_i' = w\mu_i$. Then $p\sigma' = \mu_1' + \mu_2' - \mu_3'$. The module $L_J(w_i \cdot 0)$ occurs as an L_J-composition factor of the cohomology ring $\mathrm{H}^\bullet(\mathfrak{u}_J, k)$ [UGA1, Theorem 4.2.1], which can be computed as a subquotient of the exterior algebra $\Lambda^\bullet(\mathfrak{u}_J^*)$. The weights of T in $\Lambda^\bullet(\mathfrak{u}_J^*)$ are also weights of T in the rational G-module $\Lambda^\bullet(\mathfrak{g}^*)$, which has highest weight 2ρ and lowest weight -2ρ. Then μ_i' is also weight of T in $\Lambda^\bullet(\mathfrak{g}^*)$, and $-2\rho \leq \mu_i' \leq 2\rho$. This implies that $p\sigma' \leq 6\rho$, and hence that $p(\sigma', \alpha_0^\vee) \leq 6(\rho, \alpha_0^\vee)$. Now if $\sigma \neq 0$, we get as in the proof of Lemma 3.1.4 that $2p \leq 6(h-1)$, that is, $p \leq 3(h-1)$, so if $p > 3(h-1)$, then necessarily $\sigma = 0$. □

3.2 L_J-module structure

We now compute $\mathrm{H}^\bullet((U_J)_1, L(\lambda))$ as a graded L_J-module.

Theorem 3.2.1. *Let $\lambda \in C_\mathbb{Z} \cap X^+$. Fix $J \subseteq \Pi$. There exists an isomorphism of graded L_J-modules*

$$\mathrm{H}^\bullet((U_J)_1, L(\lambda)) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, L(\lambda)), \tag{3.2}$$

where $\mathfrak{u}_J^*$ in $S^\bullet(\mathfrak{u}_J^*)$ has cohomological degree 2.

Proof. By [FP1, Proposition 1.1], there exists a spectral sequence of L_J-modules

$$E_2^{2i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}_J, L(\lambda)) \Rightarrow \mathrm{H}^{2i+j}((U_J)_1, L(\lambda)) \tag{3.3}$$

for which $E_2^{i,j} = 0$ for all odd i. We claim that the spectral sequence collapses at the E_2-page, that is, that $E_2^{i,j} \cong E_\infty^{i,j}$. Using the derivation property of the differential, it suffices to show for all $r \geq 2$ and $j \geq 0$ that $d_r^{0,j} = 0$. This we will do by showing that any L_J-composition factor of $E_r^{0,j}$ cannot be a composition factor of $E_r^{r,j+1-r}$. Since $E_r^{i,j}$ is a subquotient of $E_2^{i,j}$, it is enough to show that any composition factor of $E_2^{0,j}$ cannot be a composition factor of $E_2^{r,j+1-r}$.

By [FP1, Theorem 2.5] or [UGA1, Theorem 4.2.1], there exists an L_J-module isomorphism

$$\mathrm{H}^j(\mathfrak{u}_J, L(\lambda)) \cong \bigoplus_{\substack{w \in {}^J W \\ \ell(w) = j}} L_J(w \cdot \lambda). \tag{3.4}$$

Thus, every composition factor of $E_2^{0,j}$ has the form $L_J(w_1 \cdot \lambda)$ for some $w_1 \in {}^J W$ with $\ell(w_1) = j$. Similarly, every composition factor of $E_2^{r,j+1-r} = S^{r/2}(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^{j+1-r}(\mathfrak{u}_J, L(\lambda))$ must have the form $L_J(\sigma)^{(1)} \otimes L_J(w_2 \cdot \lambda)$ for some $w_2 \in {}^J W$ with $\ell(w_2) = j + 1 - r$ (in particular, $\ell(w_2) < \ell(w_1)$), and some $\sigma \in X_J^+ \cap \mathbb{N}(\Phi^- \backslash \Phi_J^-)$ (i.e., σ is a sum of negative roots not in Φ_J^-).

Observe that for $\alpha \in J$, $(w_2 \cdot \lambda, \alpha^\vee) = (\lambda + \rho, w_2^{-1}(\alpha^\vee)) - 1 \geq 0$, because $w_2 \in {}^J W$ implies that $w_2^{-1}(\alpha) \in \Phi^+$. Also, $(w_2 \cdot \lambda, \alpha^\vee) = (\lambda + \rho, w_2^{-1}(\alpha^\vee)) - 1 < p$, because $\lambda \in C_\mathbb{Z}$. Then $w_2 \cdot \lambda$ is a J-restricted dominant weight, so by the Steinberg Tensor Product Theorem, there exists an L_J-module isomorphism

$$L_J(\sigma)^{(1)} \otimes L_J(w_2 \cdot \lambda) \cong L_J(w_2 \cdot \lambda + p\sigma). \tag{3.5}$$

Now suppose that $L_J(w_1 \cdot \lambda) \cong L_J(w_2 \cdot \lambda + p\sigma)$. Then $w_1 \cdot \lambda = w_2 \cdot \lambda + p\sigma$, so Lemmas 3.1.2 and 3.1.1 imply that $\sigma = 0$ and $w_1 = w_2$. This contradicts the inequality $\ell(w_2) < \ell(w_1)$, so we conclude that no composition factor of $E_2^{0,j}$ can also be a composition factor of $E_2^{r,j+1-r}$, and hence that the spectral sequence (3.3) collapses at the E_2-page.

We have shown for all i and j that $E_2^{i,j} \cong E_\infty^{i,j}$. Recall that the E_∞-page of (3.3) is the associated graded module coming from some L_J-submodule filtration of $\mathrm{H}^\bullet((U_J)_1, L(\lambda))$. To finish the proof of the theorem, we must show that $\mathrm{H}^\bullet((U_J)_1, L(\lambda))$ is isomorphic as an L_J-module to its associated graded object. For this, it suffices to show for all $m \neq n$ that

$$\mathrm{Ext}^1_{L_J}(S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^n(\mathfrak{u}_J, L(\lambda)), S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^m(\mathfrak{u}_J, L(\lambda))) = 0. \tag{3.6}$$

Using the long exact sequence in cohomology, and applying the isomorphisms (3.4) and (3.5), it suffices even to show that $\mathrm{Ext}^1_{L_J}(L_J(w_1 \cdot \lambda + p\sigma), L_J(w_2 \cdot \lambda + p\mu)) = 0$ whenever $w_1, w_2 \in {}^J W$, $w_1 \neq w_2$, and $\mu, \sigma \in \mathbb{N}(\Phi^- \backslash \Phi_J^-)$. So suppose $\mathrm{Ext}^1_{L_J}(L_J(w_1 \cdot \lambda + p\sigma), L_J(w_2 \cdot \lambda + p\mu)) \neq 0$. Then by the Linkage Principle for L_J [Jan2, II.6.17], there exists $w \in W_J$ and $\gamma \in \mathbb{Z}\Phi_J$ such that

$$w_1 \cdot \lambda + p\sigma = w \cdot (w_2 \cdot \lambda + p\mu) + p\gamma = (ww_2) \cdot \lambda + p(w\mu + \gamma).$$

By Lemmas 3.1.2 and 3.1.1, this implies that $w_1 \cdot \lambda = (ww_2) \cdot \lambda$, and hence that $w_1 = ww_2$.

This is a contradiction, because w_1 and w_2 were chosen as distinct minimal length right coset representatives of W_J in W. So we conclude that $\operatorname{Ext}^1_{L_J}(L_J(w_1\cdot\lambda+p\sigma), L_J(w_2\cdot\lambda+p\mu)) = 0$, as claimed. □

Remark 3.2.2. Friedlander and Parshall observed (3.2) for $\lambda \in X^+$ satisfying $p > (\lambda, \alpha_0^\vee)+h$ [FP1, Remark 2.7(b)]. Our result strengthens this to $p \geq (\lambda, \alpha_0^\vee) + h$, i.e., for $\lambda \in C_{\mathbb{Z}} \cap X^+$.

3.3 Structure of the associated graded algebra

Given an algebra A with a multiplicative filtration indexed by the nonnegative integers, let $\operatorname{gr} A$ be the associated graded algebra.

Corollary 3.3.1. *Fix $J \subseteq \Pi$, and suppose $p > h$. Then there exists a multiplicative filtration on $\mathrm{H}^\bullet((U_J)_1, k)$ by L_J-submodules such that*

$$\operatorname{gr} \mathrm{H}^\bullet((U_J)_1, k) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, k) \tag{3.7}$$

as graded L_J-algebras, where $\mathfrak{u}_J^*$ in $S^\bullet(\mathfrak{u}_J^*)$ has cohomological degree 2, and the algebra structure on the right-hand side of the isomorphism is the standard tensor product of algebras.

Proof. Take $\lambda = 0$ in Theorem 3.2.1. Then $L(\lambda) = k$, (3.3) is a spectral sequence of algebras, and (3.7) is the algebra isomorphism between the E_2- and E_∞-pages of (3.3). □

Remark 3.3.2. The L_J-submodule filtration on $\mathrm{H}^\bullet((U_J)_1, k)$ described in Corollary 3.3.1 can be given explicitly as follows: For $i \geq 0$, let $F^i\,\mathrm{H}^\bullet((U_J)_1, k)$ denote the i-th filtered part of $\mathrm{H}^\bullet((U_J)_1, k)$. Then there exists an L_J-module isomorphism

$$F^i\,\mathrm{H}^n((U_J)_1, k) \cong \bigoplus_{j \geq i} S^{j/2}(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^{n-j}(\mathfrak{u}, k),$$

where both sides are zero if $i > n$.

Remark 3.3.3. Suppose $p \geq h-1$. Implicit in [UGA1] is the following description of the ring structure on $\mathrm{H}^\bullet(\mathfrak{u}, k)$. First, $\mathrm{H}^\bullet(\mathfrak{u}, k)$ is computed as a subquotient of the Koszul complex $\Lambda^\bullet(\mathfrak{u}^*)$ (the exterior algebra on $\mathfrak{u}^*$). Fix an ordered root vector basis $\{x_{\gamma_1}, \ldots, x_{\gamma_N}\}$ for $\mathfrak{u}$, and let $\{f_{\gamma_1}, \ldots, f_{\gamma_N}\} \subset \mathfrak{u}^*$ be the corresponding dual basis. For $w \in W$, define $\Phi(w) = w\Phi^- \cap \Phi^+$. Write $\Phi(w) = \{\beta_1, \ldots, \beta_n\}$ with $n = \ell(w)$, and set $f_{\Phi(w)} = f_{\beta_1} \wedge \cdots \wedge f_{\beta_n} \in \Lambda^n(\mathfrak{u}^*)$. For definiteness, assume that the sequence $(f_{\beta_1}, \ldots, f_{\beta_n})$ appears as a subsequence of the sequence $(f_{\gamma_1}, \ldots, f_{\gamma_N})$. Then the vectors $f_{\Phi(w)}$ for $w \in W$ are cocycles in $\Lambda^\bullet(\mathfrak{u}^*)$, and their images $[f_{\Phi(w)}]$ in $\mathrm{H}^\bullet(\mathfrak{u}, k)$ form a vector space basis for $\mathrm{H}^\bullet(\mathfrak{u}, k)$. The ring structure on $\mathrm{H}^\bullet(\mathfrak{u}, k)$ is induced by the ring structure of $\Lambda^\bullet(\mathfrak{u}^*)$. Specifically, the cup product of cohomology classes is given by

$$[f_{\Phi(w)}] \cup [f_{\Phi(w')}] = \begin{cases} (-1)^{s(w,w')}[f_{\Phi(w'')}] & \text{if } \ell(w) + \ell(w') = \ell(w'') \text{ and } \Phi(w) \cup \Phi(w') = \Phi(w''), \\ 0 & \text{otherwise.} \end{cases}$$

If $\Phi(w) = \{\beta_1, \ldots, \beta_n\}$ and $\Phi(w') = \{\beta'_1, \ldots, \beta'_m\}$ with $\{\beta_1, \ldots, \beta_n\}$ and $\{\beta'_1, \ldots, \beta'_m\}$ written as subsequences of $(\gamma_1, \ldots, \gamma_N)$, then the integer $(-1)^{s(w,w')}$ is the sign of the permutation that maps the sequence $(\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_m)$ to a subsequence of $(\gamma_1, \ldots, \gamma_N)$.

Chapter 4

The Ring Structure of $\mathrm{H}^\bullet(U_1, k)$

4.1 Un-grading the associated graded algebra

For $p > h$, the ring structure of the cohomology rings $\mathrm{H}^\bullet(G_1, k)$ and $\mathrm{H}^\bullet(B_1, k)$ for the first Frobenius kernels of G and B were computed by Friedlander and Parshall [FP2] and Andersen and Jantzen [AJ]. In this section we compute, for $J \subseteq \Pi$ and for p not too small, the ring structure of the cohomology ring $\mathrm{H}^\bullet((U_J)_1, k)$ for the first Frobenius kernel of U_J. In particular, we compute the ring structure of $\mathrm{H}^\bullet(U_1, k)$.

Theorem 4.1.1. *Fix $J \subseteq \Pi$. If $J = \emptyset$, assume $p > 2(h-1)$. If $J \neq \emptyset$, assume $p > 3(h-1)$. Then there exists a graded L_J-algebra isomorphism*

$$\mathrm{H}^\bullet((U_J)_1, k) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, k), \tag{4.1}$$

where the algebra structure on the right-hand side is the ordinary tensor product of algebras.

Proof. To simplify the notation slightly, we give the proof for the case $J = \emptyset$. For the case $J \neq \emptyset$, one simply replaces in the following argument the symbols T, U, and $\mathfrak{u}$ by L_J, U_J, and $\mathfrak{u}_J$, respectively, and applies Lemma 3.1.6 instead of Lemma 3.1.4. Our strategy is to exhibit graded T-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{H}^\bullet(U_1, k)$ isomorphic to $S^\bullet(\mathfrak{u}^*)^{(1)}$ and $\mathrm{H}^\bullet(\mathfrak{u}, k)$, respectively, and then to show that the product map $\theta : \mathcal{A} \otimes \mathcal{B} \to \mathrm{H}^\bullet(U_1, k)$ is an isomorphism of algebras.

First, set $\mathcal{A} = \bigoplus_{n \geq 0} F^n \mathrm{H}^n(U_1, k)$ where F denotes the multiplicative filtration on $\mathrm{H}^\bullet(U_1, k)$ from Corollary 3.3.1. Then $\mathcal{A}$ is a subalgebra of $\mathrm{H}^\bullet(U_1, k)$, and it follows from Remark 3.3.2 that $\mathcal{A}$ is isomorphic as an T-algebra to $S^\bullet(\mathfrak{u}^*)^{(1)}$. Next, by Theorem 3.2.1 there exists a T-submodule $\mathcal{B}$ of $\mathrm{H}^\bullet(U_1, k)$ isomorphic to $\mathrm{H}^\bullet(\mathfrak{u}, k)$. Specifically, in the T-module direct sum decomposition

$$\mathrm{H}^\bullet(U_1, k) \cong \bigoplus_{n \geq 0} \bigoplus_{2i+j=n} S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k), \tag{4.2}$$

the space $\mathcal{B}$ is the sum of all terms with $i = 0$. By Remark 3.3.2, $\mathcal{B} = \mathrm{H}^\bullet(U_1, k)/F^1 \mathrm{H}^\bullet(U_1, k)$.

We claim that $\mathcal{B}$ is a subalgebra of $\mathrm{H}^\bullet(U_1, k)$. To see this, it suffices to show that the product of two weight vectors in $\mathcal{B}$ is again a weight vector in $\mathcal{B}$. So let $z_1, z_2 \in \mathcal{B}$ be vectors of weights μ_1 and μ_2, respectively, and suppose that $z_1 z_2 \neq 0$. Then the product $z_1 z_2$ has weight $\mu_1 + \mu_2$. Suppose the product $z_1 z_2$ has a nonzero component in some summand $S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k)$ of (4.2) with $i \neq 0$. Then there exists a weight σ of $S^i(\mathfrak{u}^*)$ and a weight μ_3 of $\mathrm{H}^j(\mathfrak{u}, k)$ such that $\mu_1 + \mu_2 = \mu_3 + p\sigma$. Since $i \neq 0$, we must have $\sigma \neq 0$. But now the explicit description of the weights of $\mathrm{H}^\bullet(\mathfrak{u}, k)$ given by (3.4) together with Lemma 3.1.4 imply that $\sigma = 0$, a contradiction. Thus, we must have $z_1 z_2 \in \mathcal{B}$, and so $\mathcal{B}$ is a subalgebra of $\mathrm{H}^\bullet(U_1, k)$. To see that $\mathcal{B} \cong \mathrm{H}^\bullet(\mathfrak{u}, k)$ as an algebra, and not just as a T-module, observe that since the spectral sequence (3.3) collapses at the E_2-page, the vertical edge map $\varphi : \mathrm{H}^\bullet(U_1, k) \to E_2^{0,\bullet}$ induces a T-algebra isomorphism $\mathrm{H}^\bullet(U_1, k)/F^1 \mathrm{H}^\bullet(U_1, k) \cong E_2^{0,\bullet} \cong \mathrm{H}^\bullet(\mathfrak{u}, k)$. By the above argument, the quotient $\mathrm{H}^\bullet(U_1, k)/F^1 \mathrm{H}^\bullet(U_1, k)$ identifies not only as a space but as an algebra with $\mathcal{B}$, so we get the T-algebra isomorphism $\mathcal{B} \cong \mathrm{H}^\bullet(\mathfrak{u}, k)$.

We have now exhibited T-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathrm{H}^\bullet(U_1, k)$ isomorphic to $S^\bullet(\mathfrak{u}^*)^{(1)}$ and $\mathrm{H}^\bullet(\mathfrak{u}, k)$, respectively. It remains to show that the product map $\theta : \mathcal{A} \otimes \mathcal{B} \to \mathrm{H}^\bullet(U_1, k)$ is an algebra isomorphism, where the algebra structure on $\mathcal{A} \otimes \mathcal{B}$ is the ordinary tensor product of algebras. First, the cohomology ring $\mathrm{H}^\bullet(U_1, k)$ identifies with the cohomology ring of the finite-dimensional Hopf algebra $\mathrm{Dist}(U_1)$ (cf. [Jan2, I.7, I.9]), so is a graded-commutative ring by [ML, VIII.4][1]. Since the subalgebra $\mathcal{A}$ is concentrated in even degrees, this implies that θ is an algebra homomorphism. Next, there exist natural maps

$$\iota_1 : \mathcal{A} = \bigoplus_{n \geq 0} F^n \mathrm{H}^n(U_1, k) \hookrightarrow \bigoplus_{i \geq 0} F^i \mathrm{H}^\bullet(U_1, k)/F^{i+1} \mathrm{H}^\bullet(U_1, k) \quad \text{and}$$

$$\iota_2 : \mathcal{B} = \mathrm{H}^\bullet(U_1, k)/F^1 \mathrm{H}^\bullet(U_1, k) \hookrightarrow \bigoplus_{i \geq 0} F^i \mathrm{H}^\bullet(U_1, k)/F^{i+1} \mathrm{H}^\bullet(U_1, k).$$

By Remark 3.3.2, the images of $\mathcal{A}$ and $\mathcal{B}$ under these maps generate the associated graded algebra $\mathrm{gr}\,\mathrm{H}^\bullet(U_1, k)$. It follows then that $\mathcal{A}$ and $\mathcal{B}$ also generate the cohomology ring $\mathrm{H}^\bullet(U_1, k)$. Indeed, given a nonzero homogeneous element $z \in \mathrm{H}^n(U_1, k)$, choose i such that $z \in F^i \mathrm{H}^n(U_1, k)$ but $z \notin F^{i+1} \mathrm{H}^n(U_1, k)$. Since $\iota_1(\mathcal{A})$ and $\iota_2(\mathcal{B})$ generate $\mathrm{gr}\,\mathrm{H}^\bullet(U_1, k)$, there exists $w \in \mathcal{A} \otimes \mathcal{B}$ such that $z - \theta(w) \in F^{i+1} \mathrm{H}^n(U_1, k)$. By induction on $i - n$, we may assume that $F^{i+1} \mathrm{H}^n(U_1, k)$ is in the image of the map θ. Then $z \in \mathrm{Im}(\theta)$, so we conclude that θ is surjective. Finally, by dimension comparison in each graded degree, θ must also be injective, hence an algebra isomorphism. □

Remark 4.1.2. The bound of $3(h-1)$ in Theorem 4.1.1 for the case $J \neq \emptyset$ is not sharp. For example, suppose Φ has type A_n with $n > 1$, and that $J = \Pi - \{\alpha\}$ for some simple root α. Then U_J is abelian, so $\mathrm{H}^\bullet((U_J)_1, k) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \Lambda^\bullet(\mathfrak{u}_J^*) \cong S^\bullet(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}_J, k)$ as a ring if $p > 2$.

[1]Mac Lane does not assume a Hopf algebra to possess an antipode. In particular, the cohomology ring of a bialgebra is always graded-commutative.

4.2 Discussion and Open problems

4.2.1 Counterexample

In this subsection, we show that if $p < 2(h-1)$, the algebra isomorphism $\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k)$ of Theorem 4.1.1 need not hold, even though the isomorphism of associated graded algebras $\mathrm{gr}\,\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k)$ holds whenever $p > h$.

Example 4.2.1. Let Φ be of type B_2, so $h = 4$, and take $p = 5$. Write $\Pi = \{\alpha, \beta\}$ with α a long root. Then Example 3.1.5 shows that the weight argument in the proof of Theorem 4.1.1 fails, as there exist weights $\mu_1 = \mu_2 = (s_\beta s_\alpha) \cdot 0$ and $\mu_3 = (s_\alpha s_\beta) \cdot 0$ of $\mathrm{H}^2(\mathfrak{u}, k)$ and a weight $\sigma = -\beta$ of $S^1(\mathfrak{u}^*)$ such that $\mu_1 + \mu_2 = \mu_3 + p\sigma$ but $\sigma \neq 0$. In fact, this nontrivial solution to the weight equation $\mu_1 + \mu_2 = \mu_3 + p\sigma$ corresponds to two elements in the subspace $\mathcal{B}$ of $\mathrm{H}^\bullet(U_1, k)$ having a product not in $\mathcal{B}$. Indeed, let $z_1 = z_2$ be a nonzero weight vector in the one-dimensional weight space $\mathrm{H}^2(U_1, k)_{s_\beta s_\alpha \cdot 0} \subset \mathcal{B}$. We have been able to verify by computer calculation in MAGMA [BCP] that $z_1 z_2 \neq 0$ in $\mathrm{H}^\bullet(U_1, k)$, even though every vector in the Lie algebra cohomology ring $\mathrm{H}^\bullet(\mathfrak{u}, k)$ squares to zero. Thus, for type B_2 we cannot have an isomorphism of graded T-algebras $\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k)$ when $p = 5$.

4.2.2 Open questions

Although the condition $p \geq 2(h-1)$ in Theorem 4.1.1 can not be relaxed for arbitrary type of G, it is possible for investigating every single type. According to the discussion between Leonard Scott and the author, we have the following conjecture on type A.

Conjecture 4.2.2. *Suppose G is of type A and let $p > h$. Then there is a graded isomorphism of rings*

$$\mathrm{H}^\bullet(U_1, k) \cong S^\bullet(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^\bullet(\mathfrak{u}, k).$$

One way to approach this problem is beside considering the weight equation in Lemma 3.1.4, we must have the agreement of degrees of weight vectors on both sides. This observation leads to the study of the following system of equations

$$\begin{cases} w_1 \cdot 0 + w_2 \cdot 0 & = w_3 \cdot 0 + p\sigma \\ \ell(w_1) + \ell(w_2) & = \ell(w_3) + 2\deg(\sigma) \end{cases}$$

where $\ell(w)$ is the length of the element w in the Weyl group and $\deg(\sigma)$ is the degree of σ as of a polynomial in $S^\bullet(\mathfrak{u}^*)$. If this system has no solutions except $\sigma = 0$, then by similar argument as earlier we obtain the isomorphism. By listing all the possibilities, we already verified the conjecture for G of types A_2 and A_3. The answers are affirmative. Computations for higher cases are more complicated. We hope to have some new ideas to generalize this work in the near future.

Chapter 5

Parabolic Cohomology

5.1 T_1-invariants of Lie algebra cohomology

In this section we apply our previous results to compute the structure of the cohomology space $\mathrm{H}^n((P_J)_1, L(\lambda))$ for each $n \geq 0$ when $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$. We assume throughout this chapter that $p > h$ so that $X^+ \cap \overline{C}_{\mathbb{Z}}$ is non-empty. First we prove a combinatorial lemma.

Lemma 5.1.1. *Let $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$. Suppose that λ is weakly p-linked to 0, that is, that $\lambda = w \cdot 0 + p\sigma$ for some $\sigma \in X$. Then:*

(a) The weight σ is minuscule or zero. The weight λ uniquely determines w and σ.

