Descent on Elliptic Curves

an honours seminar by

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- We require $a, b, c \in \mathbb{Q}$ such that $a^2 + b^2 = c^2$ and $n = ab/2$. Make the change of variables $x = (c/2)^2$ and $y = (b^2 - a^2)c/8$. We can easily check that

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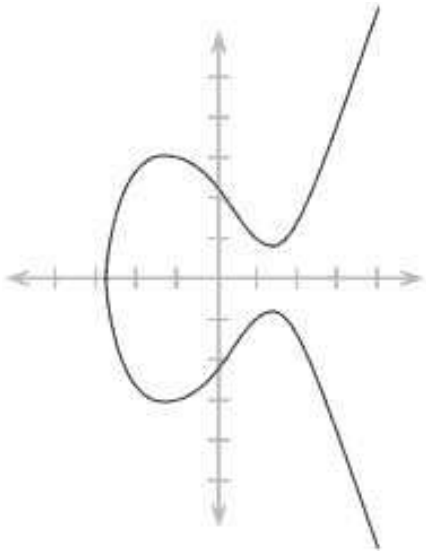
- So we have converted our problem into finding rational points on a curve – as we will see this is actually an elliptic curve.

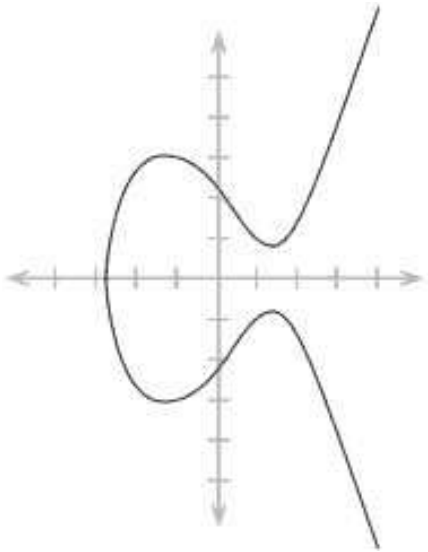
Elliptic Curves

- An elliptic curve is the set of points on a curve $y^2 = x^3 + ax + b$ together with a 'point at infinity', $\mathcal{O}$.



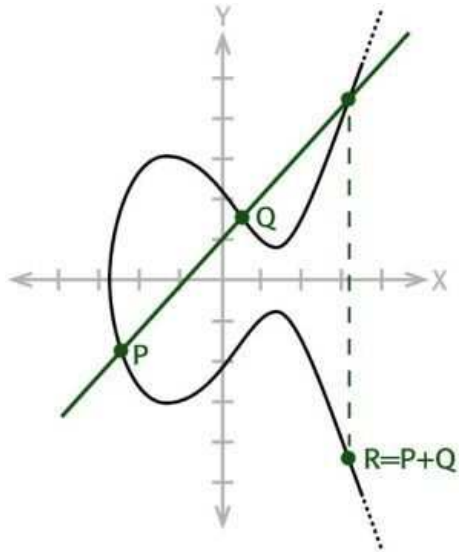
Elliptic Curves

- An elliptic curve is the set of points on a curve $y^2 = x^3 + ax + b$ together with a 'point at infinity', $\mathcal{O}$.
- We also require the curve to be non-singular – to have no cusps or self intersections. This is equivalent to requiring that $x^3 + ax + b$ has no repeated roots.



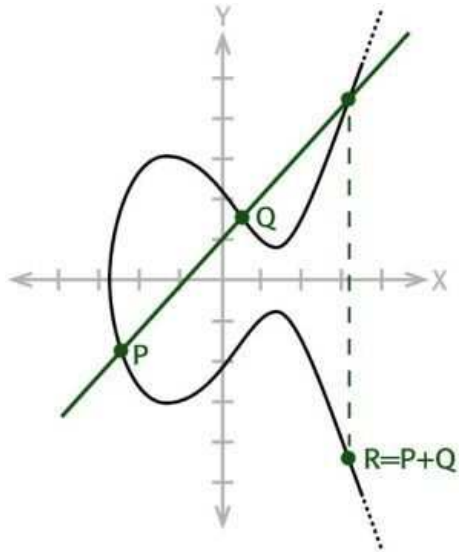
The Group Law

- Given two points on an elliptic curve draw the line from P to Q until you hit the curve again.



The Group Law

- Given two points on an elliptic curve draw the line from P to Q until you hit the curve again.
- Next draw a line from the point at infinity to this point. The point $P + Q$ is where this line intersects the elliptic curve again.



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- There is a subgroup made up of the points on the elliptic curve that have rational coordinates, denoted $E(\mathbb{Q})$.
- As we are doing number theory we are interested in finding this group.

