RESEARCH STATEMENT

MCKENZIE LAMB

1. Overview

My research lies at the intersection of Lie theory and Poisson geometry. The fundamental object of study is a Poisson Lie group, a Lie group G together with a Poisson structure (a bivector field satisfying the Jacobi identity, in a certain sense) that is compatible with the group multiplication on G: the map $G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ is a Poisson map. Thus, a Poisson Lie group is a space that is endowed with three structures, a smooth structure, a group structure, and a Poisson structure, all of which are somehow compatible.

Poisson structures can be thought of as symplectic structures for which degeneracy is allowed. Generalizing from symplectic to Poisson manifolds yields a framework in which a broader class of problems can be modeled, including many problems in physics. Since Lie groups and Lie group actions also play an important role in both physics and various fields of mathematics, it is natural to consider Lie groups with compatible Poisson structures as well as actions of Lie groups on Poisson manifolds. Furthermore, actions of Poisson Lie groups on Poisson manifolds can be thought of as generalizations of Lie group actions on symplectic manifolds, and the notion of a momentum map for a symplectic action can be extended to the Poisson setting.

Over the past twenty years, a number of results have been proved which relate actions of Poisson Lie groups on Poisson manifolds to actions of Lie groups on symplectic manifolds, the latter being generally simpler and better-understood. Reducing a Poisson action to a symplectic action is similar in spirit to reducing a nonlinear problem to a linear problem. In fact, this sort of reduction is essentially tied to the existence of isomorphisms between certain linear and nonlinear Poisson structures. The existence of such isomorphisms was first shown by Ginzburg and Weinstein in [GW92] for compact, semisimple Lie groups endowed with a particular Poisson structure. Flaschka and Ratiu ([FR96]) used this result to transfer a convexity theorem for symplectic moment maps to Poisson moment maps. Alekseev showed in [Ale97] that Poisson actions of compact groups on symplectic manifolds can be reduced to symplectic actions and indicated the relationship between this fact and the existence of Ginzburg-Weinstein isomorphisms.

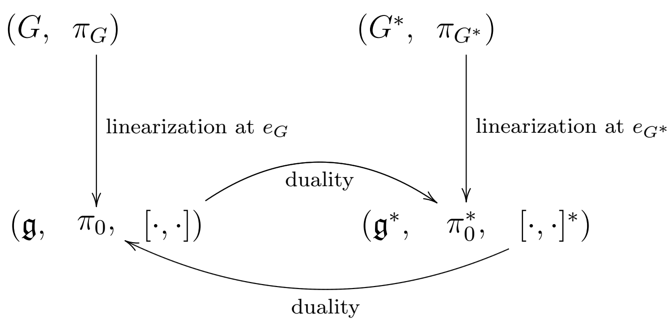
All results mentioned above, and nearly all results in this area, apply only to compact groups. My thesis (working under Dr. Philip Foth) will consist in an extension of the result of Ginzburg and Weinstein to the non-compact pseudo-unitary groups $SU(p, q)$. (The isomorphism in this case must be restricted to an open subset of $SU(p, q)$.)

Our argument is an adaptation of the Ginzburg-Weinstein proof to the non-compact case. One of the two primary components in this adaptation involves the use of a spectral sequence relating Poisson cohomology to Lie algebra cohomology. The other component involves essentially compactifying certain $SU(p,q)$ orbits by mapping them into a compact algebraic variety. Thus, the argument brings together elements of Poisson geometry, Lie theory, algebraic topology, and algebraic geometry.

2. Detailed Description

Given a Poisson Lie group, (G, π_G) with Lie algebra $\mathfrak{g}$, the dual space $\mathfrak{g}^*$ carries a linear Poisson structure π_0^* induced by the Lie bracket on $\mathfrak{g}$. Lu and Weinstein have defined (see [LW90]) the corresponding nonlinear notion of a dual Poisson Lie group. Roughly speaking, the dual of (G, π_G) is a Poisson Lie group (G, π_{G^*}) such that the linearization of π_{G^*} at the identity in G^* is the linear structure π_0^* and the linearization of π_G at the identity in G is the linear structure π_0 on $\mathfrak{g}$ induced by the Lie bracket on $\mathfrak{g}^*$. Figure 1 illustrates these relationships.

Figure 1. Duality of Poisson Lie Groups



Example 2.1. Let $K = SU(2)$ with Lie algebra $\mathfrak{k} = \mathfrak{su}(2)$. Using the pairing $\langle X, Y\rangle := \Im\mathrm{Trace}(XY)$ on $\mathfrak{sl}(2,\mathbb{C})$, the dual of $\mathfrak{k}$ can be identified with

$$\mathfrak{k}^* = \mathfrak{a} + \mathfrak{n} = \left\{ \begin{pmatrix} z & x+iy \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

with corresponding Lie group

$$AN = \left\{ \begin{pmatrix} e^{z/2} & x+iy \\ 0 & e^{-z/2} \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The Lie bracket on $\mathfrak{su}(2)$ induces the linear bracket $\{,\}^*$ on $\mathfrak{a} + \mathfrak{n}$ given by

$$\text{\{x,y\}=z}, \quad \text{\{x,z\}=-y}, \quad \text{\{y,z\}=x} \ ,$$

which corresponds to the bivector field

$$\pi_0 = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

There exists a so-called *standard* Poisson structure π_K on K such that the dual structure on AN is

$$\pi_{AN} = \sinh z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

The linearization of π_{AN} at the identity $= (0,0,0)$ is obtained by taking the linear terms in the Taylor series for the coefficient functions $\sinh z$, y, and x at 0. The result is π_0, as required.