(b) There exists a T-module isomorphism,

$$\mathrm{H}^j(\mathfrak{u}, L(\lambda))^{T_1} \cong \begin{cases} w^{-1}\sigma & \text{if } j = \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

Proof. First, since $w\rho + p\sigma = \lambda + \rho \in \rho + X^+$, and since $|(w\rho, \alpha^\vee)| = |(\rho, w^{-1}\alpha^\vee)| \leq h - 1 < p$

for all $\alpha \in \Phi$, it follows that $\sigma \in X^+$. Next, since $\lambda \in \overline{C}_{\mathbb{Z}}$, we get

$$\begin{aligned} p(\sigma, \alpha_0^\vee) &= (\lambda - w \cdot 0, \alpha_0^\vee) \\ &= (\lambda + \rho, \alpha_0^\vee) - (\rho, w^{-1}\alpha_0^\vee) \\ &\leq p + (\rho, \alpha_0^\vee) \\ &= p + h - 1 < 2p. \end{aligned}$$

It follows that $(\sigma, \alpha_0^\vee) \in \{0, 1\}$. If $(\sigma, \alpha_0^\vee) = 0$, then $\lambda = 0$ and $w = 1$. Otherwise, σ is a minuscule weight. Now suppose that $\lambda = w \cdot 0 + p\omega_i = w' \cdot 0 + p\omega_j$ for some $w, w' \in W$ and some minuscule weights ω_i, ω_j (so Φ is necessarily of type A, D or E). Then $p(\omega_j - \omega_i) = w \cdot 0 - w' \cdot 0 \in \mathbb{Z}\Phi$. Since $p > h$, we conclude that $\omega_i = \omega_j$, for otherwise $p(\omega_j - \omega_i) \notin \mathbb{Z}\Phi$. Now $w \cdot 0 = w' \cdot 0$, so $w = w'$ by Lemma 3.1.1. This proves part (a). Next, by [UGA1, Theorem 4.2.1] we have

$$\mathrm{H}^j(\mathfrak{u}, L(\lambda))^{T_1} \cong \bigoplus_{\substack{w' \in W \\ \ell(w') = j}} (k_{w' \cdot \lambda})^{T_1}.$$

Here $k_{w' \cdot \lambda}$ denotes the one-dimensional T-module of weight $w' \cdot \lambda$. Suppose $w' \cdot \lambda = p\sigma'$ for some $\sigma' \in X$. Then $0 = w' \cdot \lambda - p\sigma' = w'w \cdot 0 + p(w'\sigma - \sigma')$. By (a), this implies that $w'\sigma - \sigma' = 0$ and that $w'w = 1$, that is, that $w' = w^{-1}$. So $\sigma' = w^{-1}\sigma$ and $\ell(w) = \ell(w') = j$. □

5.2 Main result

We can now compute the structure of $\mathrm{H}^\bullet((P_J)_1, L(\lambda))$ when $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$.

Theorem 5.2.1. *Let $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$ and let $J \subseteq \Pi$.*

(a) $\mathrm{H}^\bullet((P_J)_1, L(\lambda)) = 0$ *if λ is not weakly p-linked to* 0.

(b) If $\lambda = w \cdot 0 + p\sigma$, then there exists a P_J-module isomorphism

$$\mathrm{H}^j((P_J)_1, L(\lambda))^{(-1)} \cong \begin{cases} \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] & \text{if } j \equiv \ell(w) \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Part (a) is established by the argument in [FP1, Remark 2.7(b)], using the Linkage Principle for $(L_J)_1$. Next suppose that $\lambda = w \cdot 0 + p\sigma$ for some $w \in W$ and $\sigma \in X$. We have $R^m \mathrm{ind}_B^{P_J} L(\lambda) \cong L(\lambda) \otimes R^m \mathrm{ind}_B^{P_J}(k) = 0$ for all $m > 0$, so by [Jan2, II.12.2], there exists a spectral sequence

$$E_2^{i,j} = R^i \mathrm{ind}_B^{P_J}[\mathrm{H}^j(B_1, L(\lambda))^{(-1)}] \Rightarrow \mathrm{H}^{i+j}((P_J)_1, L(\lambda))^{(-1)}. \tag{5.1}$$

Also, $\mathrm{H}^\bullet(B_1, L(\lambda)) \cong \mathrm{H}^\bullet(U_1, L(\lambda))^{T_1}$ by [Jan2, I.6.9(3)], so by Lemma 5.1.1(b),

$$\begin{aligned} R^i \mathrm{ind}_B^{P_J}[\mathrm{H}^j(B_1, L(\lambda))^{(-1)}] &\cong R^i \mathrm{ind}_B^{P_J}[(\mathrm{H}^j(U_1, L(\lambda))^{T_1})^{(-1)}] \\ &\cong R^i \mathrm{ind}_B^{P_J}[(\oplus_{2a+b=j} S^a(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^b(\mathfrak{u}, L(\lambda))^{T_1})^{(-1)}] \\ &\cong R^i \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma], \end{aligned}$$

and the spectral sequence can be rewritten as

$$E_2^{i,j} = R^i \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] \Rightarrow \mathrm{H}^{i+j}((P_J)_1, L(\lambda))^{(-1)}.$$

Next, we claim that $w^{-1}\sigma$ is antidominant, that is, that $w^{-1}\sigma \in -X^+$. To see this, first observe that for any $\alpha \in \Pi$, $(w^{-1} \cdot \lambda, \alpha^\vee) = p(w^{-1}\sigma, \alpha^\vee) \in p\mathbb{Z}$. Next observe that

$$(w^{-1} \cdot \lambda, \alpha^\vee) = (\lambda + \rho, w\alpha^\vee) - 1 < p$$

because $\lambda \in \overline{C}_\mathbb{Z}$. Then for all $\alpha \in \Pi$, $(w^{-1} \cdot \lambda, \alpha^\vee) \leq 0$, so also $(w^{-1}\sigma, \alpha^\vee) \leq 0$, i.e.,

$w^{-1}\sigma \in -X^+$.

Now set $B' = B \cap L_J$, $U' = U \cap L_J$, and $\mathfrak{u}' = \mathrm{Lie}(U')$. Then $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{u}_J$ and $S^\bullet(\mathfrak{u}^*) \cong S^\bullet(\mathfrak{u}'^*) \otimes S^\bullet(\mathfrak{u}_J^*)$ are B'-module decompositions, and the action of B' on $S^\bullet(\mathfrak{u}_J^*)$ lifts to the natural action of L_J on $S^\bullet(\mathfrak{u}_J^*)$. By [CPS, Example 4.2(a)], there exists an isomorphism of L_J-modules

$$R^i \operatorname{ind}_B^{P_J}[S^\bullet(\mathfrak{u}^*) \otimes w^{-1}\sigma] \cong R^i \operatorname{ind}_{B'}^{L_J}[S^\bullet(\mathfrak{u}^*) \otimes w^{-1}\sigma],$$

and then by the generalized tensor identity we get

$$R^i \operatorname{ind}_{B'}^{L_J}[S^\bullet(\mathfrak{u}^*) \otimes w^{-1}\sigma] \cong S^\bullet(\mathfrak{u}_J^*) \otimes R^i \operatorname{ind}_{B'}^{L_J}[S^\bullet(\mathfrak{u}'^*) \otimes w^{-1}\sigma]$$

Now $R^i \operatorname{ind}_{B'}^{L_J}[S^\bullet(\mathfrak{u}'^*) \otimes w^{-1}\sigma] = 0$ if $i > 0$ by the calculations of Kumar, Lauritzen, and Thomsen [KLT, Theorem 2].[1] Consequently, the spectral sequence (5.1) collapses at the E_2-page, and we obtain the P_J-module isomorphism $\mathrm{H}^j((P_J)_1, L(\lambda))^{(-1)} \cong \operatorname{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma]$. $\square$

[1]Kumar, Lauritzen, and Thomsen choose their Borel subgroup to correspond to the negative roots, while we have chosen ours to correspond to the positive roots; this is why we checked, for example, that $w^{-1}\sigma \in -X^+$ rather than $w^{-1}\sigma \in X^+$.

Chapter 6

Results for Quantum Groups

In this section we adapt our main results to quantum groups. New techniques are necessary for quantum groups because of the lack of a quantum analog for the spectral sequence (3.3). The arguments given here for quantum groups can be adapted to work for algebraic groups as well, and thus provide different proofs of the earlier theorems. The notation we use in this section is generally the same as that in [BNPP], though to maintain consistency with Chapters 3–5 we define our Borel subalgebras to correspond to positive root vectors and not negative root vectors. For convenience, we first recall some notation and background material.

6.1 Definitions

6.1.1 Quantum enveloping algebra

Let Φ be a finite, irreducible root system. Fix Π a set of simple roots in Φ and J a subset of Π. Let $\mathsf{A} = \mathbb{Z}[q, q^{-1}]$ be the $\mathbb{Z}$-algebra of Laurent polynomials in an indeterminate q. Let ℓ be a non negative integer satisfying the following condition:

Assumption 6.1.1. The number ℓ is odd and greater than 1. If Φ is of type E_6 or G_2, then 3 does not divide ℓ. We also assume that ℓ is not divisible by a bad prime for Φ.

Let ζ be a primitive ℓ-th root of unity. Set $k = \mathbb{Q}(\zeta)$ be the cyclotomic field generated by ζ over $\mathbb{Q}$. We consider k as A-algebra by the homomorphism $\mathbb{Z}[q, q^{-1}] \to k$ mapping $q \mapsto \zeta$. Now for given integer i, we set

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}},$$

then denote $[i]^! = [i][i-1]\dots[1]$. Note that $[0]^! = 1$ as a convention. Suppose m is a positive integer and n is an integer, we write

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n][n-1]\dots[n-m+1]}{[1][2]\dots[m]}$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$. Let $q_\alpha = q^{d_\alpha}$ where $d_\alpha = (\alpha, \alpha)/2$. Then for any element $f \in \mathsf{A}$ and $\alpha \in \Pi$, we denote $f_\alpha \in \mathsf{A}$ by replacing q by q_α in f. Next, for $\alpha \in \Pi$ and $m \geq 0$, we define the m-th divided powers as follows:

$$E_\alpha^{(m)} = \frac{E_\alpha^m}{[m]_\alpha^!}, \quad F_\alpha^{(m)} = \frac{F_\alpha^m}{[m]_\alpha^!}.$$

The quantum enveloping algebra $\mathbb{U}_q$ of $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra with generators $E_\alpha, K_\alpha^{\pm 1}, F_\alpha$ with $\alpha \in \Pi$ satisfying following relations:

1. $K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \quad K_\alpha K_\beta = K_\beta K_\alpha,$

2. $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta,$

3. $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta,$

4. $E_\alpha F_\beta - F_\beta E_\alpha = \delta \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}},$

5. $\Sigma_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-s} E_\beta E_\alpha^s = 0,$

6. $\Sigma_{s=0}^{1-a_{\alpha\beta}}(-1)^s \begin{bmatrix} 1-a_{\alpha\beta} \\ s \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0,$

where $\delta_{\alpha\beta}$ is the Kronecker and $a_{\alpha\beta} = (\beta, \alpha^\vee)$.

6.1.2 Two A-forms of $\mathbb{U}_q$

The DeConcini-Kac A-form $\mathcal{U}_q^{\mathsf{A}}$ of $\mathbb{U}_q$ is the A-subalgebra of $\mathbb{U}_q$ generated by $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ for all $\alpha \in \Pi$. Then define

$$\mathcal{U}_\zeta = \mathcal{U}_\zeta(\mathfrak{g}) = \mathcal{U}_{\mathbb{C}} / \left\langle 1 \otimes K_\alpha^\ell - 1 \otimes 1, \alpha \in \Pi \right\rangle$$

where $\mathcal{U}_{\mathbb{C}} = \mathcal{U}_q^{\mathsf{A}} \otimes_{\mathsf{A}} \mathbb{C}$. It is called the DeConcini-Kac quantum enveloping algebra.

We also define the Lusztig A-form $\mathbb{U}_q^{\mathsf{A}}$ of $\mathbb{U}_q$ as the A-subalgebra of $\mathbb{U}_q$ generated by all the symbols $E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha^{\pm}$ for all $\alpha \in \Pi$ and $m \in \mathbb{N}$. Put

$$U_\zeta = U_\zeta(\mathfrak{g}) = \mathbb{U}_q^{\mathsf{A}} / \left\langle K_\alpha^\ell - 1, \alpha \in \Pi \right\rangle \otimes_{\mathsf{A}} \mathbb{C}$$

where $\left\langle K_\alpha^\ell - 1, \alpha \in \Pi \right\rangle$ is the ideal generated by all $K_\alpha^\ell - 1$ with $\alpha \in \Pi$ and $\mathbb{C}$ is regarded as a A-algebra by specializing $q \mapsto \zeta$. We also define the small quantum group $u_\zeta = u_\zeta(\mathfrak{g})$ as a finite dimensional Hopf subalgebra of U_ζ generated by $E_\alpha, F_\alpha, K_\alpha$ with all $\alpha \in \Pi$. Here we identify $E_\alpha^{(m)} \otimes 1, F_\alpha^{(m)} \otimes 1, K_\alpha^{\pm 1} \otimes 1$ with $E_\alpha^{(m)}, F_\alpha^{(m)}, K_\alpha^{\pm 1}$ and $E_\alpha^{(1)} = E_\alpha$, etc.

Let $\mathbb{U}_q^+$ (resp. $\mathbb{U}_q^-, \mathbb{U}_q^0$) be the subalgebra of $\mathbb{U}_q$ generated by $\{E_\alpha : \alpha \in \Pi\}$ (resp. $\{F_\alpha : \alpha \in \Pi\}$, $\{K_\alpha^{\pm 1} : \alpha \in \Pi\}$). Similarly, specializing gives us $U_\zeta^+, U_\zeta^-, U_\zeta^0$ and $u_\zeta^+, u_\zeta^-, u_\zeta^0$. Each triple satifies PBW-isomorphism via the multiplication. More details can be found in [Lus, §1.4]. For each positive root $\gamma \in \Phi^+$, there exist root vectors $E_\gamma \in \mathbb{U}_q^+$ and $F_\gamma \in \mathbb{U}_q^-$, defined in terms of certain braid group operators on $\mathbb{U}_q$. Let $\mathcal{Z}$ be the subalgebra of $\mathcal{U}_\zeta$ generated by the set $\{E_\gamma^\ell, F_\gamma^\ell : \gamma \in \Phi^+\} \subset \mathcal{U}_\zeta$. Then $\mathcal{Z}$ is a central polynomial subalgebra of $\mathcal{U}_\zeta$, and $\mathcal{U}_\zeta$ is free and finitely-generated over $\mathcal{Z}$ [DCK, §3.1, 3.3]. The inclusion of A-forms

$\mathcal{U}_{\mathsf{A}} \to U_{\mathsf{A}}$ induces algebra isomorphisms $\mathcal{U}_\zeta//\mathcal{Z} \cong u_\zeta$ and $\mathcal{U}_\zeta(\mathfrak{b})//\mathcal{Z}^+ \cong u_\zeta(\mathfrak{b})$.[1] The quotient map $\mathcal{U}_\zeta(\mathfrak{b}) \to u_\zeta(\mathfrak{b})$ restricts to an isomorphism $\mathcal{U}_\zeta^0 \cong u_\zeta^0$.

6.1.3 Levi and parabolic quantum groups

Let J be a subset of Π. Then we look at the parabolic Lie subalgebra $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{u}_J$ where $\mathfrak{l}_J$ is the Levi Lie subalgebra corresponding to J. Then we can define $\mathbb{U}_q(\mathfrak{p}_J)$ ($\mathbb{U}_q(\mathfrak{l}_J)$) as a subalgebra of $\mathbb{U}_q$ generated by $\{E_\alpha : \alpha \in J\} \cup \{F_\alpha, K_\alpha^{\pm 1} : \alpha \in \Pi\}$ ($\{E_\alpha, F_\alpha : \alpha \in J\} \cup \{K_\alpha^{\pm 1} : \alpha \in \Pi\}$). Specializing as previous we can define the algebras $U_\zeta(\mathfrak{p}_J), U_\zeta(\mathfrak{l}_J), u_\zeta(\mathfrak{p}_J), u_\zeta(\mathfrak{l}_J), \mathcal{U}_\zeta(\mathfrak{p}_J), \mathcal{U}_\zeta(\mathfrak{l}_J)$.

6.1.4 Hopf algebra structure

The quantized enveloping algebra $\mathbb{U}_q$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S defined as follows:

$$\begin{aligned}
&\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, && \epsilon(E_\alpha) = 0, && S(E_\alpha) = -K_\alpha^{-1}E_\alpha,\\
&\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, && \epsilon(F_\alpha) = 0, && S(F_\alpha) = -F_\alpha K_\alpha,\\
&\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, && \epsilon(K_\alpha) = 1, && S(K_\alpha) = K_\alpha^{-1}
\end{aligned}$$

for all $\alpha \in \Pi$. Inducing from these maps, we have U_ζ and u_ζ are also Hopf algebras. Note that u_ζ is finite dimensional and $\dim u_\zeta = \ell^{\dim \mathfrak{g}}$.

6.2 Central subalgebra structure

In this section, we first show that the central algebras $\mathcal{Z}$ and $\mathcal{Z}^+$ have structure of a bialgebra. Then we recall the relationship between U_ζ and $\mathbb{U}(\mathfrak{g})$. Structure of the small quantum group u_ζ is also introduced at the end of the section.

[1]Let A be an augmented k-algebra, and let B be a subalgebra of A. Let B_+ denote the augmentation ideal of B. We say that B is normal in A if $AB_+ = B_+A$. In this case, we write $A//B$ for the quotient $A/(AB_+)$. In particular, we use this notation even if A and B are not Hopf algebras.

Lemma 6.2.1. *The coproduct Δ of $\mathcal{U}_\zeta$ satisfies, for all $\gamma \in \Phi^+$,*

$$\Delta(E_\gamma^\ell) = E_\gamma^\ell \otimes 1 + 1 \otimes E_\gamma^\ell \qquad \textit{and}$$
$$\Delta(F_\gamma^\ell) = F_\gamma^\ell \otimes 1 + 1 \otimes F_\gamma^\ell.$$

In particular, the subalgebras $\mathcal{Z}$ and $\mathcal{Z}^+$ of $\mathcal{U}_\zeta$ are bialgebras.

Proof. Let $\mathcal{Z}_\mathbb{C}$ be the subalgebra of $\mathcal{U}_\mathbb{C}$ generated by $\{E_\gamma^\ell, F_\gamma^\ell, K_\alpha^{\pm\ell} : \gamma \in \Phi^+, \alpha \in \Pi\}$. For $\alpha \in \Pi$, we have $\Delta(K_\alpha^\ell) = K_\alpha^\ell \otimes K_\alpha^\ell$. If $\gamma \in \Phi^+$ is a simple root, then also

$$\Delta(E_\gamma^\ell) = E_\gamma^\ell \otimes 1 + K_\gamma^\ell \otimes E_\gamma^\ell \quad \text{and} \tag{6.1}$$
$$\Delta(F_\gamma^\ell) = F_\gamma^\ell \otimes K_\gamma^{-\ell} + 1 \otimes F_\gamma^\ell \tag{6.2}$$

in $\mathcal{U}_\mathbb{C}$ by [Jan1, 4.9(4)]. The subalgebra $\mathcal{Z}_\mathbb{C}$ of $\mathcal{U}_\mathbb{C}$ is stable under the braid group automorphisms of $\mathcal{U}_\mathbb{C}$ by [DCK, Proposition 3.3], so the identities (6.1) and (6.2) follow for arbitrary $\gamma \in \Phi^+$ from [AJS, Proposition C.5(2)]. Thus, $\mathcal{Z}_\mathbb{C}$ and $\mathcal{Z}_\mathbb{C}^+$ are sub-bialgebras of $\mathcal{U}_\mathbb{C}$. The lemma now follows because $\mathcal{Z}$ and $\mathcal{Z}^+$ are the images of $\mathcal{Z}_\mathbb{C}$ and $\mathcal{Z}_\mathbb{C}^+$ in $\mathcal{U}_\zeta$. □

Let $\mathbb{U}(\mathfrak{g})$ be the universal enveloping algebra for $\mathfrak{g}$, and let $F_\zeta : U_\zeta \to \mathbb{U}(\mathfrak{g})$ be the quantum Frobenius morphism defined by Lusztig in [Lus, §8]. Then F_ζ induces the Hopf-algebra isomorphism $U_\zeta//u_\zeta \cong \mathbb{U}(\mathfrak{g})$. Similarly, let $\mathbb{U}(\mathfrak{b})$ be the universal enveloping algebra for $\mathfrak{b}$. Then F_ζ restricts to a map $U_\zeta(\mathfrak{b}) \to \mathbb{U}(\mathfrak{b})$, and induces the isomorphism $U_\zeta(\mathfrak{b})//u_\zeta(\mathfrak{b}) \cong \mathbb{U}(\mathfrak{b})$. Given a left $\mathbb{U}(\mathfrak{g})$- (resp. $\mathbb{U}(\mathfrak{b})$-) module M, write $M^{(1)}$ for M considered as a U_ζ- (resp. $U_\zeta(\mathfrak{b})$-) module via F_ζ.

Given $\lambda \in X^+$, let $L^\zeta(\lambda)$ be the irreducible integrable U_ζ-module (i.e., the irreducible type-1 integrable $U_\mathbb{C}$-module) of highest weight λ. Similarly, given $\lambda \in X_J^+$, let $L_J^\zeta(\lambda)$ be the simple integrable $U_\zeta(\mathfrak{l}_J)$-module of highest weight λ. Define the fundamental alcove $C_\mathbb{Z}$ for U_ζ by replacing p by ℓ in the definition for $C_\mathbb{Z}$ given in Chapter 2.

The small quantum torus $u_\zeta^0 \subset u_\zeta(\mathfrak{g})$ is a semisimple algebra, isomorphic to the group ring over $\mathbb{C}$ for the finite group $(\mathbb{Z}/\ell\mathbb{Z})^n$. The set X_ℓ of ℓ-restricted dominant weights is defined by $X_\ell = \{\mu \in X^+ : (\mu, \alpha^\vee) < \ell \text{ for all } \alpha \in \Pi\}$. The irreducible u_ζ^0-modules are parametrized by the set $X/\ell X$. Equivalently, the irreducible u_ζ^0-modules are parametrized by the set X_ℓ, which forms a set of coset representatives for ℓX in X.

6.3 Weight space decomposition

Let $\lambda \in X^+$. Then $L^\zeta(\lambda)$ is by restriction a $u_\zeta(\mathfrak{b})$-module. Since $u_\zeta(\mathfrak{b})$ is flat as a right $u_\zeta(\mathfrak{u})$-module, and since $u_\zeta(\mathfrak{u})$ is normal in $u_\zeta(\mathfrak{b})$ with quotient $u_\zeta(\mathfrak{b})//u_\zeta(\mathfrak{u}) \cong u_\zeta^0$, the cohomology space $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$ is naturally a graded left u_ζ^0-module. In this section we describe the u_ζ^0-weight space decomposition of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$.

Lemma 6.3.1. *Let $\lambda \in X^+$ and $\mu \in X$. Then $\mathrm{Hom}_{u_\zeta^0}(\mu, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))) = 0$ unless $\mu = w \cdot \lambda + \ell\sigma$ for some $w \in W$ and some $\sigma \in X$.*

Proof. The subalgebra $u_\zeta(\mathfrak{u})$ acts trivially on the one-dimensional $u_\zeta(\mathfrak{b})$-module $-\mu$, so we get

$$\begin{aligned}\mathrm{Hom}_{u_\zeta^0}(\mu, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))) &\cong \mathrm{Hom}_{u_\zeta^0}(k, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \otimes -\mu) \\ &\cong \mathrm{Hom}_{u_\zeta^0}(k, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda) \otimes -\mu)) \\ &\cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -\mu)\end{aligned}$$

The last isomorphism follows by applying the LHS spectral sequence for the algebra $u_\zeta(\mathfrak{b})$ and its normal subalgebra $u_\zeta(\mathfrak{u})$, and by using the fact that u_ζ^0 is a semisimple algebra. Now given a $u_\zeta(\mathfrak{b})$-module M, let $Z'(M) := u_\zeta(\mathfrak{g}) \otimes_{u_\zeta(\mathfrak{b})} M$ be the module obtained via tensor induction from $u_\zeta(\mathfrak{b})$ to $u_\zeta(\mathfrak{g})$. Then

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -\mu) \cong \mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{b})}(\mu, L^\zeta(\lambda)) \cong \mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{g})}(Z'(\mu), L^\zeta(\lambda)).$$

By the Linkage Principle for $u_\zeta(\mathfrak{g})$, the last Ext-group is zero unless $\mu = w \cdot \lambda + \ell\sigma$ for some $w \in W$ and some $\sigma \in X$. □

Lemma 6.3.2. *Let $\lambda \in C_{\mathbb{Z}}$. Assume that ℓ is odd, that ℓ is coprime to $n+1$ if Φ is of type A_n, and that ℓ is coprime to 3 if Φ is of type E_6 or G_2. Suppose $w_1 \cdot \lambda = w_2 \cdot \lambda + \ell\sigma$ for some $w_1, w_2 \in W$ and some $\sigma \in X$. Then $\sigma = 0$ and $w_1 = w_2$.*

Proof. Suppose $w_1 \cdot \lambda = w_2 \cdot \lambda + \ell\sigma$. Then

$$(\lambda + \rho) - w_1^{-1} w_2(\lambda + \rho) = \ell(w_1^{-1}\sigma).$$

Since $\lambda + \rho$ is a strongly dominant weight, the left-hand side of the above equation is a sum of positive roots by [Hum1, Lemma 13.2A]. In particular, $\ell(w_1^{-1}\sigma) \in \mathbb{Z}\Phi$. By assumption, ℓ does not divide the order of the finite group $X/(\mathbb{Z}\Phi)$, so $w_1^{-1}\sigma \in \mathbb{Z}\Phi$, and hence $\sigma \in \mathbb{Z}\Phi$. Then $\sigma = 0$ by Lemma 3.1.2 (the lemma remains true if the prime p is replaced by an arbitrary integer), and hence $w_1 = w_2$ by Lemma 3.1.1. □

Remark 6.3.3. If $\lambda = 0$, then the conclusion to Lemma 6.3.2 also holds if ℓ is odd, if $\ell > h$, and if ℓ is coprime to 3 if Φ is of type G_2; for details see the argument in the proof of [GK, Theorem 2.5].