Why We Study Elliptic Curves

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  - Elliptic curve cryptography.
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- Elliptic curves are projective curves of genus one with a specified basepoint. Curves of genus zero are conics and are well understood – if they have one rational point then they have infinitely many.
- Curves of genus greater than one have only finitely many rational points by a deep theorem of Falting. Thus elliptic curves are an interesting boundary case that have provided much of the motivation for modern algebraic geometry and arithmetic geometry.

The Mordell–Weil Theorem

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  Theorem (Mordell 1922)**.** *Let E be an elliptic curve defined over $\mathbb{Q}$. Then $E(\mathbb{Q})$ is finitely generated.*

- By the fundamental theorem of finitely generated abelian groups this means that

$$E(\mathbb{Q}) \cong E_{\mathrm{tors}}(\mathbb{Q}) \oplus \mathbb{Z}^r.$$

Finding the rank

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- Since we are looking for rational points you might think we can just search for them. However relatively simple curves can have very complicated points.
- For example the curve $y^2 = x^3 + 877x$ has rank one and the x–coordinate of a generator is given by

$$\left(\frac{6127760831879473681 01}{78841535860639002 10}\right)^2.$$

The Congruent number problem

- We have the following solution (if we can find ranks effectively) to our congruent number problem

  Theorem. *A positive integer n is a congruent number if and only if the elliptic curve $y^2 = x^3 - n^2x$ has rank greater than zero.*

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- The congruent numbers less than 100 are

$$5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, 38, 39, 41,$$
$$45, 46, 47, 52, 53, 54, 55, 56, 60, 61, 62, 63, 65, 69, 70, 71, 77, 78, 79,$$
$$80, 84, 85, 86, 87, 88, 92, 93, 94, 95, 96$$

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- Tunnel (1983) gave an effective procedure for determining if n is a congruent number assuming the Birch–Swinnerton-Dyer conjecture.

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- As of 2006 the curve of highest known rank is of rank at least 28. It is given by

$$y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502x+$$
$$3448161179503055646703298569039072037485594435931918036126600829629193944873224342 9.$$

The Weak Mordell–Weil Theorem

- In our proof of the Mordell–Weil Theorem we use the following

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- In fact it can be shown that if we can find generators for $E(\mathbb{Q})/2E(\mathbb{Q})$ then we have an effective procedure to find generators for $E(\mathbb{Q})$.
- Thus from now on we are only concerned with finding $E(\mathbb{Q})/2E(\mathbb{Q})$.

p–adic numbers

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- Define the p–adic integers, $\mathbb{Z}_p$, to be the set of sequences

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- Thus $(\ldots, 23, 5, 2)$ is in $\mathbb{Z}_3$ but $(\ldots, 22, 5, 2)$ isn't since $22 \neq 5 \pmod{9}$.

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- The p–adic numbers contain $\mathbb{Q}$ and are a field of characteristic zero.
- It can be shown that the only possible completions of the rationals are the p–adic numbers for each p and the reals. For this reason we denote $\mathbb{R}$ by $\mathbb{Q}_\infty$.

The Local-Global principle

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- Thus to find 'global' solutions over $\mathbb{Q}$ only need to consider the equation 'locally' at each prime p.
- However a similar theorem doesn't hold for cubic polynomials so local information doesn't give the full picture for elliptic curves.
- We are still interested in local methods but now we need to keep track of when they fail.

Some Galois Cohomology

- The algebraic closure of $\mathbb{Q}$, denoted $\overline{\mathbb{Q}}$, is the set of all solutions of polynomials with rational coefficients. The Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ is the set of isomorphisms from $\overline{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$ that fix $\mathbb{Q}$.

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- Define a G–module to be an abelian group M with an action of G ($\sigma \in G$ acts on $m \in M$ by sending $m \mapsto m^\sigma$) such that
  - $m^1 = m$
  - $(m + m')^\sigma = m^\sigma + m'^\sigma$
  - $(m^\sigma)^\tau = m^{\sigma\tau}$.

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- Examples of G–modules are $\overline{\mathbb{Q}}$ and E.

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- $H^0(G,M) = M^G = \{m \in M \mid m^\sigma = m$ for all $\sigma \in G\}$.
- $H^1(G,M) = Z^1(G,M)/B^1(G,M)$ where $Z^1(G,M)$ is the group of maps $\xi\colon G \to M$ satisfying the cocycle condition

$$\xi(\sigma\tau) = \xi(\sigma)^\tau + \xi(\tau)$$

and $B^1(G,M)$, is the group of maps $\xi\colon G \to M$ where there exists an $m \in M$ such that, for all $\sigma \in G$

$$\xi(\sigma) = m^\sigma - m.$$

Diagrams and Exact Sequences

- A commutative diagram is a diagram such that it doesn't matter what path we take. For example the next diagram is commutative if and only if $g \circ f = k \circ h$.

