Math 223 – Vector Calculus Practice Exam 4

Author: Arlo Caine Name: Solutions

Directions: Read all questions carefully. Use a pencil and erase all unnecessary marks. Show all of your work in the space provided and display your answer on the line given if requested. You will lose points if you make an approximation and fail to indicate the approximation. Be careful to use proper notation to indicate vector versus scalar quantities as well.

1. Determine whether the following are true or false (T or F). You do not need to give your reasons.

   (a) T If f is a function then $\text{div}(\text{grad} f)$ is a scalar valued function.

   (b) T If $\vec{F} = x^2 y \cos(z)\vec{i} + \arctan(z)\vec{j} + z^3\cos(z^2-1)\vec{k}$ then the flux through the rectangle in the plane $z = 1$ with $0 \le x \le 2$ and $0 \le y \le 5$ is ± 10 depending on the orientation.

   (c) T If $\text{div}\vec{F} = x$ then the flux of $\vec{F}$ through any sphere centered at the origin is zero.

   (d) F If S is a sphere of radius one and $\int\int_S \vec{F}\cdot d\vec{A} = 0$ then $\text{div}\vec{F} = 0$ at all points inside S.

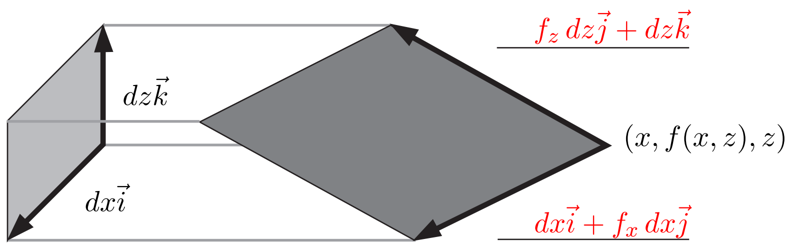
   (e) F If $\int\int_S \vec{F}\cdot d\vec{A} > \int\int_S \vec{G}\cdot d\vec{A}$ then $\|\vec{F}\| > \|\vec{G}\|$ at all points on S.

   (f) T $\text{grad}(\text{div}\vec{F}) - \text{curl}(\text{curl}(\vec{F}))$ is a vector valued function.

   (g) F If the area of a surface is doubled then it is always true that the flux through the surface is doubled.

   (h) F If $\vec{F}$ is a vector field on space and f is a scalar valued function then $\text{div}(f\vec{F}) = f\text{div}\vec{F}$.

2. In order to calculate the flux of a vector field through a surface $y = f(x,z)$ which is the graph of a function of x and z one needs to derive the vector field $d\vec{A}$. The vector field $d\vec{A}$ evaluated at a point $(x, f(x,z), z)$ on the surface gives a vector which is perpendicular to the surface at that point with magnitude equal to the area of parallelogram tangent to the surface at $(x, f(x,z), z)$ whose shadow in the xz plane is a rectangle with area $dx\,dz$. Fill in the blanks on this highly magnified picture of this parallelogram and shadow to label the vectors spanning the parallelogram and calculate an expression for $d\vec{A}$ assuming the surface is oriented away from the xz-plane.



$d\vec{A} = (-f_x\vec{i} + \vec{j} - f_z\vec{k})\,dx\,dz$

3. Compute the flux of $\vec{F} = z^2\vec{k}$ through the sphere $x^2 + y^2 + z^2 = 25$ via the definition of a flux integral. Do not apply the Divergence Theorem. (*Hint: Use spherical coordinates.*)

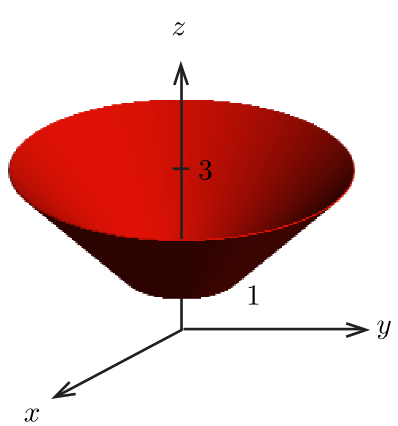
$$\begin{aligned} \int\int_S \vec{F} \cdot d\vec{A} &= \int_0^{2\pi} \int_0^{\pi} (5\cos\phi)^2 \vec{k} \cdot \frac{\vec{r}}{\|\vec{r}\|} 5^2 \sin\phi \, d\phi \, d\theta \\ &= 125 \int_0^{2\pi} \int_0^{\pi} \cos^3\phi \sin\phi \, d\phi \, d\theta \\ &= 125(2\pi) \int_0^{\pi} \cos^3\phi \sin\phi \, d\phi = 0 \end{aligned}$$

Note: This integral can be seen to be zero by geometric reasoning as well.

4. Write an expression for $d\vec{A}$ in cylindrical coordinates for the cylinder of radius 17 centered on the z-axis.

$$d\vec{A} = \frac{x\vec{i} + y\vec{j}}{\|x\vec{i} + y\vec{j}\|} = (\cos\theta\vec{i} + \sin\theta\vec{j})17 \, d\theta \, dz$$

5. Let S be the surface consisting of points (x, y, z) whose distance to the z-axis is equal to their distance to the xy-plane and which have the property that $1 \leq z \leq 3$. Assume that S is oriented away from the z-axis. Find the orientation field $\vec{n}$ for this surface.