Corollary 6.3.4. *Let $\lambda \in C_{\mathbb{Z}}$. Assume that ℓ is odd, that ℓ is coprime to $n+1$ if Φ has type A_n, and that ℓ is coprime to 3 if Φ has type E_6 or G_2. Then*

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \cong \bigoplus_{w \in W} \mathrm{Hom}_{u_\zeta^0}(w \cdot \lambda, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))).$$

Proof. The graded u_ζ^0-module $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$ decomposes as a direct sum of simple u_ζ^0-modules. By a slight abuse of notation, we write this decomposition as

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \cong \bigoplus_{\mu \in X/\ell X} \mathrm{Hom}_{u_\zeta^0}(\mu, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))).$$

By Lemma 6.3.1, the only non-zero summands are those for which $\mu \equiv w \cdot \lambda \mod \ell X$ for some $w \in W$, and by Lemma 6.3.2, the weights $w \cdot \lambda$ for $w \in W$ are all incongruent modulo ℓX. □

Remark 6.3.5. Using Remark 6.3.3, the corollary also holds if $\lambda = 0$, if ℓ is odd, if $\ell > h$, and if ℓ is coprime to 3 if Φ is of type G_2.

6.4 Kostant's theorem for quantum groups

Let $\lambda \in C_{\mathbb{Z}}$. The cohomology calculations of Chapters 4 and 5 were critically dependent on Kostant's explicit formula for the B-module structure of the ordinary Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{u}, L(\lambda))$. Equivalently, Kostant's formula computes the $\mathrm{Dist}(B)$-module structure of $\mathrm{H}^\bullet(\mathbb{U}(\mathfrak{u}), L(\lambda))$, where $\mathbb{U}(\mathfrak{u})$ is the universal enveloping algebra for $\mathfrak{u}$.

In the root-of-unity quantum setting, the correct analogs for $\mathrm{Dist}(B)$ and $\mathbb{U}(\mathfrak{u})$ are $U_\zeta(\mathfrak{b})$ and $\mathcal{U}_\zeta(\mathfrak{u})$, respectively. Just as we needed an explicit description for the $\mathrm{Dist}(B)$-module structure on $\mathrm{H}^\bullet(\mathbb{U}(\mathfrak{u}), L(\lambda))$ to compute $\mathrm{H}^\bullet(U_1, L(\lambda))$, so too will we need an explicit description of the $U_\zeta(\mathfrak{b})$-module structure on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda))$ to compute $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$. More generally, one can compute the $U_\zeta(\mathfrak{l}_J)$-module structure of the cohomology space $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda))$. This is essentially done already in [UGA2, §6], though the setup there is not quite the same as what we need here (in [UGA2], the cohomology space $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda))$ is computed as a $\mathcal{U}_\zeta(\mathfrak{l}_J)$-module instead of as a $U_\zeta(\mathfrak{l}_J)$-module). In this section we make some brief remarks to explain how the results of [UGA2, §6] can be modified to suit our needs.

Let $\lambda \in X^+$. Then the simple integrable U_ζ-module $L^\zeta(\lambda)$ is made a $\mathcal{U}_\zeta$-module via the quotient $\mathcal{U}_\zeta \twoheadrightarrow u_\zeta$ and the inclusion $u_\zeta \hookrightarrow U_\zeta$. By restriction, $L^\zeta(\lambda)$ is a module for the algebras $U_\zeta(\mathfrak{l}_J) \subset U_\zeta$ and $\mathcal{U}_\zeta(\mathfrak{u}_J) \subset \mathcal{U}_\zeta$. The right adjoint action of $\mathbb{U}_q$ on itself induces a right action of $U_\zeta(\mathfrak{p}_J)$ on $\mathcal{U}_\zeta(\mathfrak{u}_J)$ [BNPP, §2.7]. Then by [Dru2, Theorem 4.3.1], the cohomology space $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda))$ is naturally a graded left $U_\zeta(\mathfrak{p}_J)$-module, and $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ is a graded $U_\zeta(\mathfrak{p}_J)$-module algebra. By restriction, $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda))$ and $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), \mathbb{C})$ are modules for the Levi subalgebra $U_\zeta(\mathfrak{l}_J) \subset U_\zeta(\mathfrak{p}_J)$.

The $U_\zeta(\mathfrak{l}_J)$-module structure on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda))$ can be computed using the same arguments as in [UGA2, §6], once two important changes are made. First, the algebra $\mathcal{U}_\zeta(\mathfrak{l}_J)$ should be replaced by the algebra $U_\zeta(\mathfrak{l}_J)$. Second, the algebra K defined in [UGA2, §6.2] should be replaced by the algebra U_ζ^0. Then the subalgebra $U_\zeta^0 u_\zeta(\mathfrak{l}_J)$ of U_ζ generated by U_ζ and $u_\zeta(\mathfrak{l}_J)$ is the correct quantum analog of the group scheme $(L_J)_1T$. With these modifications, the main result of [UGA2, §6.4] may be stated as follows:

Theorem 6.4.1. (cf. [UGA2, Theorem 6.4.1]) *Let $\lambda \in \overline{C}_{\mathbb{Z}}$. Assume that ℓ is odd, that ℓ is coprime to 3 if Φ has type G_2, and that $\ell \geq h-1$. Then for each $n \in \mathbb{N}$, there exists a $U_\zeta(\mathfrak{l}_J)$-module isomorphism*

$$\mathrm{H}^n(\mathcal{U}_\zeta(\mathfrak{u}_J), L^\zeta(\lambda)) \cong \bigoplus_{\substack{w \in {}^JW \\ \ell(w)=n}} L_J^\zeta(w \cdot \lambda).$$

6.5 The ring structure of $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$

In this section we provide a semi-explicit description for the ring structure on $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$. The description is similar to that for $\mathrm{H}^\bullet(\mathfrak{u}, \mathbb{C})$ given in Remark 3.3.3, though significantly more work is required to obtain the ring structure in the quantum setting because of the lack of an explicit projective resolution for $\mathcal{U}_\zeta(\mathfrak{u})$.

Let $\{\gamma_1, \ldots, \gamma_N\}$ be an enumeration of Φ^+ as in [DCP, §9.3], and let $E_{\gamma_1}, \ldots, E_{\gamma_N} \in \mathcal{U}_\zeta(\mathfrak{u})$ be the corresponding positive root vectors. Set $\Lambda = \mathbb{N}^{N+1}$, viewed as a totally ordered semigroup via the reverse lexicographic order. Then by [DCP, Theorem 9.3 and §10.1], there exists a multiplicative Λ-filtration on $\mathcal{U}_\zeta(\mathfrak{u})$ such that the associated graded algebra $\mathrm{gr}^\Lambda \mathcal{U}_\zeta(\mathfrak{u})$ is generated by the symbols $\{E_{\gamma_1}, \ldots, E_{\gamma_N}\}$ subject to the relations

$$E_{\gamma_i} E_{\gamma_j} = \zeta^{(\gamma_i, \gamma_j)} E_{\gamma_j} E_{\gamma_i} \qquad \text{if } 1 \leq i < j \leq N.$$

Alternately, by [DCP, Remark 10.1], there exists a sequence of algebras $U^{(-1)}, U^{(0)}, U^{(1)}, \dots, U^{(N)}$ such that

1. $U^{(-1)} = \mathcal{U}_\zeta(\mathfrak{u})$;
2. For $0 \leq i \leq N$, $U^{(i-1)}$ admits a multiplicative $\mathbb{N}$-filtration;
3. For $1 \leq i \leq N$, $U^{(i)}$ is the associated graded algebra of $U^{(i-1)}$; and
4. $U^{(N)} \cong \operatorname{gr}^\Lambda \mathcal{U}_\zeta(\mathfrak{u})$.

It follows that for each $1 \leq i \leq N$, there exists a spectral sequence of algebras satisfying

$$E_1^{p,q} = \mathrm{H}^{p+q}(U^{(i)}, \mathbb{C})_{(p)} \Rightarrow \mathrm{H}^{p+q}(U^{(i-1)}, \mathbb{C}), \tag{6.3}$$

where the subscript on the E_1-term denotes the internal grading on $\mathrm{H}^\bullet(U^{(i)}, \mathbb{C})$ arising from the $\mathbb{N}$-grading on $U^{(i)}$. The right adjoint action of U_ζ^0 on $\mathcal{U}_\zeta(\mathfrak{u})$ passes to an action of U_ζ^0 on each $U^{(i)}$, so (6.3) is also a spectral sequence of U_ζ^0-module algebras.

By [GK, Proposition 2.1], the cohomology ring $\mathrm{H}^\bullet(U^{(N)}, \mathbb{C})$ is isomorphic to the graded ring $\Lambda_\zeta^\bullet$ generated by the symbols $\{x_{\gamma_1}, \dots, x_{\gamma_N}\}$ (each of graded degree one), subject to the relations

$$\begin{aligned} x_{\gamma_i} x_{\gamma_j} + \zeta^{-(\gamma_i, \gamma_j)} x_{\gamma_j} x_{\gamma_i} &= 0 && \text{if } 1 \leq i < j \leq N, \text{ and} \\ x_{\gamma_i}^2 &= 0 && \text{for all } 1 \leq i \leq N. \end{aligned}$$

The U_ζ^0-module structure on $\Lambda_\zeta^\bullet$ is obtained by assigning x_{γ_i} to have weight $-\gamma_i$, so there exists an equality of formal characters $\operatorname{ch} \Lambda_\zeta^\bullet = \operatorname{ch} \Lambda^\bullet(\mathfrak{u}^*)$. In particular, for each $w \in W$, the $(w \cdot 0)$-weight space of $\Lambda_\zeta^\bullet$ (and hence also of $\mathrm{H}^\bullet(U^{(i)}, \mathbb{C})$ for each $-1 \leq i \leq N$) is one-dimensional.

Given $w \in W$ with $\Phi(w) = \{\beta_1, \dots, \beta_n\}$ ordered as in Remark 3.3.3, set $f_{\Phi(w)} = x_{\beta_1} \cdots x_{\beta_n} \in \Lambda_\zeta^\bullet$. Since $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$ is spanned by weight vectors of weights $w \cdot 0$ with

$w \in W$ (by Theorem 6.4.1), and since for each $w \in W$ the $(w \cdot 0)$-weight space of $\Lambda_\zeta^\bullet$ is one-dimensional, the (image of the) vector $f_{\Phi(w)}$ must for all $1 \leq i \leq N$ be a permanent cycle in the spectral sequence (6.3), and (the image of) the set $\{f_{\Phi(w)} : w \in W\}$ must form a basis for $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$. Finally, the uni-dimensionality of the $(w \cdot 0)$-weight spaces in $\Lambda_\zeta^\bullet$ implies that the cup product between weight vectors in $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), \mathbb{C})$ is induced by the multiplication in $\Lambda_\zeta^\bullet$.[2]

6.6 Weight space structure

Now we look at the structure of the u_ζ^0-weight space

$$\mathrm{Hom}_{u_\zeta^0}(w \cdot \lambda, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda)$$

as a left module for $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ under the cup product. Note that $u_\zeta(\mathfrak{u})$ is a left coideal subalgebra in $u_\zeta(\mathfrak{b})$ (i.e., the coproduct Δ on $u_\zeta(\mathfrak{b})$ satisfies $\Delta(u_\zeta(\mathfrak{u})) \subset u_\zeta(\mathfrak{b}) \otimes u_\zeta(\mathfrak{u})$), so it makes sense to consider the left cup product action

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda) \otimes -w \cdot \lambda) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda) \otimes -w \cdot \lambda).$$

Let $\{x_{\gamma_1}, \ldots, x_{\gamma_N}\} \subset \mathfrak{u}$ and $\{f_{\gamma_1}, \ldots, f_{\gamma_N}\} \subset \mathfrak{u}^*$ be dual bases for $\mathfrak{u}$ and $\mathfrak{u}^*$ as in Remark 3.3.3. For $0 \leq i \leq N$, let $R_i \subset \mathfrak{u}^*$ be the span of the set $\{f_{\gamma_1}, \ldots, f_{\gamma_i}\}$. Then $S^\bullet(R_N) \cong S^\bullet(\mathfrak{u}^*)$. Let $\mathcal{Z}_i$ be the subalgebra of $\mathcal{U}_\zeta(\mathfrak{b})$ generated by the set $\{E_{\gamma_1}^\ell, \ldots, E_{\gamma_i}^\ell\}$. Then $\mathcal{Z}_i$ is normal in $\mathcal{U}_\zeta(\mathfrak{b})$ because the generators for $\mathcal{Z}$ are central in $\mathcal{U}_\zeta$. Set $A_i = \mathcal{U}_\zeta(\mathfrak{b})//\mathcal{Z}_i$. Then $A_0 = \mathcal{U}_\zeta(\mathfrak{b})$ and $A_N = \mathcal{U}_\zeta(\mathfrak{b})//\mathcal{Z}^+ \cong u_\zeta(\mathfrak{b})$. Let $B_i \subset A_{i-1}$ be the (normal) subalgebra generated by $E_{\gamma_i}^\ell$. Then $A_{i-1}//B_i \cong A_i$. Moreover, it follows from the description of the coproduct in Lemma 6.2.1 that the algebras A_i and B_i inherit bialgebra structures from

[2] In a spectral sequence of algebras, the product on the E_∞-page will determine the product on the abutment up to terms in lower filtered degree. In our situation, the product also respects weight spaces, and the unidimensionality of certain weight spaces implies that there are no vectors of the correct weight in lower filtered degree.

$\mathcal{U}_\zeta(\mathfrak{b})$.

Proposition 6.6.1. *Assume that ℓ is odd, that $\ell > h$, and that ℓ is coprime to 3 if Φ is of type G_2. Then for all $0 \leq i \leq N$, $\mathrm{H}^{\mathrm{odd}}(A_i, \mathbb{C}) = 0$ and $\mathrm{H}^{2\bullet}(A_i, \mathbb{C}) \cong S^\bullet(R_i)^{(1)}$ as U_ζ^0-module algebras. Now let $\lambda \in C_\mathbb{Z}$ and $w \in W$, and suppose that also ℓ is coprime to $n+1$ if Φ is of type A_n, and that ℓ is coprime to 3 if Φ is of type E_6. Then the space $\mathrm{H}^\bullet(A_i, L^\zeta(\lambda) \otimes -w \cdot \lambda)$ is free as a left $\mathrm{H}^\bullet(A_i, \mathbb{C})$-module under the cup product, generated by a vector in degree $\ell(w)$ of U_ζ^0-weight zero.*

Proof. The proof is by induction on i. For $i = 0$, we have, as in the proof of Lemma 6.3.1,

$$\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda) \cong (\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \otimes -w \cdot \lambda)^{u_\zeta^0}.$$

By Theorem 6.4.1, $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda))$ decomposes as a direct sum of one-dimensional U_ζ^0-modules

$$\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \cong \bigoplus_{w' \in W} \mathbb{C}_{w' \cdot \lambda},$$

with the summand $\mathbb{C}_{w' \cdot \lambda}$ appearing in degree $\ell(w')$. Since the $w' \cdot \lambda$ are all distinct as weights for u_ζ^0 by Lemma 6.3.2 (cf. also Remark 6.3.3), we get

$$\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda) = \mathrm{H}^{\ell(w)}(\mathcal{U}_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda) \cong \mathbb{C}.$$

In particular, if $\lambda = 0$ and $w = 1$, then $L^\zeta(\lambda) \otimes -w \cdot \lambda = \mathbb{C}$, and we get $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{b}), \mathbb{C}) \cong \mathbb{C}$. This establishes the induction hypothesis.

Now let $0 < i \leq N$, and assume that the proposition is true for $i-1$. Set $V = L^\zeta(\lambda) \otimes -w \cdot \lambda$. Since A_{i-1} is free over its central subalgebra B_i, we can consider the pair of LHS spectral sequences

$$E_2^{a,b}(\mathbb{C}) = \mathrm{H}^a(A_{i-1}//B_i, \mathrm{H}^b(B_i, \mathbb{C})) \Rightarrow \mathrm{H}^{a+b}(A_{i-1}, \mathbb{C}), \quad \text{and} \tag{6.4}$$

$$E_2^{a,b}(V) = \mathrm{H}^a(A_{i-1}//B_i, \mathrm{H}^b(B_i, V)) \Rightarrow \mathrm{H}^{a+b}(A_{i-1}, V). \tag{6.5}$$

Because $B_i \subset A_{i-1}$ are bialgebras, (6.4) is a spectral sequence of algebras, and (6.5) is a module over (6.4). The algebra B_i acts trivially on V, so $\mathrm{H}^\bullet(B_i, V) \cong V \otimes \mathrm{H}^\bullet(B_i, \mathbb{C})$ as an $A_{i-1}//B_i$-module. Moreoever, since B_i is central in A_{i-1}, the action of $A_{i-1}//B_i \cong A_i$ on $\mathrm{H}^\bullet(B_i, \mathbb{C})$ is trivial by [GK, Lemma 5.2.2]. Then the spectral sequences can be rewritten as

$$E_2^{a,b}(\mathbb{C}) = \mathrm{H}^b(B_i, \mathbb{C}) \otimes \mathrm{H}^a(A_i, \mathbb{C}) \Rightarrow \mathrm{H}^{a+b}(A_{i-1}, \mathbb{C}), \quad \text{and} \tag{6.6}$$

$$E_2^{a,b}(V) = \mathrm{H}^b(B_i, \mathbb{C}) \otimes \mathrm{H}^a(A_i, V) \Rightarrow \mathrm{H}^{a+b}(A_{i-1}, V). \tag{6.7}$$

The identification between the E_2-pages of (6.4) and (6.6) is an isomorphism of graded-commutative algebras.

From now on, write $E_2^{a,b}$ to denote generically a term in the E_2-page of either (6.6) or (6.7). Since B_i is a polynomial algebra on a generator of weight $\ell\gamma_i$ for U_ζ^0, $\mathrm{H}^\bullet(B_i, \mathbb{C})$ is an exterior algebra on the one-dimensional vector space $\mathbb{C}_i$ of weight $-\ell\gamma_i$, i.e., $\mathrm{H}^\bullet(B_i, \mathbb{C}) \cong \Lambda^\bullet(\mathbb{C}_i)$. Then $E_2^{a,1} \cong \mathbb{C}_i \otimes E_2^{a,0}$, $E_2^{a,b} = 0$ for all $b \geq 2$, and the only possible non-zero differentials have the form $d_2 : E_2^{a,1} \to E_2^{a+2,0}$. Furthermore, arguing exactly as in [BNPP, §5.3], one can use the induction hypothesis to show that $E_2^{a,0} = 0$ for $a < \ell(w)$, that $E_2^{\ell(w),0} \cong \mathbb{C}$, that $E_2^{\ell(w)+a,0} = 0$ for all odd a, and that $d_2 : E_2^{\ell(w)+a,1} \to E_2^{\ell(w)+a+2,0}$ is injective for all even a (taking $w = 1$ for (6.6)). These observations imply that $E_\infty^{a,b} = 0$ if $b \geq 1$, and hence that the edge maps $\mathrm{H}^\bullet(A_i, \mathbb{C}) \to \mathrm{H}^\bullet(A_{i-1}, \mathbb{C})$ and $\mathrm{H}^\bullet(A_i, V) \to \mathrm{H}^\bullet(A_{i-1}, V)$ of (6.6) and (6.7) are surjective.

Fix $0 \neq v \in \mathbb{C}_i$, and set $z_i = d_2(v) \in \mathrm{H}^2(A_i, \mathbb{C})$. The vector z_i is central in $\mathrm{H}^\bullet(A_i, \mathbb{C})$, because the cohomology ring of a bialgebra is always graded-commutative. Now fix elements $z_1, \ldots, z_{i-1} \in \mathrm{H}^2(A_i, \mathbb{C})$ of weights $-\ell\gamma_1, \ldots, -\ell\gamma_{i-1}$ lifting the polynomial generators for $\mathrm{H}^\bullet(A_{i-1}, \mathbb{C})$. For $r \geq 0$, let $S_i^r \subseteq \mathrm{H}^{2r}(A_i, \mathbb{C})$ be the subspace spanned by all homogenous degree-r monomials in the elements $z_1, \ldots, z_i$. Then S_i^r is a quotient of the space $S^r(R_i)$ defined prior to the statement of the theorem, with the quotient map $S^r(R_i) \to S_i^r$ defined by $f_j \mapsto z_j$. Moreover, there exists a natural multiplication map $S^r(R_i) \otimes \mathrm{H}^{\ell(w)}(A_i, V) \to$

$\mathrm{H}^{\ell(w)+2r}(A_i, V)$ induced by the composition of the quotient map $S^r(R_i) \to S_i^r$ with the natural action of $S_i^r \subseteq \mathrm{H}^{2r}(A_i, \mathbb{C})$ on $\mathrm{H}^{\ell(w)}(A_i, V)$.

We claim that the multiplication map $S^r(R_i) \otimes \mathrm{H}^{\ell(w)}(A_i, V) \to \mathrm{H}^{\ell(w)+2r}(A_i, V)$ is a bijection. Indeed, taking $\lambda = 0$ and $w = 1$, then the claim is equivalent to showing for all $r \geq 0$ that $S^r(R_i) \cong \mathrm{H}^{2r}(A_i, \mathbb{C})$, and hence that $\mathrm{H}^\bullet(A_i, \mathbb{C})$ is a polynomial algebra generated by the elements $\{z_1, \ldots, z_i\}$. Then taking arbitrary $\lambda \in C_\mathbb{Z}$ and $w \in W$, the claim implies that $\mathrm{H}^\bullet(A_i, V)$ is generated freely as a $\mathrm{H}^\bullet(A_i, \mathbb{C})$-module over the one-dimensional space $\mathrm{H}^{\ell(w)}(A_i, V)$.

We proceed to prove the claim. For $r = 0$ the claim is true by the observation $E_2^{\ell(w),0} \cong \mathbb{C}$, so let $r \geq 1$ and assume that the claim is true for $r - 1$. Then there exists a short exact sequence

$$0 \longrightarrow E_2^{\ell(w)+2(r-1),1} \xrightarrow{d_2} E_2^{\ell(w)+2r,0} \longrightarrow E_\infty^{\ell(w)+2r,0} \longrightarrow 0,$$

which may be rewritten as

$$0 \longrightarrow \mathbb{C}_i \otimes \mathrm{H}^{\ell(w)+2(r-1)}(A_i, V) \xrightarrow{d_2} \mathrm{H}^{\ell(w)+2r}(A_i, V) \longrightarrow \mathrm{H}^{\ell(w)+2r}(A_{i-1}, V) \longrightarrow 0. \tag{6.8}$$

By induction on i and r, the natural multiplication maps induce isomorphisms

$$S^{r-1}(R_i) \otimes \mathrm{H}^{\ell(w)}(A_i, V) \cong \mathrm{H}^{\ell(w)+2(r-1)}(A_i, V) \quad \text{and}$$
$$S^r(R_{i-1}) \otimes \mathrm{H}^{\ell(w)}(A_{i-1}, V) \cong \mathrm{H}^{\ell(w)+2r}(A_{i-1}, V).$$

Moreover, since (6.7) is a module over (6.6), there exists a commutative diagram

$$\begin{array}{ccc} S^r(R_{i-1}) \otimes \mathrm{H}^{\ell(w)}(A_i, V) & \longrightarrow & \mathrm{H}^{\ell(w)+2r}(A_i, V) \\ \downarrow \wr & & \downarrow \\ S^r(R_{i-1}) \otimes \mathrm{H}^{\ell(w)}(A_{i-1}, V) & \xrightarrow{\sim} & \mathrm{H}^{\ell(w)+2r}(A_{i-1}, V) \end{array}$$

where the vertical maps are induced by the edge maps of (6.7), and the horizontal maps are the natural multiplication maps. The left-hand vertical map in the diagram is an isomorphism because the edge map $E_2^{\ell(w),0} \to E_\infty^{\ell(w),0}$ is surjective and both spaces are one-dimensional. Also using the fact that (6.6) is a spectral sequence of algebras and that (6.7) is a module over (6.6), we see that the differential $d_2 : \mathbb{C}_i \otimes E_2^{a,0} \to E_2^{a+2,0}$ is just multplication on $E_2^{a+2,0}$ by f_i. Combining the above observations, we conclude that the multiplication map $S^r(R_i) \otimes \mathrm{H}^{\ell(w)}(A_i, V) \to \mathrm{H}^{\ell(w)+2r}(A_i, V)$ is surjective, and also injective by dimension count, and hence an isomorphism. □

Corollary 6.6.2. *Assume that ℓ is odd, that $\ell > h$, and that ℓ is coprime to 3 if Φ has type G_2. Then $\mathrm{H}^{\mathrm{odd}}(u_\zeta(\mathfrak{b}), \mathbb{C}) = 0$ and $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{b}), \mathbb{C}) \cong S^\bullet(\mathfrak{u}^*)^{(1)}$ as U_ζ^0-module algebras. Now let $\lambda \in C_\mathbb{Z}$ and $w \in W$, and suppose that also ℓ is coprime to $n+1$ if Φ is of type A_n, and that ℓ is coprime to 3 if Φ is of type E_6. Then $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda)$ is free as a left $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$-module under the cup product, generated by a vector in degree $\ell(w)$ of weight zero for U_ζ^0.*

Proof. This is the case $i = N$ of the proposition. □

6.7 Cohomology for the nilpotent small quantum group

Now we are ready to compute the structure of the cohomology space $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$.