Although duality of Poisson Lie group is fairly natural from an abstract point of view, the original motivation for this construction comes from physics. Under some conditions, a Poisson Lie group acts on its dual (or a subset of the dual) via the *dressing action*. This setting serves as a model for the *groups of dressing transformations* from integrable systems. For our purposes, the important property of the dressing action is that the dressing orbits in G^* are (subsets of) the symplectic leaves induced by the Poisson structure π_{G^*}.

Example 2.1 extends to the higher dimensional cases $K = SU(n)$, $n > 2$: The group K admits a Poisson structure π_K such that the dual group K^* can be identified with the group AN of upper-triangular $n \times n$ matrices with real, positive diagonal and determinant 1. The Lie algebra of AN is the set $\mathfrak{a} + \mathfrak{n}$ of upper-triangular $n \times n$ matrices with real diagonal and trace 0.

More generally, for any complex, semisimple Lie group G, there exist decompositions–called Iwasawa decompositions–of the form $G = KAN$, where K is compact, A is abelian, and N is unipotent. For a given Iwasawa decomposition, the groups K and AN can be given multiplicative Poisson structures in such a way that they are mutually dual Poisson Lie groups ([LW90]). These Poisson structures are called the *standard* or Lu-Weinstein structures on each group. On the other hand, denoting the Lie algebras of G, K, and AN by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{a} + \mathfrak{n}$, respectively, the decomposition $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$ yields an identification of $\mathfrak{a} + \mathfrak{n}$ with $\mathfrak{k}^*$ via the imaginary part of the Killing form. Ginzburg and Weinstein proved in [GW92] the existence of a Poisson isomorphism from $\mathfrak{g}^*$, endowed with the linear Poisson structure, to G^* endowed with its standard Poisson structure. For the case $G = SU(n)$, Flaschka and Ratiu conjectured the existence of a distinguished Ginzburg-Weinstein isomorphism which intertwines the Gelfand-Tsetlin coordinates (given by eigenvalues of upper-left $k \times k$ minors) on $\mathfrak{su}(n)^*$ and $SU(n)^*$ ([FR96]). This conjecture was later proved by Alekseev and Meinrenken in [AM07].

My thesis will consist in an extension of the Ginzburg-Weinstein result to the case in which the compact group $SU(n)$ is replaced by the non-compact group $SU(p,q)$. In the $SU(p,q)$ case, we have only a partial decomposition at the group level. Nonetheless, the groups $SU(p,q)$ and AN can be endowed with dual Poisson Lie group structures. As in the compact case, $\mathfrak{su}(p,q)^*$ can be identified with $\mathfrak{a}+\mathfrak{n}$, inducing a linear Poisson structure on $\mathfrak{a} + \mathfrak{n}$. Our theorem asserts the existence of a Poisson isomorphism from an open cone in $\mathfrak{su}(p,q)^* \cong \mathfrak{a} + \mathfrak{n}$ to an open cone in $SU(p,q)^* \cong AN$, endowed with the linear and standard Poisson structures, respectively, induced by $SU(p,q)$.

For the case $p = q = 1$, the required isomorphism can be computed in coordinates. Although a computation approach is not practical in the higher dimensional cases, the argument of Ginzburg and Weinstein can be adapted to the noncompact setting. The argument in [GW92] involves transferring the nonlinear structure on K^* to $\mathfrak{k}^*$ and proving the existence of a certain vector field on $\mathfrak{k}^*$. This vector field is then used to produce a flow which takes the linear structure to the nonlinear structure.

This construction relies on the compactness of K in two fundamental ways. First, a cohomological argument is used to show that the second Poisson cohomology group of K^* is trivial. This guarantees the existence of a certain vector field X on $\mathfrak{k}^*$. This can be accomplished in the noncompact case using a special case of the Hochschild-Serre spectral sequence. At the moment, our proof applies only to a dense open subset of the cone mentioned above (the set of dressing orbits of maximal dimension). However, we believe that the proof can be extended to the entire cone.