The surface is a piece of cone with slope 1 and vertex at the origin. It can be realized as the level set $g = x^2 + y^2 - z^2 = 0$. The normal vector is then parallel to $\nabla g = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$. By dividing by the magnitude of ∇g we obtain a vector field with unit magnitude along the surface S pointing away from the z-axis.

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{x\vec{i} + y\vec{j} - \vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

6. Setup and compute a flux integral which calculates the surface area of the one $z = \frac{h}{R}\sqrt{x^2+y^2}$ with $x^2+y^2 \leq R^2$. Your answer will be a formula in R and h.

This problem correlates to geometry problem 6. To find the surface area we compute the flux of the normal vector field over the surface. Since the surface is the graph of a function of x and y, $dA = \|d\vec{A}\| = \|(-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dx\,dy\|$.

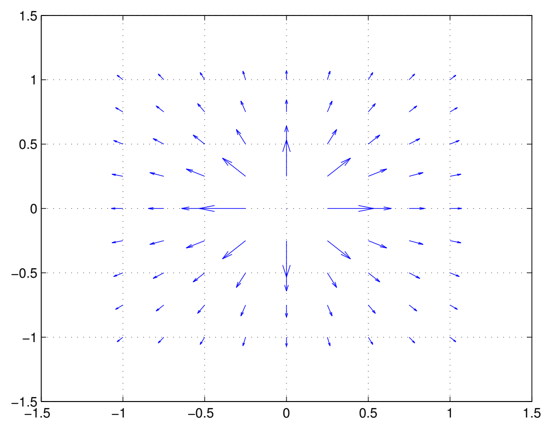
$$\int\int_S \vec{n}\cdot d\vec{A} = \int\int_S \vec{n}\cdot\vec{n}\,dA = \int\int_S dA \text{ and } d\vec{A} = \left(-\frac{h}{R}\frac{x}{\sqrt{x^2+y^2}}\vec{i} - \frac{h}{R}\frac{y}{\sqrt{x^2+y^2}}\vec{j} + \vec{k}\right)dx\,dy$$

$$\|d\vec{A}\|^2 = \left(\frac{h^2}{R^2}\frac{x^2}{x^2+y^2} + \frac{h^2}{R^2}\frac{y^2}{x^2+y^2} + 1\right)(dx\,dy)^2 = \frac{R^2+h^2}{R^2}(dx\,dy)^2$$

Thus, $dA = \frac{1}{R}\sqrt{R^2+h^2}\,dx\,dy$.

$$\begin{aligned}\int\int_S dA &= \int\int_{\text{Disk}} \frac{1}{R}\sqrt{R^2+h^2}\,dx\,dy = \int_0^{2\pi}\int_0^R \frac{1}{R}\sqrt{R^2+h^2}\,r\,dr\,d\theta \\ &= 2\pi\frac{\sqrt{R^2+h^2}}{R}\left[\frac{r^2}{2}\right]_0^R = \pi R\sqrt{R^2+h^2}\end{aligned}$$

7. The picture shows a vector field $\vec{F}$. This vector field has $\operatorname{div}(\vec{F}) = 0$ everywhere the field is defined. Give a geometric explanation of why the divergence of this field is zero at each point.



The field lines are spreading at every point but the magnitudes of the vectors are decreasing along the flow. The decreasing magnitudes along the flow contribute to make the divergence negative, but the spreading of the field lines contributes to make the divergence positive. These contributions balance for this field making the divergence zero.

8. According to Coulomb's Law, the electrostatic field $\vec{E}$ at the point $\vec{r}$ due to a charge with sign $q = \pm 1$ at the origin is given by

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{\|\vec{r}\|^3}.$$

(a) Compute the divergence of $\vec{E}$, where it is defined.

$$\begin{aligned}
\text{div}(\vec{E}) &= q\left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right) \\
&= q\left(3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}\right) \\
&= 0 \text{ except at the origin where it is undefined.}
\end{aligned}$$

(b) Compute the flux of $\vec{E}$ through S_R, the sphere of radius R centered at the origin, oriented outward.

$$\int\int_{S_R} \vec{E}\cdot d\vec{A} = \int\int_{S_R} q\frac{\vec{r}}{\|\vec{r}\|^3}\cdot\frac{\vec{r}}{\|\vec{r}\|}dA_{\text{sphere}} = \frac{q}{R^2}\int\int_{S_R} dA_{\text{sphere}} = 4\pi q.$$

(c) Let S be an arbitrary, outward-oriented sphere in space (not necessarily centered at the origin). Compute the flux of E through S. (*Hint: Your answer will depend on whether the sphere surrounds the origin or not.*)

If does not enclose the origin then we may apply the divergence theorem to the inside W of the sphere to get $\int\int_S \vec{E}\cdot d\vec{r} = \int\int\int_W \text{div}(\vec{E})\,dV = 0$. If S does enclose the origin we cannot directly apply the divergence theorem. However, we can apply the divergence theorem to the region W between sphere of very small radius R, centered at the origin and oriented inward, and the sphere S. In this case we have

$$\int\int_S \vec{E}\cdot d\vec{r} - \int\int_{S_R} \vec{E}\cdot d\vec{r} = \int\int\int_W \text{div}(\vec{E})\,dV = 0$$

From this, we see that the flux through S is the same as the flux through a sphere centered at the origin.

$$\text{Flux of } \vec{E} \text{ through } S = \begin{cases} 0 & \text{if } S \text{ does not enclose the origin} \\ 4\pi q & \text{if } S \text{ does enclose the origin} \\ \text{undefined} & \text{if } S \text{ goes through the origin} \end{cases}$$