Theorem 6.7.1. *Assume that ℓ is odd, that $\ell > h$, that ℓ is coprime to $n+1$ if Φ has type A_n, and that ℓ is coprime to 3 if Φ has type E_6 or G_2. Let $\lambda \in C_\mathbb{Z}$ and $w \in W$. Then there exists an isomorphism of left U_ζ^0-modules and of left $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$-modules*

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda)) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda)), \tag{6.9}$$

with $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ acting via the cup product on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$, and via left multiplication on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}), L^\zeta(\lambda))$.

Proof. The theorem follows by applying Corollaries 6.3.4 and 6.6.2. □

The right adjoint action of $U_\zeta(\mathfrak{b})$ on itself stabilizes the subspace $u_\zeta(\mathfrak{u})$ [BNPP, §2.7]. By [Dru2, Theorem 4.3.1], this induces a left action of $U_\zeta(\mathfrak{b})$ on the cohomology space $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$.

Theorem 6.7.2. *The isomorphism* (6.9) *is an isomorphism of left* $U_\zeta(\mathfrak{b})$*-modules.*

Proof. The subalgebra $u_\zeta(\mathfrak{u}) \subset U_\zeta(\mathfrak{b})$ acts trivially on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$, while the root vectors $\{E_\gamma^{(n\ell)} : \gamma \in \Phi^+, n \geq 1\}$ all commute with the subalgebra $u_\zeta^0 \subset U_\zeta(\mathfrak{b})$. It follows then that the action of $U_\zeta(\mathfrak{b})$ on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))$ leaves stable the u_ζ^0-weight spaces, so it suffices to show for each $w \in W$ that the U_ζ^0-module isomorphism

$$\mathrm{Hom}_{u_\zeta^0}(w \cdot \lambda, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), L^\zeta(\lambda))) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), L^\zeta(\lambda) \otimes -w \cdot \lambda) \cong \mathrm{H}^{\bullet - \ell(w)}(u_\zeta(\mathfrak{b}), \mathbb{C})$$

is an isomorphism of $U_\zeta(\mathfrak{b})$-modules.

Set $V = L^\zeta(\lambda) \otimes -w \cdot \lambda$. The theorem now follows from two observations. First, $\mathrm{H}^{\ell(w)}(u_\zeta(\mathfrak{b}), V)$ is by Corollary 6.6.2 a one-dimensional $U_\zeta(\mathfrak{b})$-module of U_ζ^0-weight zero, that is, is isomorphic to the trivial module for $U_\zeta(\mathfrak{b})$. Second, the cup product

$$\cup : \mathrm{H}^{\ell(w)}(u_\zeta(\mathfrak{b}), V) \otimes \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \to \mathrm{H}^{\bullet + \ell(w)}(u_\zeta(\mathfrak{b}), V) \tag{6.10}$$

is a homomorphism of $U_\zeta(\mathfrak{b})$-modules. To see this, use the definition for the adjoint actions of $U_\zeta(\mathfrak{b})$ on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ and $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), V)$ given in [Dru2, (4.3.1)], together with the explicit description for the cup product at the level of cocycles given in [Dru1, (5.3)]. But the cup product in (6.10) is equivalent to the cup product

$$\cup : \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) \otimes \mathrm{H}^{\ell(w)}(u_\zeta(\mathfrak{b}), V) \to \mathrm{H}^{\bullet + \ell(w)}(u_\zeta(\mathfrak{b}), V)$$

by [ML, VIII.4], which is an isomorphism by Corollary 6.6.2. □

6.8 Ring structure

Though $u_\zeta(\mathfrak{u})$ is not a Hopf algebra, $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) = \mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{u})}(\mathbb{C},\mathbb{C})$ is still a ring under the Yoneda composition of extensions.

Theorem 6.8.1. *Assume that ℓ is odd, that $\ell > 2(h-1)$, that ℓ is coprime to $n+1$ if Φ is of type A_n, and that ℓ is coprime to 3 if Φ is of type E_6 or G_2. Then there exists a graded ring isomorphism*

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}).$$

Proof. By Theorem 6.7.1, there exists an isomorphism of U_ζ^0- and $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C})$-modules

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}), \tag{6.11}$$

Also, $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \cong S^{\bullet/2}(\mathfrak{u}^*)^{(1)}$ as U_ζ^0-modules by Corollary 6.6.2. Consider the LHS spectral sequence of U_ζ^0-modules:

$$E_2^{a,b} = \mathrm{H}^a(\mathcal{U}_\zeta(\mathfrak{u})//\mathcal{Z}^+, \mathrm{H}^b(\mathcal{Z}^+,\mathbb{C})) \Rightarrow \mathrm{H}^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}).$$

Since $\mathcal{Z}^+$ is central in $\mathcal{U}_\zeta(\mathfrak{u})$ and $\mathcal{U}_\zeta(\mathfrak{u})//\mathcal{Z}^+ \cong u_\zeta(\mathfrak{u})$, the spectral sequence may be rewritten as

$$E_2^{a,b} = \mathrm{H}^b(\mathcal{Z}^+,\mathbb{C}) \otimes \mathrm{H}^a(u_\zeta(\mathfrak{u}),\mathbb{C}) \Rightarrow \mathrm{H}^{a+b}(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}). \tag{6.12}$$

Moreoever, $\mathrm{H}^\bullet(\mathcal{Z}^+,\mathbb{C}) \cong \Lambda^\bullet(\mathfrak{u}^*)^{(1)}$ as a U_ζ^0-module. We claim that the edge map $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) \to \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ of (6.12) is surjective. Indeed, by [UGA2, Theorem 6.4.1], $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ decomposes as a direct sum of one-dimensional U_ζ^0-modules, $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}) \cong \bigoplus_{w\in W} \mathbb{C}_{w\cdot 0}$. Fix $w \in W$, and suppose $w\cdot 0$ occurs as a weight of U_ζ^0 in $E_2^{a,b}$. Then by (6.11) and (6.12), $w\cdot 0 = w'\cdot 0 + \ell(\sigma_1+\sigma_2)$ for some $w' \in W$ and some weights σ_1 of $\Lambda^\bullet(\mathfrak{u}^*)$ and σ_2 of $S^\bullet(\mathfrak{u}^*)$. In particular, $\sigma_1,\sigma_2 \in \mathbb{N}\Phi^-$. Then Lemma 6.3.2 implies that $\sigma_1+\sigma_2 = 0$, and hence that $\sigma_1 = \sigma_2 = 0$. This is only possible if $b = 0$, so it follows that the edge map

$\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) \to \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ is surjective, and that under the U_ζ^0-module isomorphism of (6.11), the horizontal edge map in (6.12) is the projection onto the subspace

$$1 \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}) \subset \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}).$$

Now let $\mathcal{B} \subset \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$ be the U_ζ^0-submodule corresponding to the subspace $1 \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ under the isomorphism (6.11). Considering the weights of $\mathcal{B}$ and arguing as in the proof of Theorem 4.1.1, we see that $\mathcal{B}$ is in fact a subalgebra of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$. (In particular, Lemma 3.1.4 remains valid with the same proof if ℓ is substituted for p.) Since the edge map $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}) \to \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ maps $\mathcal{B}$ isomorphically onto $\mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$, we conclude that $\mathcal{B} \cong \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C})$ as algebras.

Recall that $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C})$ identifies with the subalgebra $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})^{u_\zeta^0}$ of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$. The inclusion $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \hookrightarrow \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$ is just the restriction map in cohomology. Then from (6.11) we see that $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$ is generated as an algebra by $\mathcal{B}$ together with $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C})$. Finally, it follows from [ML, VIII.4] and the fact that $\mathrm{H}^{\mathrm{odd}}(u_\zeta(\mathfrak{b}),\mathbb{C}) = 0$ that $\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C})$ is a central subalgebra of $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$. We conclude that multiplication in $\mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})$ induces an isomorphism of algebras

$$\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}),\mathbb{C}) \otimes \mathrm{H}^\bullet(\mathcal{U}_\zeta(\mathfrak{u}),\mathbb{C}) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C})^{u_\zeta^0} \otimes \mathcal{B} \xrightarrow{\sim} \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}),\mathbb{C}). \qquad \square$$

6.9 Parabolic computations

For completeness, we now state the quantum analog of the parabolic computations in Chapter 5. Before stating the main result, we point out that the cohomology space $\mathrm{H}^\bullet(u_\zeta(\mathfrak{p}_J),L^\zeta(\lambda))$ is naturally a rational $\mathrm{Dist}(P_J)$-module (equivalently, a rational P_J-module). The $\mathrm{Dist}(P_J)$-module structure arises as follows. First, the right adjoint action of $U_\zeta(\mathfrak{p}_J)$ on itself stabilizes the subalgebra $u_\zeta(\mathfrak{p}_J) \subset U_\zeta(\mathfrak{p}_J)$. This gives rise to a natural action of $U_\zeta(\mathfrak{p}_J)$ on $\mathrm{H}^\bullet(u_\zeta(\mathfrak{p}_J),L^\zeta(\lambda))$, which factors through the quotient $\mathrm{Dist}(P_J) \cong U_\zeta(\mathfrak{p}_J)//u_\zeta(\mathfrak{p}_J)$.

Theorem 6.9.1. *Assume that ℓ is odd, that ℓ is coprime to $n+1$ if Φ has type A_n, and that ℓ is coprime to 3 if Φ has type E_6 or G_2. Let $\lambda \in X^+ \cap \overline{C}_{\mathbb{Z}}$ (so $\ell > h$), and let $J \subseteq \Pi$.*

(a) $\mathrm{H}^\bullet(u_\zeta(\mathfrak{p}_J), L^\zeta(\lambda)) = 0$ *unless λ is weakly ℓ-linked to zero.*

(b) If $\lambda = w \cdot 0 + \ell\sigma$ for some $w \in W$ and $\sigma \in X$, then $\lambda = 0$ or σ is minuscule.

(c) Suppose $\lambda = w \cdot 0 + \ell\sigma$. Then there exists a P_J-module isomorphism

$$\mathrm{H}^j(u_\zeta(\mathfrak{p}_J), L^\zeta(\lambda)) \cong \begin{cases} \mathrm{ind}_B^{P_J}[S^{\frac{j-\ell(w)}{2}}(\mathfrak{u}^*) \otimes w^{-1}\sigma] & \text{if } j \equiv \ell(w) \text{ mod } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is analogous to that given for Theorem 5.2.1, so the details are omitted. □

Chapter 7

Low Degree Cohomology for Frobenius Kernels

7.1 Main tools

Let G be a reduced and irreducible algebraic group defined over k. In this section, we introduce some methods to compute B_r and G_r-cohomology which will be applied later to obtain results on low degree cohomology and $(SL_2)_r$-cohomology. For simplicity, we write S^i and Λ^j instead of $S^i(\mathfrak{u}^*)$ and $\Lambda^j(\mathfrak{u}^*)$. We start with the spectral sequence in [Jan2, Proposition I.9.14]. Replacing $\mathfrak{g}$ by $\mathfrak{u}$, we immediately get a new spectral sequence to compute cohomology of U_r with coefficients in a B-module M. The resulting spectral sequence can be written as follows:

$$E_1^{i,j} = \bigoplus M \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)} \Rightarrow \mathrm{H}^{i+j}(U_r, M) \qquad (7.1)$$

where the direct sum is taken over all a_i's and b_j's satisfying

$$\begin{cases} i+j &= 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r \\ i &= \sum_{n=1}^{r}(a_n p^n + b_n p^{n-1}). \end{cases} \qquad (7.2)$$

This spectral sequence will play an important role in our calculations for cohomology in later sections.

7.1.1 Spectral sequence for B_r-cohomology

In order to set up an inductive proof and some intuition for the proofs, we first look at a simple case when $r = 2$. As T_2 is diagonalisable, the fixed point functor $(-)^{T_2}$ is exact. Hence,

$$\mathrm{H}^{i+j}(U_2, M)^{T_2} \cong \mathrm{H}^{i+j}(B_2, M).$$

So applying the fixed point functor on both sides of the spectral sequence (7.1), we have

$$\begin{aligned}
\bigoplus \left(M \otimes S^{a_1(1)} \otimes S^{a_2(2)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)}\right)^{T_2} &\Rightarrow \mathrm{H}^{i+j}(U_2, M)^{T_2} \\
\bigoplus \left(S^{a_1(1)} \otimes S^{a_2(2)} \otimes (M \otimes \Lambda^{b_1})^{T_1} \otimes \Lambda^{b_2(1)}\right)^{T_2/T_1} &\Rightarrow \mathrm{H}^{i+j}(B_2, M) \\
\bigoplus \left(S^{a_1} \otimes S^{a_2(1)} \otimes (M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes \Lambda^{b_2}\right)^{T_1^{(1)}} &\Rightarrow \mathrm{H}^{i+j}(B_2, M) \\
\bigoplus S^{a_2(2)} \otimes \left(S^{a_1} \otimes \Lambda^{b_2} \otimes (M \otimes \Lambda^{b_1})^{T_1(-1)}\right)^{T_1^{(1)}} &\Rightarrow \mathrm{H}^{i+j}(B_2, M)
\end{aligned}$$

where each direct sum is taken over all a_i, b_i satisfying the condition (7.2). This computation can be generalized for arbitrary r as follows.

Theorem 7.1.1. *There exists, for each B-module M, a spectral sequence converging to $\mathrm{H}^\bullet(B_r, M)$ as a $B^{(r)}$-module with the first page:*

$$E_1^{i,j} = \bigoplus S^{a_r(r)} \otimes \left[\left[(M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}} \tag{7.3}$$

where the direct sum is taken over all a_i, b_j satisfying condition (7.2). Alternatively, we can write

$$\bigoplus_{n=2(a_1+..+a_r)+b_1+..+b_r} S^{a_r(r)} \otimes \left[\left[(M \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}} \Rightarrow \mathrm{H}^n(B_r, M)$$

Proof. We first consider the spectral sequence for U_1-cohomology as follows:

$$\bigoplus M \otimes S^{a_1(1)} \otimes \Lambda^{b_1} \Rightarrow \mathrm{H}^n(U_1, M)$$

Taking T_1-invariant functor on both sides, we have

$$\begin{aligned}
&\bigoplus (M \otimes S^{a_1(1)} \otimes \Lambda^{b_1})^{T_1} \Rightarrow \mathrm{H}^n(U_1, M)^{T_1} \cong \mathrm{H}^n(B_1, M) \\
&\bigoplus S^{a_1(1)} \otimes (M \otimes \Lambda^{b_1})^{T_1} \Rightarrow \mathrm{H}^n(B_1, M).
\end{aligned}$$

This verifies the theorem for $r = 1$. Suppose it is true for r. Apply the invariant functor $(-)^{T_{r+1}}$ on the U_{r+1}-spectral sequence, we obtain

$$\bigoplus \left(M \otimes S^{a_1(1)} \otimes .. \otimes S^{a_{r+1}(r+1)} \otimes \Lambda^{b_1} \otimes .. \otimes \Lambda^{b_{r+1}(r)}\right)^{T_{r+1}} \Rightarrow \mathrm{H}^{i+j}(U_{r+1}, M)^{T_{r+1}} \cong \mathrm{H}^{i+j}(B_{r+1}, M)$$

with a_i, b_j satisfying

$$\begin{cases} i + j &= 2(a_1 + \cdots + a_{r+1}) + b_1 + \cdots + b_{r+1} \\ i &= \sum_{n=1}^{r+1} (a_n p^n + b_n p^{n-1}). \end{cases}$$

In order to complete our induction proof on r, we show that the E_1-page of this above spectral sequence can be rewritten in the form of (7.3). First note that $T_j/T_i \cong T_{j-i}^{(i)}$ for $0 \leq i \leq j$ where $T_0 = T$. So if M is a T-module, it can be identified with a T/T_i-module for each i via the aforementioned isomorphism. In particular, we have for each $0 \leq i \leq j$

$$M^{T_j/T_i} \cong M^{T_{j-i}^{(i)}} \cong \left(M^{(-i)}\right)^{T_{j-i}}.$$

This observation gives us the following isomorphisms on each direct summand of LHS:

$$\begin{aligned}
&\Big[\big(M\otimes S^{a_1(1)}\otimes\cdots\otimes S^{a_r(r)}\otimes\Lambda^{b_1}\otimes\cdots\otimes\Lambda^{b_r(r-1)}\big)^{T_r}\otimes S^{a_{r+1}(r+1)}\otimes\Lambda^{b_{r+1}(r)}\Big]^{T_{r+1}/T_r}\\
&\cong\Big[\big(M\otimes S^{a_1(1)}\otimes\cdots\otimes S^{a_r(r)}\otimes\Lambda^{b_1}\otimes\cdots\otimes\Lambda^{b_r(r-1)}\big)^{T_r(-r)}\otimes S^{a_{r+1}(1)}\otimes\Lambda^{b_{r+1}}\Big]^{T_1^{(r)}}\\
&\cong\left[\left(S^{a_r(r)}\otimes\Big[\big[(M\otimes\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes\Lambda^{b_2}\big]^{T_1(-1)}\otimes..\otimes\Lambda^{b_r}\Big]^{T_1^{(r-1)}}\right)^{(-r)}\otimes S^{a_{r+1}(1)}\otimes\Lambda^{b_{r+1}}\right]^{T_1^{(r)}}\\
&\cong\left[S^{a_r}\otimes\Big[\big[(M\otimes\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes\Lambda^{b_2}\big]^{T_1(-1)}\otimes..\otimes\Lambda^{b_r}\Big]^{T_1^{(-1)}}\otimes S^{a_{r+1}(1)}\otimes\Lambda^{b_{r+1}}\right]^{T_1^{(r)}}\\
&\cong\left[\Big[\big[(M\otimes\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes\Lambda^{b_2}\big]^{T_1(-1)}\otimes..\otimes\Lambda^{b_r}\Big]^{T_1(-1)}\otimes S^{a_r}\otimes\Lambda^{b_{r+1}}\right]^{T_1^{(r)}}\otimes S^{a_{r+1}(r+1)}
\end{aligned}$$

where the second isomorphism is by inductive hypothesis. This completes our proof. □

7.1.2 Spectral sequence for G_r-cohomology

Recall from [Jan2, II.12.2] that if $R^i \operatorname{ind}_B^G M = 0$ for all $m > 0$, then G_r-cohomology can be computed from the following spectral sequence

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, M)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \operatorname{ind}_B^G M)^{(-r)}.$$

In particular, for every dominant weight $\lambda \in X^+$, we always have

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, \lambda)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \mathrm{H}^0(\lambda))^{(-r)}.$$

So our strategy is to compute $\mathrm{H}^m(B_r, \lambda)$ first by Theorem 7.1.1, then use this spectral sequence to get G_r-cohomology of $\mathrm{H}^0(\lambda)$.

7.2 Some results on low degree cohomology

In this section, we apply our previous calculations to show that $\mathrm{H}^n(B_r, k)$ is isomorphic to $\mathrm{H}^n(B_1, k)$ as a vector space for all degree $n \leq \frac{p}{c}$ with p a very good prime for the system Φ (see Section 2.1); and c will be defined at the beginning of Section 7.2.2. This implies similar results on the cohomology $\mathrm{H}^n(G_r, k)$ with same restrictions on n.

7.2.1 Injective inflation maps

Recall that G is defined over $\mathbb{F}_p$. Hence we can identify the algebraic group $G^{(r)}$ with G via the F^r as in [Jan2, Remark I.9.5], so that $B_r/B_{r-1} \cong B_1$. Therefore, the Lyndon-Hochschild-Serre spectral sequence gives us the inflation maps of vector spaces

$$\mathrm{H}^n(B_1, k)^{(r-1)} \cong \mathrm{H}^n(B_r/B_{r-1}, k) \to \mathrm{H}^n(B_r, k)$$

for each $n \geq 0$. In fact, these maps are injective. Before giving a proof for this fact, we state some results on the Steinberg representations. They played a key role in the proof of Kempf's Vanishing Theorem and other topics. We will exploit the injectivity of these module to show injectivity of our base maps. Further study can be found in [Jan2, Chapter II.10].

For each $r \geq 1$, we define $St_r = L((p^r - 1)\rho)$ the r-th *Steinberg module*. Here we recall that ρ is a half sum of all positive roots in Φ^+. Then denote

$$St'_r = St_r \otimes (p^r - 1)\rho.$$

We also recall some properties of the two modules which will be useful later.

Proposition 7.2.1. *[Jan2, Proposition II.10.2, Remark and Lemma II.10.11(3)]*
For each $r \geq 1$, we have

(a) the Steinberg module St_r is injective and projective as both G_rT-module and G_r-module.

(b) As a B_rT-module, St'_r is an injective hull of k. Moreover,

$$(St'_r)^B \cong k \cong (St'_r)^{B_r}$$

Lemma 7.2.2. *For each $r \geq 1$, the module St'_r is an injective B_r-module.*

Proof. As St'_r is a B-module, it is naturally a B_r-module. The proof immediately follows from Proposition 7.2.1(b) and Lemma II.9.3(ii) in [Jan2]. □

Now we are ready to show our inflation maps are injective. First we recall a general fact.

Theorem 7.2.3. *[Jan2, Proposition I.6.10] Let N be exact in G. Suppose Q is a G-module, E is a G-submodule of Q such that E is finitely generated and projective over k. If Q is injective as an N-module and if $\mathrm{Hom}_N(E, Q) = k$, then the base maps*

$$\mathrm{Ext}^n_{G/N}(M_1, M_2) \to \mathrm{Ext}^n_G(E \otimes M_1, E \otimes M_2)$$

are injective for each n and all G/N-modules M_1, M_2.

Set $E = M_1 = M_2 = k$, the theorem can be written as follow:

Theorem 7.2.4. *[Jan2, Remark I.6.10(1)] If there is a G-module Q which is injective for N and satisfying $Q^G = Q^N \cong k$ then the inflation maps*

$$\mathrm{H}^n(G/N, k) \to \mathrm{H}^n(G, k)$$

are injective.

Theorem 7.2.5. *For each $s < r$, there are embedding maps of vector spaces $\mathrm{H}^n(B_s, k) \hookrightarrow \mathrm{H}^n(B_r, k)$.*

Proof. We apply Theorem 7.2.4 to the B_r-injective module $Q = St'_{r-s}$ (from Lemma 7.2.2) with $G = B_r$ and $N = B_{r-s}$. Note that as B_r/B_s is isomorphic to B_{r-s} for each $0 \leq s \leq r$, it is an affine scheme. So by [Jan2, I.6.5(2)], B_s is exact in B_r for each $0 \leq s \leq r$. It follows the injectivity as desired. $\square$

7.2.2 Main results

Every root $\beta \in \Phi^+$ can be written as a sum of simple roots with non-negative coefficients. Let c be the largest coefficient in all the expressions of positive roots in terms of simple roots. By studying Table 2 in [Hum1, 12.2], one can see that if G is of type A_n (resp. $B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$) then $c = 1$ (resp. $2, 2, 2, 3, 4, 6, 4, 3$).