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- An *exact sequence* is a set of G–modules with homomorphisms between them

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{j-1}} A_i \xrightarrow{f_j} A_{i+1} \longrightarrow \cdots$$

such that $\operatorname{Im} f_{j-1} = \operatorname{Ker} f_j$ for all j.

- A *short exact sequence* is an exact sequence of the form

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Note that in a short exact sequence f is injective, g is surjective and $\mathrm{Im}\, f = \mathrm{Ker}\, g$.

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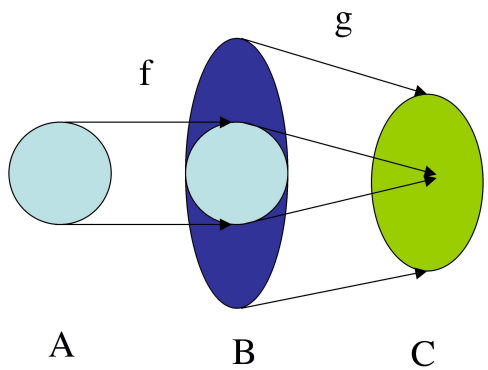
- Short exact sequences give us lots of information. We can see that every element of B either is an element of an isomorphic copy of A or maps nontrivially into C.



2-Descent

- For an abelian group A we denote the 2–torsion by $A[2] = \{a \in A \mid 2a = 0\}$. Set $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Using Galois cohomology and a generalisation of Hilbert Theorem 90 we can find a commutative diagram with exact rows (all products are over $p = 2, 3, 5, \ldots, \infty$).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(\mathbb{Q})/2E(\mathbb{Q}) & \longrightarrow & H^1(G, E[2]) & \longrightarrow & H^1(G, E)[2] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \searrow^{\lambda} & \downarrow{\scriptstyle \theta} & & \\
0 & \to & \prod E(\mathbb{Q}_p)/2E(\mathbb{Q}_p) & \to & \prod H^1(G_p, E[2]) & \to & \prod H^1(G_p, E)[2] & \to & 0
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- We are interested in finding $E(\mathbb{Q})/2E(\mathbb{Q})$ but to do this we use local information – Hensel's lemma tells us that calculating the bottom row can be reduced to a finite amount of computation.

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$$0 \longrightarrow E(\mathbb{Q})/2E(\mathbb{Q}) \longrightarrow S^{(2)} \longrightarrow Ш[2] \longrightarrow 0.$$

- The group $S^{(2)}$ is finite and computable (this means that $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite: the weak Mordell–Weil theorem). Figuring out whether a point in $S^{(2)}$ comes from $E(\mathbb{Q})/2E(\mathbb{Q})$ or maps nontrivially into $Ш[2]$ is hard.

Homogenous spaces

- We have an exact correspondence between elements of $H^1(G, E)$ and equivalence classes of curves that we call ‘homogenous spaces’.

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- Thus $Ш$ measures the failure of the local–global principle.

Calculating the rank

- If $Ш[2] = 0$ then $E(\mathbb{Q})/2E(\mathbb{Q}) \cong S^{(2)}$ is able to be calculated. If $Ш[2]$ is nontrivial then we only get a bound on the rank.

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- Programs like mwrank can calculate ranks quite well practically but there is no algorithm that always works.
- Going back to our motivating problem consider the question of the congruent number problem for $n = 5$. By running mwrank we can see that the curve $y^2 = x^3 - 25x$ has rank one. In fact with a little more work we can find a point $(1681/144, 62279/1728)$ on our curve. This corresponds to a triangle $(a, b, c) = (3/2, 20/3, 41/6)$.

Future Work

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- What I'm doing: looking at an idea of Flynn. He carries out descent without using homogenous spaces.
- What other people are doing: Higher descent – 3-descent, second 2-descent, 5 descent,...
- Trying to win a million dollars: The Clay Institute is offering one million dollars for a proof of the Birch–Swinnerton-Dyer conjecture. Associated to an elliptic curve E is an L–series $L_E(s)$. The BSD conjecture is that
  - $L_E(s)$ has a zero at $s = 1$ of order equal to the rank of $E(\mathbb{Q})$; and
  - $$\lim_{s \to 1} \frac{L_E(s)}{(s-1)^r} = \frac{\Omega R |Ш| \prod_{p|\Delta} c_p}{|E_{\text{tors}}(\mathbb{Q})|^2}.$$

The End

- Any questions?