Second, compactness of the dressing orbits in K^* is used to guarantee that the vector field X, which is tangent to the dressing orbits in K^*, can be globally integrated. In the $SU(p,q)$ case, the dressing orbits of AN are not compact. However, Evens and Lu have shown in [EL01] (building on a result of Drinfeld's) that each symplectic leaf of AN can be mapped in an equivariant, one-to-one, Poisson fashion onto an $SU(p,q)$ orbit in a compact algebraic variety $\mathcal{L}$, which will be described below. The closure of this orbit is then a compact space. In the 2×2 case, this closure is a manifold with boundary– and the argument used by Ginzburg and Weinstein applies. For the higher-dimensional cases, the *Hausforff* closures of orbits may not be smooth. However, it can be shown that the *algebraic (Zariski)* closures of $SU(p,q)$-orbits in $\mathcal{L}$ are smooth. Since the Poisson structure on $\mathcal{L}$ is algebraic and has a quadratic Casimir, the algebraic closure of each orbit is a Poisson manifold. It follows that the vector field X is tangent to each closure, which implies that X can be globally integrated.

The compact space into which the dressing orbits in AN can be mapped is the variety $\mathcal{L}$ of subalgebras of $\mathfrak{sl}(n,\mathbb{C})$ which are Lagrangian (half-dimensional, pair to 0 with themselves) with respect to the imaginary part of the Killing form. The space of Lagrangian *subspaces* of $\mathfrak{sl}(n,\mathbb{C})$ can be identified with $O(n^2-1)$, but the subalgebra requirement is much more difficult to handle. In the $n=2$ case, $\mathcal{L}$ has two connected components. The component $\mathcal{L}_0$ containing $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, denoted by $\mathcal{L}_0$, can be identified with $SO(3) \cong \mathbb{R}P^3$. In this case, the $SU(p,q)$ orbits and their closures can be computed explicitly: Hausdorff closures are hemispheres, and algebraic closures are spheres. The fact that the algebraic closures of $SU(p,q)$ orbits are smooth is obtained by identifying $\mathcal{L}_0$ with a real form of the so-called *wonderful compactification* of the complex group $SL_n(\mathbb{C})$ (see [EJ08]).

3. Goals

First, although the current argument only applies to $SU(p,q)$, it should extend fairly easily to a large class of non-compact Lie groups, particularly certain real forms of complex, semisimple Lie groups.

Second, as mentioned above, the possibility of reducing Poisson actions to symplectic actions for *compact* Lie groups is related to the existence of Ginzburg-Weinstein isomorphisms. Given that the Ginzburg-Weinstein theorem has been extended to $SU(p,q)$, I hope also to extend the notion of Poisson-to-symplectic reduction to these groups (and perhaps to other non-compact groups).

This sort of reduction has some surprising applications in linear algebra. In particular, Alekseev, Meinrenken and Woodward gave a proof of Thompson's conjecture ([AMW01]), which relates singular values of complex matrices to eigenvalues of Hermitian matrices, using precisely this machinery. If Poisson-to-symplectic reduction can be carried out for actions of $SU(p,q)$ or other non-compact groups, it should be possible to obtain similar results for pseudo-Hermitian matrices.

Finally, for a compact real form K of a complex semisimple Lie group G, the standard Poisson structure on the dual group K^* is invariant under the action of the torus $T \subset K$. Therefore, this Poisson structure descends to the quotient K/T. Lu proved in [Lu99] that each symplectic leaf in K/T can be decomposed into a finite product of 2-dimensional submanifolds. Each leaf can thus be endowed with coordinates, which in turn make possible various useful computations. Constructing a similar decomposition for $SU(p,q)$ may yield many interesting results.

References

[Ale97] A. Alekseev. On Poisson actions of compact Lie groups on symplectic manifolds. *J. Differential Geom*, 45(2):241–256, 1997.

[AM07] A. Alekseev and E. Meinrenken. Ginzburg-Weinstein via Gelfand-Zeitlin. *Journal of Differential Geometry*, 76(1):1–34, 2007.

[AMW01] A. Alekseev, E. Meinrenken, and C. Woodward. Linearization of Poisson actions and singular values of matrix products. *Annales de l'institut Fourier*, 51(6):1691–1717, 2001.

[EJ08] Sam Evens and Benjamin F. Jones. On the wonderful compactification, 2008.

[EL01] S. Evens and J.H. Lu. On the variety of Lagrangian subalgebras, I. *Annales scientifiques de l'Ecole normale supérieure*, 34(5):631–668, 2001.

[FR96] H. Flaschka and T. Ratiu. A convexity theorem for Poisson actions of compact Lie groups. *Annales Scientifiques de l'École Normale Supérieure Sér. 4*, 29(6):787–809, 1996.

[GW92] V.L. Ginzburg and A. Weinstein. Lie-Poisson Structure on Some Poisson Lie Groups. *Journal of the American Mathematical Society*, 5(2):445–453, 1992.

[Lu99] J.H. Lu. Coordinates on Schubert cells, Kostant's harmonic forms, and the Bruhat Poisson structure onG/B. *Transformation Groups*, 4(4):355–374, 1999.

[LW90] J.H. Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Diff. Geom*, 31(2):501–526, 1990.