Lemma 7.2.6. *Let p be a very good prime for Φ. Let a, b be non-negative integers satisfying $a + b < \frac{p}{c}$. Then we have*

$$(S^a \otimes \Lambda^b)^{T_1} \cong \begin{cases} k & \text{if } a = b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $(S^a \otimes \Lambda^b)^{T_1} \neq 0$. Then, there is a weight $\tau \in X$ such that $p\tau = \sigma + \lambda$ where σ, λ are respectively weights of S^a and Λ^b. Since $\sigma + \lambda \in \mathbb{N}\Phi$, so is $p\tau$. As p is a very good prime, it does not divide the order of finite group $X/(\mathbb{N}\Phi)$; hence τ must be in the root lattice $\mathbb{N}\Phi$. Note that $\sigma + \lambda$ is the sum of $a + b$ positive roots (not necessarily distinct), we can write $\sigma + \lambda = \Sigma_{i=1}^{m} d_i \alpha_i$ with $m = |\Pi|$, simple root α_i and non-negative integers d_i for $1 \leq i \leq m$. Now observe that, for each $1 \leq i \leq m$, we have the coefficient

$$d_i \leq (a + b)c < \frac{p}{c} c = p.$$

In other words, $\sigma + \lambda$ can not be a p-multiple of any nonzero root. Therefore, $\tau = 0$ and so $a = b = 0$. $\square$

Theorem 7.2.7. *Let p be a very good prime for Φ. Then for each $n < \frac{p}{c}$, there is a $B^{(r)}$-*

module isomorphism

$$\mathrm{H}^n(B_r,k)\cong \mathrm{H}^n(B_1,k)^{(r-1)}\cong \begin{cases}(S^{\frac{n}{2}})^{(r)} & \textit{if } n \textit{ even},\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We first prove for the case $r=1$. Note that this has been verified for $p>h$ (see [AJ]). Taking T_1-invariant functor on the Friedlander-Parshall spectral sequence for U_1-cohomology [FP1, Proposition 1.1], we obtain that

$$E_2^{2i,j}=(S^i)^{(1)}\otimes(\mathrm{H}^j(\mathfrak{u},k))^{T_1}\Rightarrow \mathrm{H}^{2i+j}(B_1,k). \tag{7.4}$$

Now we consider all indices i,j such that $2i+j\le \frac{p}{c}$. Since we can identify $\mathrm{H}^j(\mathfrak{u},k)$ with a T-submodule of Λ^j, and by Lemma 7.2.6

$$(\Lambda^j)^{T_1}\cong\begin{cases}k & \text{if } j=0,\\ 0 & \text{otherwise,}\end{cases}$$

it follows that $\mathrm{H}^j(\mathfrak{u},k)^{T_1}=k$ if $j=0$, and is 0 otherwise. We then have

$$E_2^{2i,j}\cong\begin{cases}(S^i)^{(1)} & \text{if } j=0,\\ 0 & \text{otherwise.}\end{cases}$$

This implies that all the differentials of the spectral sequence $d_h^{2i,j}:E_h^{2i,j}\to E_h^{2i+h,j+1-h}$ are 0 for all $0\le j\le\frac{p}{c}$ since either $E^{2i,j}=0$ or $E_h^{2i+h,j+1-h}=0$ with $h\ge 2$. Therefore, the spectral sequence (7.4) has $E_\infty^{2i,j}=E_2^{2i,j}$ for all $2i+j<\frac{p}{c}$. This implies that $\mathrm{H}^n(B_1,k)\cong(S^{\frac{n}{2}})^{(1)}$ for even degree $n<\frac{p}{c}$.

Now assume that $r>1$ and look at the spectral sequence in Theorem 7.1.1

$$\bigoplus S^{a_r(r)}\otimes\left[\left[(\Lambda^{b_1})^{T_1(-1)}\otimes S^{a_1}\otimes\Lambda^{b_2}\right]^{T_1(-1)}\otimes..\otimes S^{a_{r-1}}\otimes\Lambda^{b_r}\right]^{T_1^{(r-1)}}\Rightarrow \mathrm{H}^n(B_r,k)$$

where the direct sum is taken over all a_i, b_j satisfying $n = 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r$. Since $n < \frac{p}{c}$, we have $b_i, a_i + b_{i+1} < \frac{p}{c}$ for each $1 \leq i \leq r-1$. Lemma 7.2.6 shows that

$$\left[\left[(\Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2}\right]^{T_1(-1)} \otimes .. \otimes S^{a_{r-1}} \otimes \Lambda^{b_r}\right]^{T_1^{(r-1)}} \cong \begin{cases} k & \text{if } a_i = b_i = 0 \text{ for} \\ & 1 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we look at the spectral sequence at the first page with $i + j < \frac{p}{c}$, we can see that $E_1^{i,j} = 0$ except $E_1^{p^r(\frac{i+j}{2}),(2-p^r)(\frac{i+j}{2})} = (S^{\frac{i+j}{2}})^{(r)}$. It follows that $\mathrm{H}^n(B_r, k)$ is a subquotient of $(S^{\frac{n}{2}})^{(r)}$ for $n < \frac{p}{c}$. On the other hand, Theorem 7.2.5 gives us that as a vector space

$$(S^{\frac{n}{2}})^{(r)} = (S^{\frac{n}{2}})^{(1)} \cong \mathrm{H}^n(B_1, k) \hookrightarrow \mathrm{H}^n(B_r, k).$$

As $S^{\frac{n}{2}}$ is a finite dimensional vector space, $\mathrm{H}^n(B_r, k)$ is isomorphic to $(S^{\frac{n}{2}})^{(r)}$ as a B-module when n is even. □

Remark 7.2.8. Apply this Theorem for $n = 1$ and $p \geq 3$, we immediately get that $\mathrm{H}^1(B_r, k) = 0$ for G of type A, B, C, and D. Some argument is needed for the other types. This result agrees with the one in [BNP1] and [Jan2, II.12].

Note that for each $m < \frac{p}{c}$, by [KLT] we have for each $n > 0$

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, k)^{(-r)}) \cong R^n \operatorname{ind}_B^G(S^{\frac{m}{2}}(\mathfrak{u}^*)) = 0.$$

So the same technique can be applied to the spectral sequence

$$R^n \operatorname{ind}_B^G(\mathrm{H}^m(B_r, k)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, k)^{(-r)}$$

to get G_r-cohomology for degree up to $\frac{p}{c}$. Hence, we obtain the following result.

Theorem 7.2.9. *Let p be a very good prime for Φ. For each $n < \frac{p}{c}$, there are isomorphisms* *of G-modules*

$$\mathrm{H}^n(G_r,k)^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^n(B_r,k)^{(-r)}) \cong \begin{cases} \mathrm{ind}_B^G(S^{\frac{n}{2}}) & \text{if } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for each $n < \frac{p}{c}$, *we have*

$$\mathrm{H}^n(G_r,k) \cong \mathrm{H}^n(G_1,k)^{(r-1)}.$$

Remark 7.2.10. We notice that Friedlander and Parshall also studied this problem in the paper [FP1, Theorem 1.8]. They showed that if G is a simple, simply connected algebraic group of type A with $p > h$, then for each $r > 1$, the inflation map $\mathrm{H}^i(G_r/G_{r-1},k) \to \mathrm{H}^i(G_r,k)$ is an isomorphism for all $i < 2p-1$. Recently, Bendel-Nakano-Pillen [BNP2, Theorem 6.1] computed the second cohomology of Frobenius kernels for arbitrary type of G and prime p that for each $r \geq 1$,

$$\mathrm{H}^2(G_r,\mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G(\mathrm{H}^2(B_r,\lambda)^{(-r)})$$

for $\lambda \in X^+$. Hence, it could be seen that our result generalized both theorems in a certain sense.

Chapter 8

Cohomology for Frobenius kernels of SL_2

We assume in this chapter that $G = SL_2$ and p is an arbitrary odd prime. Let α be the only simple root in the root system Φ of G. Denote by ω the fundamental weight in the weight lattice X [Hum1, 13.2]. Then we have $\omega = \frac{\alpha}{2}$. Note also that as $\mathfrak{u}$ is a one-dimensional vector space, we have the following T-module identifications on each degree of the exterior algebra

$$\Lambda^i = \Lambda^i(\mathfrak{u}^*) = \begin{cases} k & \text{if } i = 0, \\ \alpha & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Our overall goal is computing the cohomology $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))$ for every dominant weight $\lambda \in X^+$. Following the strategy in Subsection 7.1.2, we start with U_r-cohomology.

8.1 Cohomology of U_r

Proposition 8.1.1. *Let λ be a dominant weight. For each $r \geq 1$, there is a B-isomorphism*

$$\mathrm{H}^n(U_r, \lambda) \cong \bigoplus_{n=2(a_1+\cdots+a_r)+b_1+\cdots+b_r} \lambda \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)}.$$

Proof. Recall from Section 7.1 that U_r-cohomology can be computed by the following spectral sequence

$$E_1^{i,j} = \bigoplus \lambda \otimes S^{a_1(1)} \otimes \cdots \otimes S^{a_r(r)} \otimes \Lambda^{b_1} \otimes \Lambda^{b_2(1)} \otimes \cdots \otimes \Lambda^{b_r(r-1)} \Rightarrow \mathrm{H}^{i+j}(U_r, \lambda)$$

where the direct sum is taken over all tuples $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying $i + j = 2(a_1 + \ldots + a_r) + b_1 + \ldots + b_r$ and $i = \sum_{n=1}^{r}(a_n p^n + b_n p^{n-1})$. Observe that as a B-module, $S^m = m\alpha$ and $\Lambda^0 = k, \Lambda^1 = \alpha$, and $\Lambda^m = 0$ for $m > 1$. For each $n > 0$, we have $d_n^{i,j} : E_n^{i,j} \to E_n^{i+n,j-n+1}$ where the B-module $E_n^{i,j}$ has weight

$$\begin{aligned} \lambda + pa_1\alpha + \cdots + p^r a_r\alpha + b_1\alpha + \cdots + p^{r-1}b_r\alpha &= \lambda + (pa_1 + \cdots + p^r a_r + b_1 + \cdots + p^{r-1}b_r)\alpha \\ &= \lambda + i\alpha. \end{aligned}$$

Likewise, $E_n^{i+n,j-n+1}$ is of weight $\lambda + (i+n)\alpha$. As all the differentials respect to T-action, we must have $\lambda + i\alpha = \lambda + (i+n)\alpha$ if the map is nonzero. This implies $n = 0$, and so $d_n^{i,j} = 0$ for all i, j and $n > 0$. Hence the spectral sequence collapses at the first page. The result therefore follows. □

When $\lambda = 0$, the isomorphism is also compatible with the ring structure. This computation was completely done by Andersen-Jantzen in [AJ, 2.4]. We recall their result as follows.

Corollary 8.1.2. *For each $r \geq 1$, there is an isomorphism of B-algebra*

$$\mathrm{H}^\bullet(U_r, k) \cong S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^\bullet \otimes \cdots \otimes \Lambda^{\bullet(r-1)}.$$

Consequently,

$$\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}} \cong S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)}.$$

Remark 8.1.3. Another way to obtain the isomorphism is to exploit the fact that U_r is an abelian group which is described as the product $U_1 \times U_1^{(1)} \times \cdots \times U_1^{(r-1)}$. The theory of group cohomology gives us a complete description of $\mathrm{H}^\bullet(U_r, k)$ as a ring. Our proof actually has more flavors than that. It guarantees the isomorphism is compatible with the B-module structure which will become handy in computing G_r-cohomology later.

8.2 Cohomology of B_r

For later convenience, we identify $S^{\bullet(i)}$ with the polynomial ring $k[x]^{(i)} = k[x^{(i)}]$ where $x^{(i)}$ of weight $p^i\alpha$ and degree 2. We also denote by $y^{(i)}$ the generator of the exterior algebra $\Lambda^{(i)}$. In particular, we have

$$S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^\bullet \otimes \cdots \otimes \Lambda^{\bullet(r-1)} = k[x_1^{(1)}, \ldots, x_r^{(r)}] \otimes \Lambda(y_1, y_2^{(1)}, \ldots, y_r^{(r-1)}).$$

Now we make use of this notation to describe the B_r-cohomology.

Proposition 8.2.1. *Suppose λ is a dominant weight in X^+, i.e., $\lambda = m\omega$ for some non-negative integer m. For each $r \geq 1$, there is a B-isomorphism*

$$\mathrm{H}^n(B_r, \lambda)^{(-r)} \cong \bigoplus \left\langle (x_1^{(1)})^{a_1} y_1^{b_1} (x_2^{(2)})^{a_2} (y_2^{(1)})^{b_2} \cdots (x_r^{(r)})^{a_r} (y_r^{(r-1)})^{b_r} \right\rangle$$

where the direct sum is taken over all a_i, b_j satisfying the following conditions

$$\begin{cases} a_i \in \mathbb{N} \text{ and } b_i \in \{0,1\} \text{ for all } 1 \leq i \leq r \\ n = 2(a_1 + \cdots + a_r) + b_1 + \cdots + b_r \\ \frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + a_r p^r \in p^r X^+. \end{cases} \tag{8.1}$$

Proof. It is observed that collapsing of the spectral sequence (7.1) implies the collapse of the one in Theorem 7.1.1. So Proposition 8.1.1 implies that

$$\mathrm{H}^n(B_r, \lambda) \cong \bigoplus S^{a_r(r)} \otimes \left[\left[(\lambda \otimes \Lambda^{b_1})^{T_1(-1)} \otimes S^{a_1} \otimes \Lambda^{b_2} \right]^{T_1(-1)} \otimes .. \otimes S^{a_{r-1}} \otimes \Lambda^{b_r} \right]^{T_1^{(r-1)}}$$

where the direct sum is taken over all tuples $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying $n = 2(a_1 + \ldots + a_r) + b_1 + \ldots + b_r$. Using the identifications earlier, we can explicitly write out the cohomology for B_r as a decomposition of weight spaces

$$\mathrm{H}^n(B_r, \lambda)^{(-r)} \cong \bigoplus \left\langle (x_1^{(1)})^{a_1} y_1^{b_1} (x_2^{(2)})^{a_2} (y_2^{(1)})^{b_2} \cdots (x_r^{(r)})^{a_r} (y_r^{(r-1)})^{b_r} \right\rangle$$

where each monomial weights

$$\left[a_r\alpha + \frac{\frac{\frac{\frac{\lambda + b_1\alpha}{p} + (a_1+b_2)\alpha}{p} + (a_2+b_3)\alpha}{p} + \cdots + (a_{r-1} + b_r)\alpha}{p} \right] \in X^+.$$

This is equivalent to $\frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + a_r p^r \in p^r X^+$. Now as U is an abelian group in this case, it trivially acts on both sides of the above isomorphism. Hence, this isomorphism is compatible with B-action via the identification $B/U \cong T$. $\square$

Remark 8.2.2. For each tuple $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0,1\}^r$ satisfying (8.1), there is $\gamma = n\omega$ for some non-negative integer n such that

$$\frac{m}{2} + b_1 + (a_1 + b_2)p + \cdots + (a_{r-1} + b_r)p^{r-1} + a_r p^r = \frac{n}{2}p^r. \tag{8.2}$$

Let $N_r(p, m, n)$ denote the number of solutions $(a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{N}^r \times \{0, 1\}^r$ for the equation. As a consequence, we establish our goal of this subsection.

Theorem 8.2.3. *Suppose $\lambda = m\omega \in X^+$. Then there is a B-module isomorphism*

$$\mathrm{H}^\bullet(B_r, \lambda)^{(-r)} \cong \bigoplus_{n=0}^{\infty} (n\omega)^{N_r(p,m,n)}$$

and so

$$\mathrm{ch}\, \mathrm{H}^\bullet(B_r, \lambda)^{(-r)} = \sum_{n \in \mathbb{N}} N_r(p, m, n) e(n\omega).$$

Remark 8.2.4. Investigating the number $N_r(p, m, n)$ is an interesting problem. One can compute it by the algorithm for counting integral lattice points in a polytope. In particular, let P_r be a polytope in $\mathbb{R}^r$ determined by the following equations:

$$\begin{cases} \frac{m}{2} + y_1 + (x_1 + y_2)p + \cdots + (x_{r-1} + y_r)p^{r-1} + x_r p^r = \frac{n}{2}p^r, \\ 0 \leq x_i \leq \frac{n}{2}p^r \text{ for each } 1 \leq i \leq r, \\ 0 \leq y_i \leq 1 \text{ for each } 1 \leq i \leq r. \end{cases}$$

Then we have $N_r(p, m, n) = |P_r \cap \mathbb{Z}^r|$ for each $r \geq 1$. So one can use Barvinok's algorithm to compute the right-hand side. However, this algorithm is getting slow when r is big due to many complicated subalgorithms and computations involving Complex Analysis. Barvinok actually proved that the algorithm terminates after a polynomial time for a fixed dimension depending on the data for vertices of the polytope (see [Bar] for further details). An alternative program to compute $N_r(p, m, n)$ is written in Appendix 12.2.

8.3 Reduced ring for B_r-cohomology

In Theorem 8.2.3, we computed $\mathrm{H}^\bullet(B_r, k)$ as a B-module. Describing its ring structure, however, is extremely hard for big r (see [AJ]). By looking at the reduced part, we can compute $\mathrm{H}^\bullet(B_r, k)$ as a finitely generated $\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}$-module. We first need the following observation.

Lemma 8.3.1. *Given any T-algebra M, there is an isomorphism*

$$(M^{T_r})_{\mathrm{red}} \cong (M_{\mathrm{red}})^{T_r}.$$

Proof. Note that $M_{\mathrm{red}} = M/\operatorname{Nilrad} M$, we consider the short exact sequence

$$0 \to \operatorname{Nilrad} M \to M \to M_{\mathrm{red}} \to 0.$$

As T_r is diagonalisable, $(-)^{T_r}$ is exact, so we obtain that

$$0 \to (\operatorname{Nilrad} M)^{T_r} \to (M)^{T_r} \to (M_{\mathrm{red}})^{T_r} \to 0.$$

On the other hand, we know that $(M^{T_r})_{\mathrm{red}} = M^{T_r}/\operatorname{Nilrad}(M^{T_r})$. So we just need to check that $(\operatorname{Nilrad} M)^{T_r} = \operatorname{Nilrad}(M^{T_r})$ which is true since both equal to $\operatorname{Nilrad}(M) \cap M^{T_r}$. $\square$

From Corollary 8.1.2, we can identify $\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}$ with the polynomial algebra $k[x_1^{(1)}, \ldots, x_r^{(r)}]$ where each $x_i^{(i)}$ is of weight $p^i\alpha$. The preceding lemma and Corollary 8.1.2 imply that

$$\begin{aligned}
\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} &\cong \left(\mathrm{H}^\bullet(U_r, k)^{T_r}\right)_{\mathrm{red}} \cong \left(\mathrm{H}^\bullet(U_r, k)_{\mathrm{red}}\right)^{T_r} \\
&\cong \left(S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)}\right)^{T_r} \\
&\cong \left(k[x_1^{(1)}, \ldots, x_r^{(r)}]\right)^{T_r}.
\end{aligned}$$

As a B-module, this reduced cohomology ring can be represented in terms of monomials as follows.

Theorem 8.3.2. *For $r \geq 1$, there is a $B^{(r)}$-module isomorphism*

$$\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \cong \bigoplus \left\langle x_1^{(1)a_1} x_2^{(2)a_2} \dots x_r^{(r)a_r} \right\rangle \tag{8.3}$$

where a_i's are non-negative integers satisfying

$$a_1 + a_2 p + \cdots + a_r p^{r-1} = np^{r-1} \tag{8.4}$$

for some $n \in \mathbb{N}$. Furthermore, let $R = k[x_1^{(1)p^{r-1}}, x_2^{(2)p^{r-2}}, \dots, x_r^{(r)}]$, then the reduced ring $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ is a finitely free R-module with the basis $\mathfrak{B}_r = \{x_1^{(1)a_1} x_2^{(2)a_2} \dots x_{r-1}^{(r-1)a_{r-1}}\}$ where for each $i = 1, \dots, r-1$, $0 \leq a_i < p^{r-i}$ and satifies the equation (8.4).

Proof. The right-hand side is obtained by setting $b_1 = b_2 = \cdots = b_r = 0$ in the Theorem 8.2.1. Now observe that every tuple $(m_1 p^{r-1}, m_2 p^{r-2}, \dots, m_r)$ with m_i a non-negative integer is a solution for (8.4). It follows that $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ contains R as a subring. Moreover, it can be verified that every monomial in the right-hand side of the isomorphism (8.3) is uniquely written as a product of an element in R and a monomial in $\mathfrak{B}_r$. The fact that $\mathfrak{B}_r$ is finite completes our proof.

□

Remark 8.3.3. As in Remark 8.2.4, we can make use of Barvinok algorithm to compute the cardinality of $\mathfrak{B}_r$. An explicit code is implemented in Appendix.

Corollary 8.3.4. *For each $r \geq 1$, there is a homeomorphism from* Spec $k[x_1^{(1)}, x_2^{(2)}, \dots, x_r^{(r)}]$ *onto* Spec $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$.

Proof. We recall the r-th Frobenius homomorphism of rings

$$\mathcal{F} : k[x_1^{(1)}, x_2^{(2)}, \dots, x_r^{(r)}] \to k[x_1^{(1)}, x_2^{(2)}, \dots, x_r^{(r)}]$$

$$\mathcal{F}(f) \longmapsto f^{p^{r-1}}$$

for all $f \in k[x_1^{(1)}, x_2^{(2)}, \ldots, x_r^{(r)}]$. Observe that

$$\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \subseteq k[x_1^{(1)}, x_2^{(2)}, \ldots, x_r^{(r)}]$$

are finitely generated commutative algebras. Note further that $\operatorname{Im}\mathcal{F} = k[x_1^{(1)p^{r-1}}, x_2^{(2)p^{r-1}}, \ldots, x_r^{(r)p^{r-1}}]$ lies in the ring $R = k[x_1^{(1)p^{r-1}}, x_2^{(2)p^{r-2}}, \ldots, x_r^{(r)}]$; hence is a subalgebra of $\mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$. The inclusions

$$\operatorname{Im}\mathcal{F} \subseteq \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} \subseteq k[x_1^{(1)}, x_2^{(2)}, \ldots, x_r^{(r)}]$$

implies that the morphism $\mathfrak{i} : \operatorname{Spec} k[x_1^{(1)}, x_2^{(2)}, \ldots, x_r^{(r)}] \to \operatorname{Spec}\ \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}$ is an F-isomorphism (see [Ben, 5.4.7]). Hence, it is a homeomorphism. □

Remark 8.3.5. One can easily construct an example where this morphism is not an isomorphism. From Appendix 12.1, we have $\operatorname{Spec}\ \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}} = \operatorname{Spec} k[x_1^p, x_2]$ which is obviously not isomorphic to $\operatorname{Spec} k[x_1, x_2]$ as there is no degree 1 morphism from one to the other.

8.4 Cohomology of G_r

We are now ready to compute G_r-cohomology.

Theorem 8.4.1. *Suppose $\lambda = m\omega \in X^+$. Then there is a G-module isomorphism*

$$\mathrm{H}^\bullet(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \operatorname{ind}_B^G(\mathrm{H}^\bullet(B_r, \lambda)^{(-r)}) \cong \bigoplus_{n=0}^{\infty} \operatorname{ind}_B^G (n\omega)^{N_r(p,m,n)}.$$

where each $N_r(p, m, n)$ is defined in Subsection 8.2.4.

Proof. From the spectral sequence in Section 7.1.2, we have

$$E_2^{m,n} = R^m \operatorname{ind}_B^G(\mathrm{H}^n(B_r, \lambda)^{(-r)}) \Rightarrow \mathrm{H}^{n+m}(G_r, \mathrm{H}^0(\lambda))^{(-r)}.$$

Note that all weights of $\mathrm{H}^\bullet(B_r, \lambda)$ are also weights of $\mathrm{H}^\bullet(U_r, \lambda)$ which are in turn weights of $\lambda \otimes S^{\bullet(1)} \otimes \cdots \otimes S^{\bullet(r)} \otimes \Lambda^{\bullet} \otimes \Lambda^{\bullet(1)} \otimes \cdots \otimes \Lambda^{\bullet(r-1)}$ by the spectral sequence (7.1). So all weights of $\mathrm{H}^n(B_r, \lambda)$ are dominant for each $n \geq 0$. It follows from Kempf's vanishing that $R^m \operatorname{ind}_B^G(\mathrm{H}^n(B_r, \lambda)) = 0$ for each $n \geq 0$ and $m > 0$. Hence the spectral sequence collapses at the first page and we obtain

$$\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \operatorname{ind}_B^G \left(\mathrm{H}^n(B_r, \lambda)^{(-r)}\right).$$

The other isomorphism follows by Theorem 8.2.3. □

Remark 8.4.2. This theorem gives us an explicit good filtration of $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))^{(-r)}$ for each n, r. This result implies the one in [AJ, 4.5(1)] and [VdK, Corollary 2.2]. In fact, for arbitrary simple algebraic group G, there is a conjecture that $\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))$ has a good filtration [Jan2, 12.15].

Following Theorem 8.2.3, we immediately obtain the analogous result for G_r-cohomology.

Corollary 8.4.3. *Suppose $\lambda = m\omega \in X^+$. Then we have*

$$\operatorname{ch} \mathrm{H}^{\bullet}(G_r, \mathrm{H}^0(\lambda))^{(-r)} = \sum_{n \in \mathbb{N}} N_r(p, m, n) \operatorname{ch} \mathrm{H}^0(n\omega) = \sum_{n \in \mathbb{N}} N_r(p, m, n) \chi(n\omega).$$

8.5 Reduced G_r-cohomology

Again, we want to investigate the reduced part of G_r-cohomology ring. We first introduce a link between the reduced parts of B_r- and of G_r-cohomology.

Lemma 8.5.1. *Let k be a perfect field. Suppose G is a split reductive group over k and let M be a k-algebra. Then we have* $\operatorname{Nilrad}(k[G] \otimes M) = k[G] \otimes \operatorname{Nilrad} M$.

Proof. Consider the short exact sequence

$$0 \to \operatorname{Nilrad} M \to M \to M_{\mathrm{red}} \to 0.$$

As G is reductive, the coordinate algebra $k[G]$ is free over k [Jan2, II.1.1]. So we have the following sequence

$$0 \to k[G] \otimes \operatorname{Nilrad} M \to k[G] \otimes M \to k[G] \otimes M_{\text{red}} \to 0$$

is exact. It follows that

$$k[G] \otimes M_{\text{red}} \cong \frac{k[G] \otimes M}{k[G] \otimes \operatorname{Nilrad} M}.$$

On the other hand, since $k[G]$ is reduced, it is well-known that the ring $k[G] \otimes M_{\text{red}}$ is reduced when k is perfect. This implies that $k[G] \otimes \operatorname{Nilrad} M = \operatorname{Nilrad}(k[G] \otimes M)$. □

Lemma 8.5.2. *Suppose the same assumptions on k and G as in Lemma 8.5.1. Given a B-algebra M and suppose that all weights of* Nilrad M *are dominant. Then, as an algebra, we always have* $\left[\operatorname{ind}_B^G M\right]_{\text{red}} \cong \operatorname{ind}_B^G(M_{\text{red}})$.

Proof. We first show that $\left[(k[G] \otimes M)^B\right]_{\text{red}} \cong \left[(k[G] \otimes M)_{\text{red}}\right]^B$. Let $A = k[G] \otimes M$. Then we need to prove $(A^B)_{\text{red}} \cong (A_{\text{red}})^B$, i.e., $\frac{A^B}{\operatorname{Nilrad}(A^B)} \cong \left(\frac{A}{\operatorname{Nilrad}(A)}\right)^B$. This is equivalent to showing that the following sequence

$$0 \to \operatorname{Nilrad}(A)^B \to A^B \to (A_{\text{red}})^B$$

is right exact; hence equivalent to $\mathrm{H}^1(B, \operatorname{Nilrad}(A)) = 0$. Indeed, the preceding lemma shows that

$$\operatorname{Nilrad}(A) = \operatorname{Nilrad}(k[G] \otimes M) = k[G] \otimes \operatorname{Nilrad}(M).$$

It follows by [Jan2, Prop.I.4.10] and Kempf's vanishing that

$$\mathrm{H}^1(B, \operatorname{Nilrad}(A)) = \mathrm{H}^1\left(B, k[G] \otimes \operatorname{Nilrad}(M)\right) \cong R^1 \operatorname{ind}_B^G\left(\operatorname{Nilrad}(M)\right) = 0$$

since all weights of Nilrad M are dominant. Finally, we have

$$\begin{aligned}\left(\operatorname{ind}_B^G M\right)_{\mathrm{red}} &\cong \left[(k[G]\otimes M)^B\right]_{\mathrm{red}}\\ &\cong \left[(k[G]\otimes M)_{\mathrm{red}}\right]^B\\ &\cong \left[k[G]\otimes M_{\mathrm{red}}\right]^B\\ &= \operatorname{ind}_B^G(M_{\mathrm{red}}).\end{aligned}$$

□

Theorem 8.5.3. *For each $r \geq 1$, there is a G-isomorphism*

$$\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}^{(-r)} \cong \operatorname{ind}_B^G \mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}^{(-r)}.$$

Proof. From the Theorem 8.4.1 and the above Lemma, we have

$$\mathrm{H}^m(G_r,k)_{\mathrm{red}}^{(-r)} \cong \left[\operatorname{ind}_B^G(\mathrm{H}^m(B_r,k)^{(-r)})\right]_{\mathrm{red}} \cong \operatorname{ind}_B^G(\mathrm{H}^m(B_r,k)_{\mathrm{red}}^{(-r)}).$$ □

Proposition 8.5.4. *For each $r \geq 1$, there is a homeomorphism from* Spec $k[G\times^B \mathfrak{u}^r]$ *onto* Spec $\mathrm{H}^\bullet(G_r,k)_{\mathrm{red}}$.

Proof. Let $\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)} = \operatorname{Spec} k[x_1^{(1)}, x_2^{(2)},\ldots,x_r^{(r)}]$ where each $\mathfrak{u}^{(i)}$ is identified with the weight space $k_{p^i\alpha}$ for each $1\leq i\leq r$; hence we consider it an affine space equipped with the B-action. In Corollary 8.3.4, we can see that the F-isomorphism $\mathfrak{i}$ arises from the inclusion $\mathrm{H}^\bullet(B_r,k)_{\mathrm{red}} \subseteq k[x_1^{(1)}, x_2^{(2)},\ldots,x_r^{(r)}]$, so it is compatible with the B-action. It follows the B-equivariant homeomorphism

$$\mathfrak{i}^{(-r)} : \left(\mathfrak{u}^{(1)}\times\cdots\times\mathfrak{u}^{(r)}\right)^{(-r)} \to \operatorname{Spec}\ \mathrm{H}^\bullet(B_r,k)_{\mathrm{red}}^{(-r)}.$$

Apply the fibered product with G over B on both sides, we have a homeomorphism

$$id_G \times_B \mathfrak{i}^{(-r)} : G \times_B \left(\mathfrak{u}^{(1)} \times \cdots \times \mathfrak{u}^{(r)}\right)^{(-r)} \to G \times_B \text{Spec } \mathrm{H}^\bullet(B_r, k)_{\mathrm{red}}^{(-r)}.$$

On the other hand, define a map:

$$\begin{aligned}\Phi : \mathfrak{u}^{(1-r)} \times \cdots \times \mathfrak{u} &\to \mathfrak{u}^r \\ (y_1, y_2, \ldots, y_r) &\mapsto (y_1^{p^{r-1}}, y_2^{p^{r-2}}, \ldots, y_r)\end{aligned}$$

for all $y_i \in \mathfrak{u}$. It is easy to see that Φ is a B-equivariant continuous map which is also a homeomorphism. It follows that the fibered products $G \times_B \left(\mathfrak{u}^{(1)} \times \cdots \times \mathfrak{u}^{(r)}\right)^{(-r)}$ and $G \times_B \mathfrak{u}^r$ are homeomorphic as a topological space. Now combine two above homeomorphisms and apply Theorem 8.5.3, we establish a homeomorphism from Spec $k[G \times^B \mathfrak{u}^r]$ onto Spec $\mathrm{H}^\bullet(G_r, k)_{\mathrm{red}}^{(-r)}$; hence completes our proof. □

8.6 Open problems

There are some remaining problems in SL_2-case. Recall that Andersen-Jantzen computed the B_2-cohomology ring in [AJ]. Although we have done computing B_r and G_r-cohomology as a module in this thesis, their ring structure remains an open problem for larger r.

Problem 1. For each $r > 2$, determine ring structures of B_r- and G_r-cohomology.

Another interesting problem one might be interested in is that if we can generalize some facts in this chapter for $\mathfrak{sl}_n$ or other types. One of them should be the following.

Problem 2. Under which conditions of G and $\lambda \in X^+$ do we have

$$\mathrm{H}^n(G_r, \mathrm{H}^0(\lambda))^{(-r)} \cong \mathrm{ind}_B^G[\mathrm{H}^n(B_r, \lambda)^{(-r)}] \ ?$$

From works of Jantzen [Jan2, II.12.2] and Bendel-Nakano-Pillen [BNP2], the isomorphism holds for $n = 1, 2$ and arbitrary prime p with G a simple algebraic group and arbitrary $\lambda \in X^+$. This problem is a rephrasing of *Induction Conjecture*.

Chapter 9

Geometry

In the present chapter we introduce concepts in commutative algebra and geometry which play important roles in the later chapters. In particular, we recall the definition of Cohen-Macaulay varieties and their properties. We also show that the generalized moment map is always surjective for arbitrary type of G. In addition, we review a number of well-known results in algebraic geometry.

9.1 Background material in commutative algebra

We review in this section definitions and important properties of Cohen-Macaulay rings. Readers can refer to [E] and [H] for more details. We begin with regular sequences of a ring.

9.1.1 Cohen-Macaulay Rings

Definition 9.1.1. Let R be a commutative ring and M be an R-module. A sequence $x_1, \dots, x_n \in R$ is called a *regular sequence* on M (or M-sequence) if it satisfies

(1) $(x_1, \dots, x_n)M \neq M$,

(2) For each $1 \leq i \leq n$, x_i is not a zero-divisor of $M/(x_1, \dots, x_{i-1})M$.

Consider R as a left R-module. For a given ideal I of R, it is well-known that the length of any maximal regular sequence in I is unique. It is called the *depth* of I and denoted depth(I). The *height* or *codimension* of a prime ideal J of R is the supremum of lengths of chains of prime ideals descending from J. Equivalently, it is defined as the Krull dimension of R/J. In particular, if R is an integral domain which is finitely generated over a field, then we have $\operatorname{codim}(J) = \dim R - \dim(R/J)$. We are now ready to define a Cohen-Macaulay ring.

Definition 9.1.2. A ring R is called Cohen-Macaulay if $\operatorname{depth}(I) = \operatorname{codim}(I)$ for each maximal ideal I of R. Also, a variety V is Cohen-Macaulay if its coordinate ring $k[V]$ is a Cohen-Macaulay ring.

Example 9.1.3. Smooth varieties are Cohen-Macaulay.

Example 9.1.4. The nilpotent cone $\mathcal{N}$ of a simple Lie algebra $\mathfrak{g}$ over an algebraically closed field of a good characteristic is Cohen-Macaulay [Jan3, 8.5].

9.1.2 Minors and Determinantal rings

Let $U = (u_{ij})$ be an $m \times n$-matrix over a ring R. For indices $a_1, \ldots, a_t, b_1, \ldots, b_t$ such that $1 \leq a_i \leq m, 1 \leq b_i \leq n, i = 1, \ldots, t$, we put

$$[a_1, \ldots, a_t \mid b_1, \ldots, b_t] = \det \begin{pmatrix} u_{a_1 b_1} & \cdots & u_{a_1 b_t} \\ \vdots & \ddots & \vdots \\ u_{a_t b_1} & \cdots & u_{a_t b_t} \end{pmatrix}.$$

We do not require that $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ are given in ascending order. Note that

$$[a_1, \ldots, a_t \mid b_1, \ldots, b_t] = 0$$

if $t > \min(m, n)$. To be convenient, we let $[\emptyset \mid \emptyset] = 1$. If $a_1 \leq \cdots \leq a_t$ and $b_1 \leq \cdots \leq b_t$ we call $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ a t-minor of U.

Definition 9.1.5. Let B be a commutative ring, and consider an $m \times n$ matrix

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

whose entries are independent indeterminates over R. Let $R(X)$ be the polynomial ring over all the indeterminates of X, and $I_t(X)$ be the ideal generated by all t-minors of X. The ring

$$R_t(X) = \frac{R(X)}{I_t(X)}$$

is called a *determinantal ring.*

For readers' convenience, we recall a nice property of determinantal rings as follows.

Proposition 9.1.6. *[BV, 1.11, 2.10, 2.11, 2.12] If R is a reduced ring, then for every $1 \leq t \leq \min(m, n)$, $R_t(X)$ is a reduced, Cohen-Macaulay, normal domain of dimension $(t-1)(m+n-t+1)$.*

9.2 The moment morphism

Suppose G is a simple algebraic group. It is well-known that $\mathcal{N} = G \cdot \mathfrak{u}$ where $\mathfrak{u}$ is the Lie algebra of the unipotent radical subgroup U of the Borel subgroup B of G, and the dot is the adjoint action of G on the Lie algebra $\mathfrak{g}$. Note that if u_1, u_2 are commuting in $\mathfrak{u}$, then so are $g \cdot u_1, g \cdot u_2$ in $\mathcal{N}$ for each $g \in G$. This observation can be generalized to give the following moment map

$$m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$$

by setting $m[g, (u_1, \ldots, u_r)] = (g \cdot u_1, \ldots, g \cdot u_r)$ for all $g \in G$, and $(u_1, \ldots, u_r) \in C_r(\mathfrak{u})$. In the case $r = 1$, this the moment map is called *Springer resolution.* We also call it the moment morphism for each $r \geq 1$. The following proposition shows surjectivity of this morphism.

Theorem 9.2.1. [1] *The moment morphism* $m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$ *is always surjective.*

Proof. Suppose $(v_1, \ldots, v_r) \in C_r(\mathcal{N})$. Let $\mathfrak{b}'$ be the vector subspace of $\mathfrak{g}$ spanned by all v_i's. As $[v_i, v_j] = 0$ for all $1 \leq i, j \leq r$, it can be checked that $\mathfrak{b}'$ is a nilpotent (hence solvable) Lie subalgebra of $\mathfrak{g}$. Thus, there is a maximal solvable subalgebra $\mathfrak{b}''$ of $\mathfrak{g}$ containing $\mathfrak{b}'$. By [Hum1, Theorem 16.4], $\mathfrak{b}''$ and our Borel subalgebra $\mathfrak{b}$ are conjugate under some inner automorphism $Ad(g)$ with $g \in G$. So there are $u_1, \ldots, u_r \in \mathfrak{b}$ such that

$$(v_1, \ldots, v_r) = Ad(g^{-1})(u_1, \ldots, u_r) = g^{-1} \cdot (u_1, \ldots, u_r) = m[g^{-1}, (u_1, \ldots, u_r)].$$

As all v_i's are nilpotent and commuting, so are u_i's. This shows that m is surjective. □

As a corollary, we establish the connection between irreducibility of $C_r(\mathfrak{u})$ and $C_r(\mathcal{N})$.

Theorem 9.2.2. *For each* $r \geq 1$, *if* $C_r(\mathfrak{u})$ *is irreducible then so is* $C_r(\mathcal{N})$.

9.3 Zariski's Main Theorem

9.3.1 Introduction

Zariski's Main Theorem is one of the powerful tools to study structure sheaves of two schemes. In this section, we state the version for varieties and show that the moment map satisfies the hypothesis of Zariski's Main Theorem. We first look at proper morphisms. As defining properness requires terminologies in algebraic geometry, we refer readers to [H, Section II.4] for the details. We only introduce some important characterizations of proper morphisms which will be useful later.

Proposition 9.3.1. *In the following properties, all the morphisms are taken over Noetherian schemes.*

(a) A closed immersion is proper.

[1] The author would like to thank Christopher M. Drupieski for the idea in this proof

(b) The composition of two proper morphims is proper.

(c) A projection $X \times Y \to X$ is proper if and only if Y is projective.

We also recall that a rational map $\varphi : X \to Y$ (which is a morphism only defined on some open subset) is birational if it has an inverse rational map.

There are many versions of Zariski's Main Theorem. Here we state a "pre-version" of the theorem since the Main Theorem immediately follows from this result (cf. [H, Corollary III.11.3 and III.11.4]).

Theorem 9.3.2. *Let $f : X \to Y$ be a birational proper morphism of varieties and suppose Y is normal. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

9.3.2 Moment map vs Zariski's Main Theorem

We now verify that the morphism $m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$ satisfies the hypothesis of Zariski's Main Theorem. In other words, we have

Proposition 9.3.3. *For each $r \geq 1$, the moment morphism $m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$ is birational proper.*

Proof. We generalize the proofs of Lemma 1 and 2 in [Jan3, 6.10]. For the properness, we first look at the map

$$\epsilon : G \times^B C_r(\mathfrak{u}) \hookrightarrow G/B \times C_r(\mathcal{N})$$

with $\epsilon[g, u_1, \ldots, u_r] = (gB, g \cdot (u_1, \ldots, u_r))$ for all $g \in G$ and $(u_1, \ldots, u_r) \in C_r(\mathfrak{u})$. By the same argument as in [Jan3, 6.4], we can show that this map is actually a closed embedding; hence a proper morphism by Proposition 9.3.1(a). Next, as G/B is projective, the projection map $p : G/B \times C_r(\mathcal{N}) \to C_r(\mathcal{N})$ is proper by Proposition 9.3.1(c). Therefore, part (b) of Proposition 9.3.1 implies that $m = p \circ \epsilon$ is also proper.

Consider the projection of $C_r(\mathcal{N})$ onto the first factor $p_1 : C_r(\mathcal{N}) \to \mathcal{N}$. Recall that $z(v_{\text{reg}})$ is the centralizer of v_{reg} where v_{reg} is a fixed regular element in $\mathcal{N}$. Then as $z(v_{\text{reg}})$ is abelian, we have

$$p_1^{-1}(\mathcal{O}_{\text{reg}}) = C(\mathcal{O}_{\text{reg}}, \mathcal{N}, \ldots, \mathcal{N}) = G \cdot (v_{\text{reg}}, z(v_{\text{reg}}), \ldots, z(v_{\text{reg}})).$$

As $\mathcal{O}_{\text{reg}}$ is an open subset in $\mathcal{N}$, the preimage $p_1^{-1}(\mathcal{O}_{\text{reg}})$ is open in $C_r(\mathcal{N})$. Let $V = p_1^{-1}(\mathcal{O}_{\text{reg}})$. Since $Z_G(v_{\text{reg}}) \subseteq B$, we have m induces an isomorphism from $m^{-1}(V)$ onto V. It follows that m is a birational morphism. □

Remark 9.3.4. This morphism might not be a dominant as $C_r(\mathcal{N})$ is not always irreducible. In fact, Premet already proved in [Pr] that $C_2(\mathcal{N})$ is irreducible if and only if G is of type A. It follows that if G is not of type A then $C_r(\mathcal{N})$ is reducible for each $r \geq 2$. In the next two chapters, we prove for arbitrary $r \geq 1$, the variety $C_r(\mathcal{N})$ is irreducible for type A_1 and A_2. The result for type A_n with arbitrary $n, r > 2$ remains an open problem.

Conjecture 9.3.5. *If G is of type A, then $C_r(\mathcal{N})$ is irreducible. Moreover, the morphism*

$$m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$$

is a dominant birational proper map.

9.4 Singularities and Resolutions

In this section, we first give an observation on determinning singular points of an affine variety defined by homogeneous polynomials. Then we define certain resolutions of singularities which will be used in the later chapters.

Lemma 9.4.1. *Let V be an affine variety whose defining radical ideal is generated by a non-empty set of homogeneous polynomials of degree at least 2. Then 0 is always a singular point of V.*

Proof. Suppose V is an affine subvariety of the ambient space $\mathbb{A}^m$ associated with the coordinate ring $k[x_1, \ldots, x_m]$. Let $f_1, \ldots, f_n$ be the set of polynomials defining V. Note that $\dim V < m$ since $n \geq 1$. Consider the Jacobian matrix

$$\begin{pmatrix} \frac{df_1}{x_1} & \cdots & \frac{df_1}{x_m} \\ \vdots & \ddots & \vdots \\ \frac{df_n}{x_1} & \cdots & \frac{df_n}{x_m} \end{pmatrix}$$

As all f_i's are homogeneous of degree ≥ 2, we have $\frac{df_i}{x_j}(0, \ldots, 0) = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$. It follows that the tangent space at 0 has dimension m which is greater than $\dim V$. Thus 0 is singular. □

Definition 9.4.2. A variety X has a resolution of singularities if we can find a non-singular variety Y such that there is a proper birational morphism from Y to X.

Definition 9.4.3. A variety X has rational singularities if it is normal and has a resolution of singularities

$$f : Y \to X$$

such that the higher direct image $R^i f_* \mathcal{O}_Y$ vanishes for $i \geq 1$. (Sometimes one calls f admits a rational resolution.)

The problem of finding resolution of singularities is classical in algebraic geometry. For a one or two-dimensional variety, one showed that it has a unique minimal resolution. There are a list of methods to resolve singularities in these cases due to Kollar, Jung, Hironaka, Lipman, Albanese, to name a few. For higher dimensional varieties, De Jong generalized Jung's method to give a good algorithm for varieites over a field of characteristic 0. A weaker result holds in case of characteristic p.

In Lie theory, there is a famous resolution of singularites, namely *Springer resolution* which also has rational singularities. Our goal in this thesis is generalizing this resolution for commuting varieties.

Chapter 10

Commuting Varieties over 2 by 2 matrices

According to Vasconcelos [Vas], for a given Lie algebra A over a field k and a subvariety V of A, an *ordinary commuting variety* is defined as

$$C(V) = \{(v_1, v_2) \in V^2 \mid [v_1, v_2] = 0\}.$$

As we already indicated in Chapter 1, a great deal is known about ordinary commuting varieties. In this dissertation, we shall study the commuting variety of r-tuple of elements in V, i.e.,

$$C_r(V) = \{(v_1, \ldots, v_r) \in V^r \mid [v_i, v_j] = 0,\ 1 \leq i \leq j \leq r\}.$$

The commuting variety $C_r(\mathfrak{gl}_n)$ is a classical object studied by Gerstenhaber, Guralnick-Sethuraman, and Kirillov-Neretin. All of these works focused on irreducibility. In particular, Gerstenhaber claimed in his paper that it is trivial to see the ideals generated by necessary polynomials defining $C_r(\mathfrak{gl}_2)$ and $C_r(\mathfrak{gl}_3)$ are radical [G]. Guralnick and Sethuraman proved that the variety $C_3(\mathfrak{gl}_4)$ is irreducible and studied some properties of $C_r(\mathfrak{gl}_n)$ in general [GS, Proposition 4]. On the other hand, Kirillov and Neretin showed that $C_r(\mathfrak{gl}_2)$ and $C_r(\mathfrak{gl}_3)$ are irreducible while $C_4(\mathfrak{gl}_4)$ contains at least two components [KN, Theorem 4].

The problem of showing that the variety of commuting r-tuples of $n \times n$ matrices is Cohen-Macaulay and normal is still difficult to verify. With ordinary commuting varieties, computer verification works up to $n = 4$ [Hr]. There are also some studies on the Cohen-Macaulayness of other structures closely similar to ordinary commuting varieties by Knutson, Mueller, Zolbanin-Snapp and Zoque [K],[Mu],[MS],[Z]. Very little appears to be known for commuting varieties of r-tuples.

10.1 Nice properties of $C_r(\mathfrak{gl}_2)$

In this section, we confirm the properties of being Cohen-Macaulay and normal for $C_r(\mathfrak{gl}_2)$ and $C_r(\mathfrak{sl}_2)$ with arbitrary $r \geq 1$. Let k be an algebraically closed field of characteristic p. We first show a general result connecting between the commuting varieties over $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$.

Theorem 10.1.1. [1] *For each n and $r \geq 1$, if p does not divide n, then there is an isomorphism from $C_r(\mathfrak{gl}_n)$ to $C_r(\mathfrak{sl}_n) \times \mathbb{A}^r$ defined by setting*

$$(v_1, \ldots, v_r) \mapsto \left(v_1 - \frac{\mathrm{Tr}(v_1)}{n} I_n, \ldots, v_r - \frac{\mathrm{Tr}(v_r)}{n} I_n\right) \times (\mathrm{Tr}(v_1), \ldots, \mathrm{Tr}(v_r)) \tag{10.1}$$

for v_i's in $\mathfrak{gl}_n$.

Proof. It is easy to see that adding or subtracting cI_n from v_i's does not change the commuting conditions of v_i's. So the morphism φ is well-defined and its inverse is

$$\varphi^{-1} : (u_1, \ldots, u_r) \times (a_1, \ldots, a_r) \mapsto \left(u_1 + \frac{a_1}{n} I_n, \ldots, u_r + \frac{a_r}{n} I_n\right).$$

This completes the proof. □

This result implies that our work for $C_r(\mathfrak{gl}_2)$ will be done if we can prove that $C_r(\mathfrak{sl}_2)$ is

[1] The author would like to thank Professor William Graham for his assistance in generalizing the result to $\mathfrak{gl}_n$.

Cohen-Macaulay. Notice that Popov proved the normality for this variety in the case $r = 2$ [Po, 1.10]. However, his proof depends on computer calculations to verify that the defining ideal is radical. Here we propose another approach that completely solves the problem for arbitrary r. Let $\mathfrak{sl}_2^r$ be the affine space defined as

$$\left\{ \begin{pmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{pmatrix}, \dots, \begin{pmatrix} x_r & y_r \\ z_r & -x_r \end{pmatrix} \;\middle|\; x_i,\ y_i,\ z_i \in k, 1 \leq i \leq r \right\}.$$

Then the variety $C_r(\mathfrak{sl}_2)$ can be defined as the zero locus of the following ideal

$$J = \langle x_i y_j - x_j y_i \ , \ y_i z_j - y_j z_i \ , \ x_i z_j - x_j z_i \ \mid \ 1 \leq i \leq j \leq r \rangle. \tag{10.2}$$

Proposition 10.1.2. *For each $r \geq 1$, the variety $C_r(\mathfrak{sl}_2)$ is:*

(a) irreducible of dimension $r + 2$,

(b) Cohen-Macaulay and normal.

Proof. It was shown by Kirillov and Neretin that $C_r(\mathfrak{gl}_2)$ and $C_r(\mathfrak{gl}_3)$ are irreducible for all r in characteristic 0 [KN, Theorem 4]. The irreducibility of $C_r(\mathfrak{sl}_2)$ follows by Theorem 10.1.1 in this case. Here we provide a general proof for any characteristic $p \neq 2$.

Consider J above the ideal $I_2(\mathcal{X})$ generated by all 2-minors of following matrix

$$\mathcal{X} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ y_1 & y_2 & \ddots & y_r \\ z_1 & z_2 & \cdots & z_r \end{pmatrix}.$$

Then we can identify $k[C_r(\mathfrak{sl}_2)]$ with the ring $R_2(\mathcal{X}) = \frac{k[\mathcal{X}]}{I_2(\mathcal{X})}$. It follows immediately from Proposition 9.1.6 that $R_2(\mathcal{X})$ is a Cohen-Macaulay and normal domain; hence completing the proof. $\square$

Corollary 10.1.3. *Suppose $p \neq 2$. Then for each $r \geq 1$, the variety $C_r(\mathfrak{gl}_2)$ is* *(a) irreducible of dimension* $2r+2$,

(b) Cohen-Macaulay and normal.

Proof. Follows immediately from Theorem 10.1.1 and Proposition 10.1.2. □

10.2 Nilpotent Commuting Varieties over $\mathfrak{sl}_2$

With the nilpotency condition, problems involving commuting varieties turn out to be more difficult. One of the reasons is that the defining ideals of these varieties are no longer radical which creates additional complications. The irreducibility of ordinary nilpotent commuting varieties was studied by Baranovsky, Premet, Basili and Iarrobino (cf. [Ba],[Pr],[B],[BI]). However, there has not been any successful work on normality and Cohen-Macaulayness even in simple cases. In this section, we completely prove this conjecture for the variety of r-tuples of commuting nilpotent elements in $\mathfrak{sl}_2$.

We are back to the assumption $G = SL_2$ and k is an algebraically closed field of characteristic $p \neq 2$. The nilpotent cone of $\mathfrak{g}$ then can be written as follows.

$$\mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mid x^2 + yz = 0 \text{ with } x, y, z \in k \right\}$$

Recall that we denote the nilpotent commuting variety of $\mathfrak{g}$ by

$$C_r(\mathcal{N}) = \{(v_1, \ldots, v_r) \in \mathcal{N}^r \mid [v_i, v_j] = 0,\ 1 \leq i \leq j \leq r\}.$$

Note for each $r \geq 1$ that $C_r(\mathfrak{u}) = \mathfrak{u}^r$ so that the moment map in Section 9.2 is rewritten

$$m : G \times^B \mathfrak{u}^r \to C_r(\mathcal{N}). \tag{10.3}$$

As G/B and $\mathfrak{u}^r$ are smooth varieties, so is the vector bundle $G \times^B \mathfrak{u}^r$. In addition, the moment map m is known to be proper birational from Proposition 9.3.3. It follows that $G\times^B \mathfrak{u}^r$ is a resolution of singularities for $C_r(\mathcal{N})$ through the morphism m. Our goal in this chapter is to show that (10.3) has in fact rational singularities. In order to make use of the Zariski's Main Theorem 9.3.2, we need to study properties of $C_r(\mathcal{N})$.

Proposition 10.2.1. *For every $r \geq 1$, we have*

(a) $C_r(\mathcal{N})$ is an irreducible variety of dimension $r+1$.

(b) The only singular point in $C_r(\mathcal{N})$ is the origin 0.

Proof. (a) Define the morphism $\varphi : G\times \mathfrak{u}^r \to C_r(\mathcal{N})$ by setting for all $g \in G, v = (v_1,\ldots,v_r) \in \mathfrak{u}^r$, $\varphi(g,v) = (g\cdot v_1,\ldots,g\cdot v_r)$. Similar to the moment map m, this morphism is surjective. Then part (a) follows immediately from the fact that $G\times\mathfrak{u}^r$ is irreducible. Now since $G\times^B\mathfrak{u}^r$ is a vector bundle over the base G/B with each fiber isomorphic to $\mathfrak{u}^r$ we have

$$\dim G\times^B \mathfrak{u}^r = \dim G/B + \dim \mathfrak{u}^r = r+1.$$

As we have shown earlier that m is a birational morphism, $C_r(\mathcal{N})$ has the same dimension as $G\times^B\mathfrak{u}^k$.

(b) It is enough to show that every non-zero element in $C_r(\mathcal{N})$ belongs to a smooth open subset of dimension $r+1$. Let $0\neq v = (v_1,\ldots,v_r)\in C_r(\mathcal{N})$. We can assume that $0\neq v_1\in\mathcal{N}$. Then consider the projection on the first factor $p : C_r(\mathcal{N})\to\mathcal{N}$, we see that $v\in G\cdot(v_1,\mathfrak{u}^{r-1}) = p^{-1}(G\cdot v_1)$ which is open in $C_r(\mathcal{N})$. Now we define an action of the reductive group $G\times k^{r-1}$ on $C_r(\mathcal{N})$ as follows,

$$\left(G\times k^{r-1}\right)\times C_r(\mathcal{N})\to C_r(\mathcal{N})$$
$$(g\ ,\ a_1\ ,\ldots,\ a_{r-1})\bullet(v_1\ ,\ldots,\ v_r)\longmapsto g\cdot(v_1\ ,\ a_1v_2\ ,\ldots,\ a_{r-1}v_r).$$

It is easy to see that $p^{-1}(G\cdot v_1) = G\times k^{r-1}\bullet(v_1, v_1, \ldots, v_1)$. As every orbit is itself a smooth variety, we obtain that $p^{-1}(G\cdot v_1)$ is smooth of dimension $r+1$.

□

10.3 Cohen-Macaulayness of $C_r(\mathcal{N})$

Throughout this section, for each $r \geq 1$, let $R = k[\mathfrak{sl}_2^r] = k[x_i, y_i, z_i | 1 \leq i \leq r]$. We denote by $\cap^*$ the scheme-theoretic intersection in order to distinguish with the regular intersection of varieties. We show in this section that $C_r(\mathcal{N})$ is Cohen-Macaulay. First we need some lemmas related to Cohen-Macaulayness of varieties; the first one is an exercise in [E, Exercise 18.13] (see also [BV, Lemma 5.15]).

Lemma 10.3.1. [2] *Let X, Y be two Cohen-Macaulay varieties of the same dimension. Suppose the scheme-theoretic intersection $X \cap^* Y$ is of codimension 1 in both X and Y. Then $X\cap^* Y$ is Cohen-Macaulay if and only if $X \cup Y$ is Cohen-Macaulay.*

For each $r \geq 1$, the variety $C_r(\mathcal{N})$ is defined as the zero locus of the family of polynomials

$$\{x_i^2 + y_i z_i\ ,\ x_i y_j - x_j y_i\ ,\ x_i z_j - x_j z_i\ ,\ y_i z_j - y_j z_i \mid 1 \leq i \leq j \leq r\}.$$

It is easy to check that this ideal is not radical. This causes some difficulties for us to investigate the algebraic properties of this variety like Cohen-Macaulayness. Let I_r be the radical ideal generated by the family of polynomials above in R. We first reduce the problem to checking a certain property in commutative algebra.

Lemma 10.3.2. *If $I_s + \langle y_1 + z_1\rangle$ is a radical ideal for each $s \geq 1$, then $C_r(\mathcal{N})$ is Cohen-Macaulay for each $r \geq 1$.*

Proof. We prove by induction. When $r = 1$, $C_1(\mathcal{N}) = \mathcal{N}$ which is a well-known Cohen-Macaulay variety. Suppose that $C_{r-1}(\mathcal{N})$ is Cohen-Macaulay for some $r \geq 2$. As we have

[2] In [E], although the author did not state the word "scheme-theoretic intersection", we implicitly understand from the exercise the intersection needs to be scheme-theoretic.

seen earlier, $C_r(\mathcal{N})$ is irreducible of dimension $r+1$ and (y_1+z_1) is not zero in the ring $\frac{R}{I_r}$. The hypothesis and Lemma 5.15 in [BV] imply that it suffices to show the variety $C_r(\mathcal{N}) \cap V(y_1+z_1)$ is Cohen-Macaulay of dimension r.

Let $V = C_r(\mathcal{N}) \cap V(y_1+z_1)$. Solving from the constraint $x_1^2 + y_1 z_1 = 0$, we have either $x_1 = y_1 = -z_1$ or $x_1 = -y_1 = z_1$. Then we decompose $V = V_1 \cup V_2 \cup V_3$ where V_i's are irreducible algebraic varieties defined as below

$$\begin{aligned} V_1 &= V \cap V(x_1 = y_1 = z_1 = 0), \\ V_2 &= \overline{V \cap V(x_1 = y_1 = -z_1 \neq 0)}, \\ V_3 &= \overline{V \cap V(x_1 = -y_1 = z_1 \neq 0)}. \end{aligned}$$

Moreover, we can explicitly describe these varieties as follows:

$$\begin{aligned} V_1 &= 0 \times 0 \times 0 \times C_{r-1}(\mathcal{N}) \\ V_2 &= \overline{\{(x_1\ ,\ y_1\ ,\ z_1\ ,\ x_2,\ \frac{x_2}{x_1}y_1\ ,\ \frac{x_2}{x_1}z_1\ ,\ldots,\ x_r\ ,\ \frac{x_r}{x_1}y_1\ ,\frac{x_r}{x_1}z_1) \mid 0 \neq x_1 = y_1 = -z_1 \in k\}} \\ &= \{(x_1\ ,\ x_1\ ,\ -x_1\ ,\ x_2\ ,\ x_2\ ,\ -x_2\ ,\ldots,\ x_r\ ,\ x_r\ ,\ -x_r) \mid x_i \in k\}, \\ V_3 &= \overline{\{(x_1\ ,\ y_1\ ,\ z_1\ ,\ x_2\ ,\frac{x_2}{x_1}y_1\ ,\frac{x_2}{x_1}z_1\ ,\ldots,\ x_{r+1}\ ,\frac{x_r}{x_1}y_1\ ,\frac{x_r}{x_1}z_1) \mid 0 \neq x_1 = -y_1 = z_1 \in k\}} \\ &= \{(x_1\ ,\ -x_1\ ,\ x_1\ ,\ x_2\ ,\ -x_2\ ,\ x_2\ ,\ldots,\ x_r\ ,\ -x_r\ ,\ x_r) \mid x_i \in k\}. \end{aligned}$$

Observe that V_1 is Cohen-Macaulay of dimension r by inductive hypothesis; V_2, V_3 are affine r-spaces so they are Cohen-Macaulay. As $\{(0,0,0)\}$ is a subvariety of $\{(x_1, x_1, -x_1)\}$, we have

$$\{(0,0,0)\} \cap^* \{(x_1, x_1, -x_1)\} = \{(0,0,0)\} \cap \{(x_1, x_1, -x_1)\} = \{(0,0,0)\}.$$

Similarly, we have

$$\{(x_2, x_2, -x_2, \ldots, x_r, x_r, -x_r) \mid x_i \in k\} \cap^* C_{r-1}(\mathcal{N}) =$$
$$= \{(x_2, x_2, -x_2, \ldots, x_r, x_r, -x_r) \mid x_i \in k\} \cap C_{r-1}(\mathcal{N})$$
$$= \{(x_2, x_2, -x_2, \ldots, x_r, x_r, -x_r) \mid x_i \in k\}.$$

It follows that the scheme-theoretic intersection

$$V_1 \cap^* V_2 = \{(0, 0, 0, x_2, x_2, -x_2, \ldots, x_r, x_r, -x_r) \mid x_i \in k,\ 2 \le i \le r\}$$

is an affine $(r-1)$-space. Hence by Lemma 10.3.1 the union $V_1 \cup V_2$ is a Cohen-Macaulay variety of dimension r. Next we consider the scheme-theoretic intersection $(V_1 \cup V_2) \cap^* V_3$. Note that $V_2 \cap^* V_3 = \{0\}$, we have

$$(V_1 \cup V_2) \cap^* V_3 = V_1 \cap^* V_3 = \{(0, 0, 0, x_2, -x_2, x_2, \ldots, x_r, -x_r, x_r) \mid x_i \in k\}$$

for the same reason as earlier. Then again Lemma 10.3.1 implies that $V_1 \cup V_2 \cup V_3$ is Cohen-Macaulay of dimension r. This completes our proof. □

We are now interested in the conditions under which the sum of two radical ideals is again radical. One of the well-known concepts in commutative algebra related to this problem is Principal Radical System introduced by Hochster which shows certain class of ideals are radical.

Theorem 10.3.3. *[BV, Theorem 12.1] Let A be a Noetherian ring, and F a family of ideals in A, partially ordered by inclusion. Suppose that for every member $I \in F$ one of the following assumptions is fulfilled:*

(a) I is a radical ideal; or

(b) There exists an element $x \in A$ such that $I + Ax \in F$ and

- *x is not a zero-divisor $\frac{A}{\sqrt{I}}$ and $\bigcap_{i=0}^{\infty}(I + Ax^i)/I = 0$, or*
- *there exists an ideal $J \in F$ with $I \subsetneq J$, such that $xJ \subseteq I$ and x is not a zero-divisor of $\frac{A}{\sqrt{J}}$.*

Then all the ideals in F are radical.

Such a family of ideals is called a principal radical system. This type of ideals play an important role in the proof of Hochster and Eagon showing that determinantal rings are Cohen-Macaulay [HE]. Before applying this theorem, we need to set up some notation.

Fix $r \geq 1$. For each $1 \leq m \leq r$, let I_m be the radical ideal associated to the variety $0 \times \cdots \times 0 \times C_m(\mathcal{N}) \subseteq C_r(\mathcal{N})$ where 0 denotes the zero matrix in $\mathcal{N}$. It is easy to see that each ideal I_m is prime and

$$I_m = I_r + \sum_{j=1}^{r-m} \langle x_j, y_j, z_j \rangle .$$

We also let for each $1 \leq m \leq r$

$$P_m = \sum_{i=1}^{m} \langle x_i, y_i, z_i \rangle + \sum_{j=m+1}^{r} \langle x_j - y_j \ , \ y_j + z_j \rangle .$$

Note that each P_m is a prime ideal since R/P_m is isomorphic to $k[x_{m+1}, \ldots, x_r]$. Now we consider the following family of ideals in R

$$\mathcal{F} = \{\{I_j\}_{j=1}^r \ , \ \{P_j\}_{j=1}^r \ , \ \{I_r + \sum_{i=1}^{m} \langle y_i + z_i \rangle\}_{m=1}^r \ ,$$
$$\{I_r + \sum_{i=1}^{r} \langle y_i + z_i \rangle + \sum_{i=1}^{n} \langle x_j + y_j \rangle\}_{n=1}^r \ , \ \mathfrak{m} = \langle x_1, y_1, z_1, \ldots, x_r, y_r, z_r \rangle\}.$$

Proposition 10.3.4. *The family $\mathcal{F}$ is a principal radical system.*

Proof. It is obvious that $\{I_j\}_{j=1}^r$, $\{P_j\}_{j=1}^r$ and $\mathfrak{m}$ are radical. So we just have to consider two following cases:

(a) $I = I_r + \sum_{i=1}^{m} \langle y_i + z_i \rangle$ for some $1 \leq m \leq r-1$. Observe that $I + \langle y_{m+1} + z_{m+1} \rangle$ is an element in $\mathcal{F}$. Let $J = I_{r-m}$. It is easy to see $y_{m+1} + z_{m+1} \notin J$ so that $y_{m+1} + z_{m+1}$ is not zero-divisor in the domain $R/\sqrt{J} = R/J$. It remains to show that $(y_{m+1}+z_{m+1})J \subseteq I$. Recall that $I_{r-m} = I_r + \sum_{j=1}^{m} \langle x_j, y_j, z_j \rangle$; hence it suffices to prove that

$$(y_{m+1} + z_{m+1})\, x_j \in I,$$

$$(y_{m+1} + z_{m+1})\, y_j \in I,$$

$$(y_{m+1} + z_{m+1})\, z_j \in I$$

for all $1 \leq j \leq m$. This is done in the Appendix 12.3.1.

(b) $I = I_r + \sum_{i=1}^{r} \langle y_i + z_i \rangle + \sum_{i=1}^{n} \langle x_j + y_j \rangle$ for some $0 \leq n \leq r-1$, where if $n = 0$, we set $I = I_r + \sum_{i=1}^{r} \langle y_i + z_i \rangle$. It is clear that $I + \langle x_{n+1} + y_{n+1} \rangle \in \mathcal{F}$. Choose $J = P_n$, then the same argument as in the previous case gives us $x_{n+1} + y_{n+1}$ is not a zero-divisor of $R/\sqrt{J}$. We also refer the reader to the Appendix 12.3.2 for the proof of $(x_{n+1} + y_{n+1})J \subseteq I$.

□

Here comes the main result of this section.

Theorem 10.3.5. *For each $r \geq 1$, the variety $C_r(\mathcal{N})$ is Cohen-Macaulay and therefore normal.*

Proof. This immediately follows from Lemma 10.3.2 and Proposition 10.3.4. □

10.4 Rational singularities of $C_r(\mathcal{N})$

In this section, we prove that the moment map

$$m : G \times^B \mathfrak{u}^r \to C_r(\mathcal{N})$$

is in fact a rational resolution of $C_r(\mathcal{N})$. Since m is already a resolution of singularities, it is equivalent to show the following.

Proposition 10.4.1.

(a) $\mathcal{O}_{C_r(\mathcal{N})} = m_*\mathcal{O}_{G\times^B\mathfrak{u}^r}$.

(b) The higher direct image $R^i m_*(\mathcal{O}_{G\times^B\mathfrak{u}^r}) = 0$ *for all* $i > 0$. *Hence* $G\times^B \mathfrak{u}^r$ *is a rational resolution of* $C_r(\mathcal{N})$ *via* m.

Proof. (a) This follows from Theorem 10.3.5, and Zariski's Main Theorem 9.3.2.

(b) By [H, Proposition 8.5], we have $R^i m_*(\mathcal{O}_{G\times^B\mathfrak{u}^r}) \cong \mathcal{L}\left(\mathrm{H}^i(G\times^B\mathfrak{u}^r, \mathcal{O}_{G\times^B\mathfrak{u}^r})\right)$ for each $i\geq 0$ where $\mathcal{L}(M)$ is the associated sheaf of an A-module M. Note that $\mathcal{L}(k) = \mathcal{O}_{G\times^B\mathfrak{u}^r}$ so we have

$$\mathrm{H}^i(G\times^B\mathfrak{u}^r, \mathcal{O}_{G\times^B\mathfrak{u}^r}) \cong \bigoplus_{j=0}^{\infty}\mathrm{H}^i(G/B, \mathcal{L}_{G/B}S^j(\mathfrak{u}^{r*})) = \bigoplus_{j=0}^{\infty} R^i\operatorname{ind}_B^G(S^j(\mathfrak{u}^{*r})).$$

As we are assuming $\mathfrak{u}$ a one-dimensional space of weight corresponding to the positive root α, $S^j(\mathfrak{u}^{*r})$ can be considered a direct sum of one-dimensional weight space $(k_{-j\alpha})^{P_r(j)}$ where $P_r(j)$ is the number of partitions of j in r. This weight is antidominant so by Kempf's vanishing theorem we obtain $R^i\operatorname{ind}_B^G(S^j(\mathfrak{u}^{*r})) = 0$ for all $i>0$ and $j\geq 0$. It follows that

$$R^i m_*(\mathcal{O}_{G\times^B\mathfrak{u}^r}) = 0$$

for all $i\geq 1$.

□

10.5 Applications

We establish, in the present section, the connection with the reduced cohomology ring of G_r which we computed in the earlier chapter. This gives an alternative proof for the main result in [BFS1] for the special case $G = SL_2$.

Theorem 10.5.1. *For each $r > 0$, there is a G-equivariant isomorphism of algebras*

$$k[G \times^B \mathfrak{u}^r] \cong k[C_r(\mathcal{N})].$$

Consequently, there is a homeomorphism between $\operatorname{Spec} \mathrm{H}^\bullet(G_r, k)_{\mathrm{red}}$ *and* $C_r(\mathcal{N})$.

Proof. Note that the moment map m is G-equivariant. This implies the comorphism m^* is compatible with G-action. The isomorphism follows from part (a) of the previous proposition. Now combine this with Proposition 8.5.4, we have the homeomorphism between the spectrum of the reduced G_r-cohomology ring and $C_r(\mathcal{N})$. □

The isomorphism in the theorem above also gives us a way to compute the character formula for the coordinate algebra of $C_r(\mathcal{N})$ as follows

$$\mathrm{ch}_t\left(k[C_r(\mathcal{N})]\right) = \mathrm{ch}_t\left(k[G \times^B \mathfrak{u}^r]\right). \tag{10.4}$$

Theorem 10.5.2. *For each $r > 0$, we have*

$$\mathrm{ch}_t\, k[C_r(\mathcal{N})] = \sum_{n \geq 0} \sum_{a_1 + \cdots + a_r = n} \chi(n\alpha) t^n. \tag{10.5}$$

where the latter sum is taken over all r-tuple $(a_1, \ldots, a_r)$ of non-negative integers satisfying $a_1 + \cdots + a_r = n$.

Proof. From [Jan3, 8.11(4)], we have for each $n \geq 0$,

$$\mathrm{ch}\, k[G \times^B \mathfrak{u}^r]_n = \mathrm{ch}\, \mathrm{H}^0(G/B, S^n(\mathfrak{u}^{r*})).$$

The argument in Proposition 10.4.1 gives us

$$\mathrm{H}^i(G/B, S^n(\mathfrak{u}^{*r})) = 0$$

for all $i > 0, n \geq 0$. Hence, it follows by [Jan3, 8.14(6)] that

$$\chi(\mathrm{ch}(S^n(\mathfrak{u}^{*r}))) = \mathrm{ch}\,\mathrm{H}^0(G/B, S^n(\mathfrak{u}^{*r}))$$

for each $n \geq 0$. Here we recall that for each finite-dimensional B-module M, the Euler characteristic of M is defined as

$$\chi(M) = \chi(\mathrm{ch}\,M) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,\mathrm{H}^i(G/B, M).$$

By some abuse of notation, we can write $\mathrm{ch}(S^n(\mathfrak{u}^*)) = e(n\alpha)$; hence for each n we have

$$\begin{aligned}
\mathrm{ch}(S^n(\mathfrak{u}^{r*})) &= \mathrm{ch}(S^n(\mathfrak{u}^*)^{\otimes r}) \\
&= \mathrm{ch}\left(\sum_{a_1+\cdots+a_r=n} S^{a_1}(\mathfrak{u}^*)\otimes\cdots\otimes S^{a_r}(\mathfrak{u}^*)\right) \\
&= \sum_{a_1+\cdots+a_r=n} \mathrm{ch}\left(S^{a_1}(\mathfrak{u}^*)\otimes\cdots\otimes S^{a_r}(\mathfrak{u}^*)\right) \\
&= \sum_{a_1+\cdots+a_r=n} \mathrm{ch}\,e(n\alpha).
\end{aligned}$$

Thus, by the linear property of χ, we obtain

$$\chi\left(\mathrm{ch}(S^n(\mathfrak{u}^{*r}))\right) = \sum_{a_1+\cdots+a_r=n}\chi(\mathrm{ch}\,e(n\alpha)) = \sum_{a_1+\cdots+a_r=n}\chi(n\alpha).$$

Combining all the formulas, we get

$$\mathrm{ch}_t\,k[G\times^B\mathfrak{u}^r] = \sum_{n\geq 0}\mathrm{ch}\left(k[G\times^B\mathfrak{u}^r]_n\right)t^n = \sum_{n\geq 0}\sum_{a_1+\cdots+a_r=n}\chi(n\alpha)t^n.$$

Therefore, the identity (10.4) completes our proof. □

Remark 10.5.3. The above formula can be more specific by making use of partition functions. Let S be a set of r-tuples $(a_1,\ldots,a_r)\in\mathbb{N}^r$. Set $P_S:\mathbb{N}\to\mathbb{N}$ with $P_S(m)$ the number of ways getting m as a linear combination of elements in S with non-negative coefficients. This is called a *partition function*. Kostant used this notation with $S = \Pi$, the set of simple roots, to express the number of ways of writing a weight λ as a linear combination of simple roots (see [SB] for more details). In our case, set $S = \{(1, \ldots, 1)\} \subset \mathbb{N}^r$ and denote $P_r = P_S$. Then from (10.5) we can have

$$\operatorname{ch} k[C_r(\mathcal{N})]_n = P_r(n)\chi(n\alpha).$$

As a consequence, we obtain a result on the multiplicity of $\mathrm{H}^0(\lambda)$ in $k[C_r(\mathcal{N})]$.

Corollary 10.5.4. *For each $\lambda = m\alpha \in X^+$, we have*

$$\big[k[C_r(\mathcal{N})] : \mathrm{H}^0(\lambda)\big] = P_r(m).$$

This result shows that the coordinate algebra of $C_r(\mathcal{N})$ has a good filtration as a G-module.

Chapter 11

Commuting Variety of 3 by 3 matrices

Motivated by the results in the last chapter, there are many interesting questions about commuting varieties of r-tuples. Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field k of characteristic p, and V be a subvariety of $\mathfrak{g}$. Then for each $r \geq 2$, the following questions are still open.

1. What is the dimension of $C_r(V)$?
2. What is the singular locus of $C_r(V)$?
3. Is $C_r(V)$ Cohen-Macaulay?
4. What are the irreducible components of $C_r(V)$?

In this chapter, we answer the first two questions for commuting varieties over various subsets of $\mathfrak{gl}_3$. In particular, we investigate some properties on the dimension, irreducibility and singular locus of commuting varieties over the nilpotent cone $\mathcal{N}$ of $\mathfrak{gl}_3$.

11.1 Irreducibility of $C_r(\mathcal{N})$

We apply, in this section, strategies on determinantal rings to study behaviors of the smaller variety $C_r(\mathfrak{u})$, then use the moment morphism to obtain results on $C_r(\mathcal{N})$. First, we identify

$\mathfrak{u}^r$ with the affine space as follows.

$$\mathfrak{u}^r = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ y_1 & z_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ x_r & 0 & 0 \\ y_r & z_r & 0 \end{pmatrix} \mid x_i \ , \ y_i \ , \ z_i \in k, 1 \leq i \leq r \right\}.$$

The commutator equation is $x_i z_j - x_j z_i \ , \ 1 \leq i \leq j \leq r$ for each pair of elements in $\mathfrak{u}$. It follows that

$$k[C_r(\mathfrak{u})] = \frac{k[x_1, y_1, z_1, \ldots, x_r, y_r, z_r]}{\sqrt{\langle x_i z_j - x_j z_i \mid 1 \leq i \leq j \leq r \rangle}}.$$

Proposition 11.1.1. *For each $r \geq 1$, we have*

(a) The variety $C_r(\mathfrak{u})$ is irreducible of dimension $2r+1$.

(b) All singular points in $C_r(\mathfrak{u})$ are of the form

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 & 0 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_r & 0 & 0 \end{pmatrix}$$

for some $y_1, \ldots, y_r \in k$.

(c) The variety $C_r(\mathfrak{u})$ is Cohen-Macaulay and normal.

(d) The variety $C_r(\mathcal{N})$ is an irreducible variety of dimension $2r+4$.

Proof. We first have the following identification

$$k[C_r(\mathfrak{u})] \cong k[y_1, \ldots, y_r] \otimes \frac{k[x_1, z_1, \ldots, x_r, z_r]}{\sqrt{\langle x_i z_j - x_j z_i \mid 1 \leq i \leq j \leq r \rangle}}. \tag{11.1}$$

Let V be the variety associated to the latter ring. Then parts (a) and (c) follows if we are able to show that V is irreducible, Cohen-Macaulay and normal. Indeed, these properties can be obtained from determinantal varieties as follows.

Consider the matrix

$$\mathcal{X} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ z_1 & z_2 & \cdots & z_r \end{pmatrix}.$$

It is easy to see that

$$k[C_r(\mathfrak{u})] \cong \frac{k[\mathcal{X}]}{I_2(\mathcal{X})}$$

where $I_2(\mathcal{X})$ is the ideal generated by 2-minors of the matrix $\mathcal{X}$. Hence by Theorem 9.1.6, we obtain the variety $C_r(\mathfrak{u})$ is Cohen-Macaulay, normal and irreducible of dimension $r+1$.

(b) Note that the group $H = GL_r \times GL_2$ acts on V under the following action:

$$\left(\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \times \left[\begin{pmatrix} 0 & 0 & 0 \\ x_i & 0 & 0 \\ 0 & z_i & 0 \end{pmatrix}\right]_{i=1}^{r} \mapsto \left(\sum_{i=1}^{r} a_{1i}v_i \ , \ldots , \ \sum_{i=1}^{r} a_{1i}v_j\right)$$

where $v_i = \begin{pmatrix} 0 & 0 & 0 \\ ax_i + bz_i & 0 & 0 \\ 0 & cx_i + dz_i & 0 \end{pmatrix}$ for each $1 \leq i \leq r$. Now consider a nonzero element v in V. Without loss of generality, we can assume that the entry $x_1 \neq 0$ in v. Then v belongs to the open set $W = V \cap V(x_1 \neq 0)$. It is easy to see that

$$W = H \cdot \left[\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

which is a smooth orbit of dimension $r+1$. It follows that v is non-singular. From the isomorphism (11.1), we have all singular points of the right-hand side are $V(k[y_1, \ldots, y_r]) \times \{0\}$. This determines the singular locus as desired.

(d) Irreducibility follows from Theorem 9.2.2. Then, the birational properness of the moment morphism

$$m : G \times^B C_r(\mathfrak{u}) \to C_r(\mathcal{N})$$

implies that

$$\dim C_r(\mathcal{N}) = \dim G \times^B C_r(\mathfrak{u}) = \dim C_r(\mathfrak{u}) + \dim G/B = 2r + 4.$$

□

11.2 Singularities of $C_r(\mathcal{N})$

Serre's Criterion stated that a variety V is normal if and only if the set of singularities has codimension ≥ 2 and the depth of V at every point is $\geq \min(2, \dim V)$. This makes the task of determining dimension of singular locus of a variety necessary in order to get its normality.

Note that the problem on the singular locus of ordinary commuting varieties was studied by Popov [Po]. He showed that the codimension of singularities for $C_2(\mathfrak{g})$ is greater than or equal to 2 for an arbitrary reductive Lie algebra $\mathfrak{g}$. In other words, the variety $C_2(\mathfrak{g})$ has the necessary condition for being normal. Again, the author has not seen any literary work for commuting varieties (of r-tuples).

We prove in this section that the set of all singularities has codimension ≥ 2. Let $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$ be the set of positive roots of Φ, the underlying root system of G. We have shown in the Proposition 11.1.1(b) that the vector space $\mathfrak{u}^r_{\alpha+\beta}$ includes all singularities of $C_r(\mathfrak{u})$. The following lemmas shows that the singular locus of $C_r(\mathcal{N})$ is what we expect.

Lemma 11.2.1. *Suppose $r \geq 2$, the subset $G \cdot (z_{\text{reg}}\ , \ldots ,\ v_{\text{reg}}\ , \ldots ,\ z_{\text{reg}})$ of $C_r(\mathcal{N})$ is a smooth open subvariety of dimension $2r+4$ for each position i-th of v_{reg}, $1 \leq i \leq r$.*

Proof. As a linear combination of commuting nilpotents is also nilpotent, we have an action from $G \times (G_a^r)^{r-1}$ on $C_r(\mathcal{N})$ as follows

$$\begin{aligned}&\big(G \times (G_a^r)^{r-1}\big) \times C_r(\mathcal{N}) \to C_r(\mathcal{N})\\ &(g, (a_{ij})) \bullet (v_1, \ldots, v_r) \longmapsto g \cdot (\sum_{j=1}^{r} a_{1j} v_j \ , \ldots, v_i \ , \ldots, \ \sum_{j=1}^{r} a_{r-1,j} v_j).\end{aligned}$$

for all $(g, (a_{ij})) \in G \times (G_a^r)^{r-1}$ and $(v_1, \ldots, v_r) \in C_r(\mathcal{N})$. Now observe that $z(v_{\text{reg}})$ is a vector space of dimension 2, choose $\{v_{\text{reg}}, w\}$ be the basis of this space. It is easy to check that for $r \geq 2$ we have

$$G \cdot (z(v_{\text{reg}}) \ , \ldots, \ v_{\text{reg}} \ , \ldots, \ z(v_{\text{reg}})) = (G \times (G_a^r)^{r-1}) \bullet (w \ , \ 0 \ , \ldots, \ v_{\text{reg}} \ , \ldots, \ 0).$$

So, by [Hum2, Proposition 8.3], it is smooth.

Consider the projection map from $C_r(\mathcal{N})$ to the i-th factor $p : C_r(\mathcal{N}) \to \mathcal{N}$. Then we have

$$G \cdot (z_{\text{reg}} \ , \ldots, \ v_{\text{reg}} \ , \ldots, \ z_{\text{reg}}) = p^{-1}(\mathcal{O}_{\text{reg}})$$

which is an open subset in $C_r(\mathcal{N})$. Hence, the dimension immediately follows. □

Lemma 11.2.2. *The intersection of* z_{sub} *and* $\overline{\mathcal{O}_{\text{subreg}}}$ *is exactly the union of* $\mathfrak{u}_\alpha \times \mathfrak{u}_{\alpha+\beta}$ *and* $\mathfrak{u}_\alpha \times \mathfrak{u}_{-\beta}$*. Hence, we have* $\dim\, (z_{\text{sub}} \cap \overline{\mathcal{O}_{\text{subreg}}}) = 2$.

Proof. It can be computed that z_{sub} consists of all matrices of the form

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & x_1 & t_2 \\ x_3 & 0 & -2x_1 \end{pmatrix}$$

where x_1 , x_2 , x_3 , t_2 are in k. As the determinant of a nilpotent matrix is always 0, we obtain

$$z_{\text{sub}} \cap \mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_2 & 0 & t_2 \\ x_3 & 0 & 0 \end{pmatrix} \mid x_2, x_3, t_2 \in k \right\}.$$

On the other hand, it is well-known that $\mathcal{O}_{\text{subreg}}$ consists of all matrices of rank one and $\overline{\mathcal{O}_{\text{subreg}}} = \mathcal{O}_{\text{subreg}} \cup \{0\}$. It turns out that

$$\begin{aligned} z_{\text{subreg}} \cap \overline{\mathcal{O}_{\text{subreg}}} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_2 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \right\} \\ &= \mathfrak{u}_\alpha \times \mathfrak{u}_{\alpha+\beta} \cup \mathfrak{u}_\alpha \times \mathfrak{u}_{-\beta}. \end{aligned}$$

Hence, we have $\dim\,(z_{\text{sub}} \cap \overline{\mathcal{O}_{\text{subreg}}}) = 2$. □

Theorem 11.2.3. *For each $r \geq 1$, let C^{sing} be the singular locus of $C_r(\mathcal{N})$. Then we have $C^{\text{sing}} \subseteq C_r(\overline{\mathcal{O}_{\text{subreg}}})$. Moreover,* $\operatorname{codim} C^{\text{sing}} \geq 2$.

Proof. It is obvious that our result is true for $r = 1$. Assume now that $r \geq 2$. It suffices to prove that any point in the complement of $C_r(\overline{\mathcal{O}_{\text{subreg}}})$ in $C_r(\mathcal{N})$ is smooth. Let $V = C_r(\overline{\mathcal{O}_{\text{subreg}}})$ and suppose $w \in C_r(\mathcal{N})\backslash V$. Say $w = (w_1, \ldots, w_r)$ with some $w_n \notin \overline{\mathcal{O}_{\text{subreg}}}$, i.e., w_n is a regular element of $\mathfrak{sl}_3$. Consider the projection onto n-th factor $p : C_r(\mathcal{N}) \to \mathcal{N}$. As $G \cdot w_n = \mathcal{O}_{\text{reg}}$ is open in $\mathcal{N}$, the preimage $p^{-1}(G \cdot w_n)$ is an open set of $C_r(\mathcal{N})$. Note that

$$p^{-1}(G \cdot w_n) = G \cdot p^{-1}(w_n) = G \cdot (z_{\text{reg}}\ , \ldots ,\ v_{\text{reg}}\ , \ldots ,\ z_{\text{reg}}).$$

By Lemma 11.2.1, w is non-singular. Therefore, V contains all the singularities of $C_r(\mathcal{N})$.

Consider the projection of $C_r(\overline{\mathcal{O}_{\text{subreg}}})$ on i-th factor $p'_i : C_r(\overline{\mathcal{O}_{\text{subreg}}}) \to \overline{\mathcal{O}_{\text{subreg}}}$. Set

$$\mathfrak{O} = \bigcup_{i=1}^{r} p'^{-1}_i(\mathcal{O}_{\text{subreg}}),$$

the union of open sets in $C_r(\overline{\mathcal{O}_{\text{subreg}}})$. It is observed that $\mathfrak{O}$ contains all nonzero elements of $C_r(\overline{\mathcal{O}_{\text{subreg}}})$, i.e., $C_r(\overline{\mathcal{O}_{\text{subreg}}})$ is the closure of $\mathfrak{O}$. Now let $V_1 = \mathfrak{u}_\alpha \times \mathfrak{u}_{\alpha+\beta}$ and $V_2 = \mathfrak{u}_\alpha \times \mathfrak{u}_{-\beta}$. By preceding lemma, we have $z_{\text{sub}} \cap \overline{\mathcal{O}_{\text{subreg}}} = V_1 \cup V_2$. Also observe that for $u, v \in V_1 \cup V_2$ we have

$$[u, v] = 0 \quad \Leftrightarrow \quad u, v \in V_1 \quad \text{or} \quad u, v \in V_2.$$

It follows that for each $1 \leq i \leq r$ we have

$$p'^{-1}_i(\mathcal{O}_{\text{subreg}}) = G \cdot (V_1\ , \ldots,\ v_{\text{subreg}}\ , \ldots,\ V_1) \cup G \cdot (V_2\ , \ldots,\ v_{\text{subreg}}\ , \ldots,\ V_2).$$

Then we get

$$C_r(\overline{\mathcal{O}_{\text{subreg}}}) = \overline{\bigcup_{i=1}^{r} p'^{-1}_i(\mathcal{O}_{\text{subreg}})} = \bigcup_{i=1}^{r} \overline{G \cdot (V_1\ , \ldots,\ v_{\text{subreg}}\ , \ldots,\ V_1)} \cup \overline{G \cdot (V_2\ , \ldots,\ v_{\text{subreg}}\ , \ldots,\ V_2)}.$$

Apply the strategy of Premet in [Pr], there is an action of GL_r on $C_r(\mathcal{N})$ which implies that permutations of tuples in every element stablize in the irreducible component containing that element. As $\overline{G \cdot (v_{\text{subreg}}\ , \ldots,\ V_j)}$ is irreducible in $C_r(\mathcal{N})$ for each $j = 1, 2$, we can rewrite the union above as follows

$$C_r(\overline{\mathcal{O}_{\text{subreg}}}) = \overline{G \cdot (v_{\text{subreg}}\ , V_1\ , \ldots,\ V_1)} \cup \overline{G \cdot (v_{\text{subreg}}\ ,\ V_2\ , \ldots,\ V_2)}.$$

Using theorem of dimension on fibers, we can compute that $\overline{G \cdot (v_{\text{subreg}},\ V_j\ , \ldots,\ V_j)}$ has dimension $2r+2$ for each $j = 1, 2$. Thus $\dim C_r(\overline{\mathcal{O}_{\text{subreg}}}) = 2r+2$ so that $\operatorname{codim} C_r(\overline{\mathcal{O}_{\text{subreg}}}) = 2$. □

Chapter 12

Appendix

12.1 Reduced B_r-cohomology with small r

- Case $r = 2$

$$k[x_1^{(1)}, x_2^{(2)}]^{T_2} \cong k[x_1, x_2^{(1)}]^{T_1^{(1)}} \cong k[x_1^p, x_2^{(1)}]^{(1)} \cong k[x_1^{p(-1)}, x_2]^{(2)}.$$

- Case $r = 3$

$$k[x_1^{(1)}, x_2^{(2)}, x_3^{(3)}]^{T_3} \cong \bigoplus_{i+j=p \ or \ 0} \left(k[x_1^{p^2}, x_2^p, x_3] \otimes x_1^{pi} x_2^j\right)^{(3)}$$
$$\cong \bigoplus_{i+j=p \ or \ 0} \left(k[x_1^{p^2}, x_2^p, x_3] \otimes (\frac{i+j}{p})\alpha\right)^{(3)}$$

- Case $r = 4$

$$k[x_1^{(1)}, x_2^{(2)}, x_3^{(3)}, x_4^{(4)}]^{T_4} \cong \bigoplus_{i+j=0 \ or \ p} \ \bigoplus_{\frac{i_1+i_2+i_3+\frac{i+j}{p}}{p}<3} \left(k[x_1^{p^3}, x_2^{p^2}, x_3^p, x_4] \otimes x_1^{p^2 i_1} x_2^{p i_2} x_3^{i_3} x_1^{pi} x_2^j\right)^{(4)}$$
$$\cong \bigoplus \left(k[x_1^{p^3}, x_2^{p^2}, x_3^p, x_4] \otimes \left(\frac{i_1 + i_2 + i_3 + \frac{i+j}{p}}{p}\right)\alpha\right)^{(4)}$$

12.2 MAGMA programs

```
NN:=function(p,r,m,S)
if r eq 1 or m eq 0 then
        return 1;
else
        N:=0;
        for j:=1 to m-1 do
           N:=N+$$(p,r-1,j*p,S);
        end for;
        if IsDisjoint({1..r-1},S) then
           N:=N+1;
        end if;
        if r notin S then
           N:=N+$$(p,r-1,m*p,S);
        end if;
        return N;
end if;
end function;

CC:=function(p,r,m)
if m eq 0 then
        return 1;
else
        if m lt 0 then
           return 0;
        else
           L:=Subsets({1..r-1});
           N:=0;
        for S in L do
           N:=N+NN(p,r,m,S);
        end for;
           return N;
        end if;
end if;
end function;

NNWithLambda:=function(p,r,m,L,S)
if L[1] gt m then
  return 0;
else
  if r eq 1 then
     return 1;
  else
     N:=0;
     L1:=Remove(L,1);
     for j:=L[1]+1 to m-1 do
       N:=N+$$(p,r-1,(m-j)*p,L1,S);
     end for;
       flag:=0;
     for i:=1 to #L1 do
       flag:=flag+L1[i];
     end for;
             if IsDisjoint({1..r-1},S) and flag eq 0 then
```

```
              N:=N+1;
           end if;
           if r notin S then
              N:=N+$$(p,r-1,(m-L[1])*p,L1,S);
           end if;

     return N;
  end if;
end if;
end function;

Formula:=function(p,r,m,L)
if m lt 0 then
   return 0;
else
   SS:=Subsets({1..r-1});
   N:=0;
   for S in SS do
      N:=N+NNWithLambda(p,r,m,L,S);
   end for;
      return N;
end if;
end function;

MultiplicityEvenCase:=function(p,r,m,n)
if n gt m*p^r then
return 0;
else
 if n eq 0 then
  return CC(p,r,m);
 else
  if n mod p eq 0 then
   L:=Intseq(n,p);
  else
    if (n+1) mod p eq 0 then
      L:=Intseq(n+1,p);
    else
      return 0;
    end if;
 end if;
 k:=#L;
 if k ge r+2 then
     A:=0;
       for i:=r+2 to k do
       A:=A+L[i]*p^(i-1-r);
       end for;
     if A gt m then
       return 0;
     else
       m:=m-A;
       L[2]:=L[2]+L[1]/p;
       List:=Reverse([L[i]:i in [2..r+1]]);
       return Formula(p,r,m,List);
     end if;
```

```
 else
     for j:=k+1 to r+1 do
     L[j]:=0;
     end for;
     L[2]:=L[2]+L[1]/p;
 end if;
 List:=Reverse([L[i]:i in [2..r+1]]);
 return Formula(p,r,m,List);
 end if;
end if;
end function;


intrinsic CharacterMultiplicitySL2(p::RngIntElt,r::RngIntElt,m::RngIntElt,n::RngIntElt)->RngIntElt
{Let G=SL_2 and B be the Borel subgroup of G. Suppose \alpha is the only simple root in the root
system of G. Then \omega=\alpha/2 is the fundamental weight in weight lattice. Given non-negative
integers m,n and p>2. This function returns the multiplicity of weight m*\omega in
H^\bullet(B_r,n*\omega) over the field k of characteristic p.}

if IsOdd(m) and IsEven(n) then
return 0;
else
        if IsOdd(n) and IsEven(m) then
        return 0;
        else
                if IsEven(m) and IsEven(n) then
                m:=Quotrem(m,2);
                n:=Quotrem(n,2);
                return MultiplicityEvenCase(p,r,m,n);
                else

                        if n gt m*p^r then
                        return 0;
                        else
                                if n eq 0 then
                                return CC(p,r,m);
                                else
                                n:=Quotrem(n+p^r,2);
                                m:=Quotrem(m+1,2);
                                return MultiplicityEvenCase(p,r,m,n);
                                end if;
                        end if;
                end if;
        end if;
end if;
end intrinsic;

----------------------------------------------------------------------------------------------------

intrinsic CardinalityOfReducedBasis(p::RngIntElt,r::RngIntElt) -> RngIntElt
{Let G=SL_2 and B be the Borel subgroup of G. Let U be the unipotent radical of B. Given an odd prime
p and a positive integer r, this function returns the number of elements in the basis of the reduced
cohomology ring H^\bullet(B_r,k)_{red} as a module over the ring H^\bullet(U_r,k)_{red}.}
```

```
I:=IdentityMatrix(IntegerRing(),r-1);
K:=HalfspaceToPolyhedron([I[1][j]:j in [1..r-1]],0);
for i:=2 to r-1 do
K:=K meet HalfspaceToPolyhedron([I[i][j]:j in [1..r-1]],0);
end for;
I1:=-1*I;
for i:=1 to r-1 do
K:=K meet HalfspaceToPolyhedron([I1[i][j]:j in [1..r-1]],1-p^(r-i));
end for;
v:=[p^(i): i in [0..r-2]];
cc:=1;
for i:=1 to r-2 do
M:=K meet HyperplaneToPolyhedron(v,i*p^(r-1));
cc:=cc+EhrhartCoefficient(M,1);
end for;
return cc;
end intrinsic;
```

12.3 Principal Radical System

In this section, we verify the condition of Theorem 10.3.3(b) for I and J are in the context of Proposition 10.3.4.

12.3.1 Case 1

We check the following

$$(1) \quad \langle y_{m+1} + z_{m+1} \rangle \, x_j \in I,$$

$$(2) \quad \langle y_{m+1} + z_{m+1} \rangle \, y_j \in I,$$

$$(3) \quad \langle y_{m+1} + z_{m+1} \rangle \, z_j \in I.$$

1) For the first one, we consider for each $1 \leq j \leq m$,

$$
\begin{aligned}
(y_{m+1} + z_{m+1})x_j + I &= y_{m+1}x_j + z_{m+1}x_j + I \\
&= x_{m+1}y_j + x_{m+1}z_j + I \\
&= x_{m+1} \langle y_j + z_j \rangle + I \\
&= I
\end{aligned}
$$

where the second identity is provided by $x_{m+1}y_j - x_j y_{m+1}$, $x_{m+1}z_j - x_j z_{m+1} \in I_r \subseteq I_m$; and the last identity is provided by $y_j + z_j \in I$.

Same technique for (2) and (3) as follows:

$$
\begin{aligned}
(y_{m+1} + z_{m+1})y_j + I &= y_{m+1}y_j + z_{m+1}y_j + I \\
&= y_{m+1}y_j + y_{m+1}z_j + I \\
&= y_{m+1}(y_j + z_j) + I \\
&= I, \\
(y_{m+1} + z_{m+1})z_j + I &= y_{m+1}z_j + z_{m+1}z_j + I \\
&= z_{m+1}y_j + z_{m+1}z_j + I \\
&= z_{m+1}(y_j + z_j) + I \\
&= I.
\end{aligned}
$$

12.3.2 Case 2

We need to check the following

$$
\begin{aligned}
&(1) \quad (x_{n+1} + y_{n+1})\, x_j \in I, \\
&(2) \quad (x_{n+1} + y_{n+1})\, y_j \in I, \\
&(3) \quad (x_{n+1} + y_{n+1})\, z_j \in I, \\
&(4) \quad (x_{n+1} + y_{n+1})\, (x_h - y_h) \in I,
\end{aligned}
$$

for all $1 \leq j \leq n$ and $n+1 \leq h \leq r$. Verifying (1), (2), and (3) is the same as Case 1. Lastly, we look at

$$
\begin{aligned}
(x_{n+1} + y_{n+1})(x_h - y_h) + I &= x_{n+1}x_h + y_{n+1}x_h - x_{n+1}y_h - y_{n+1}y_h + I \\
&= x_{n+1}x_h - y_{n+1}y_h + I \\
&= x_{n+1}x_h + y_{n+1}z_h + I.
\end{aligned}
$$

Now in order to complete our verifying, we will show that $x_{n+1}x_h + y_{n+1}z_h \in I_r$. Indeed, we have

$$
\begin{aligned}
(x_{n+1}x_h + y_{n+1}z_h)^2 + I_r &= x_{n+1}^2 x_h^2 + 2x_{n+1}x_h y_{n+1}z_h + y_{n+1}^2 z_h^2 + I_r \\
&= -x_{n+1}^2 y_h z_h + 2x_{n+1}^2 y_h z_h + y_{n+1}^2 z_h^2 + I_r \\
&= x_{n+1}^2 y_h z_h + y_{n+1}^2 z_h^2 + I_r \\
&= -y_{n+1}z_{n+1}y_h z_h + y_{n+1}^2 z_h^2 + I_r \\
&= y_{n+1}z_h(-z_{n+1}y_h + y_{n+1}z_h) + I_r \\
&= I_r.
\end{aligned}
$$

As I_r is radical, we obtain $x_{n+1}x_h + y_{n+1}z_h \in I_r$ as desired.

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