Math 223 – Vector Calculus Practice Exam 4

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Directions: Read all questions carefully. Use a pencil and erase all unnecessary marks. Show all of your work in the space provided and display your answer on the line given if requested. You will lose points if you make an approximation and fail to indicate the approximation. Be careful to use proper notation to indicate vector versus scalar quantities as well.

1. Determine whether the following are true or false (T or F). You do not need to give your reasons.

   (a) ___ If f is a function then $\text{div}(\text{grad}f)$ is a scalar valued function.

   (b) ___ If $\vec{F} = x^2 y\cos(z)\vec{i} + \arctan(z)\vec{j} + z^3\cos(z^2-1)\vec{k}$ then the flux through the rectangle in the plane $z = 1$ with $0 \le x \le 2$ and $0 \le y \le 5$ is ± 10 depending on the orientation.

   (c) ___ If $\text{div}\vec{F} = x$ then the flux of $\vec{F}$ through any sphere centered at the origin is zero.

   (d) ___ If S is a sphere of radius one and $\int\int_S \vec{F}\cdot d\vec{A} = 0$ then $\text{div}\vec{F} = 0$ at all points inside S.

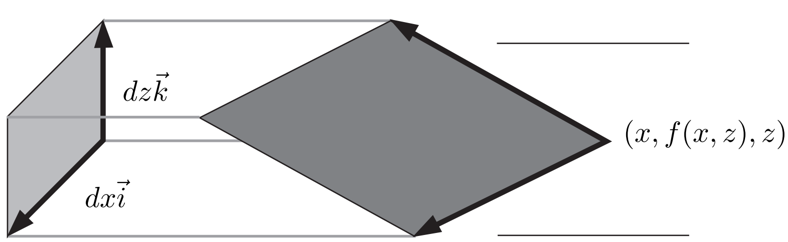
   (e) ___ If $\int\int_S \vec{F}\cdot d\vec{A} > \int\int_S \vec{G}\cdot d\vec{A}$ then $\|\vec{F}\| > \|\vec{G}\|$ at all points on S.

   (f) ___ $\text{grad}(\text{div}\vec{F}) - \text{curl}(\text{curl}(\vec{F}))$ is a vector valued function.

   (g) ___ If the area of a surface is doubled then it is always true that the flux through the surface is doubled.

   (h) ___ If $\vec{F}$ is a vector field on space and f is a scalar valued function then $\text{div}(f\vec{F}) = f\text{div}\vec{F}$.

2. In order to calculate the flux of a vector field through a surface $y = f(x,z)$ which is the graph of a function of x and z one needs to derive the vector field $d\vec{A}$. The vector field $d\vec{A}$ evaluated at a point $(x, f(x,z), z)$ on the surface gives a vector which is perpendicular to the surface at that point with magnitude equal to the area of parallelogram tangent to the surface at $(x, f(x,z), z)$ whose shadow in the xz plane is a rectangle with area $dx\,dz$. Fill in the blanks on this highly magnified picture of this parallelogram and shadow to label the vectors spanning the parallelogram and calculate an expression for $d\vec{A}$ assuming the surface is oriented away from the xz-plane.



$d\vec{A} =$ ______________________

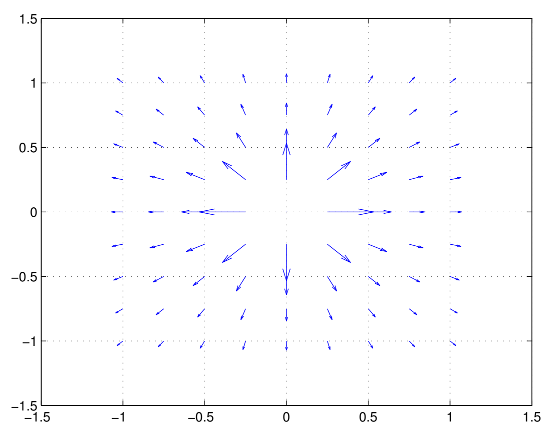
3. Compute the flux of $\vec{F} = z^2\vec{k}$ through the sphere $x^2 + y^2 + z^2 = 25$ via the definition of a flux integral. Do not apply the Divergence Theorem. (*Hint: Use spherical coordinates.*)

4. Write an expression for $d\vec{A}$ in cylindrical coordinates for the cylinder of radius 17 centered on the z-axis.

5. Let S be the surface consisting of points (x, y, z) whose distance to the z-axis is equal to their distance to the xy-plane and which have the property that $1 \leq z \leq 3$. Assume that S is oriented away from the z-axis. Find the orientation field $\vec{n}$ for this surface.

6. Setup and compute a flux integral which calculates the surface area of the one $z = \frac{h}{R}\sqrt{x^2+y^2}$ with $x^2 + y^2 \le R^2$. Your answer will be a formula in R and h.

7. The picture shows a vector field $\vec{F}$. This vector field has $\text{div}(\vec{F}) = 0$ everywhere the field is defined. Give a geometric explanation of why the divergence of this field is zero at each point.



8. According to Coulomb's Law, the electrostatic field $\vec{E}$ at the point $\vec{r}$ due to a charge with sign $q = \pm 1$ at the origin is given by

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{\|\vec{r}\|^3}.$$

(a) Compute the divergence of $\vec{E}$, where it is defined.

(b) Compute the flux of $\vec{E}$ through S_R, the sphere of radius R centered at the origin, oriented outward.

(c) Let S be an arbitrary, outward-oriented sphere in space (not necessarily centered at the origin). Compute the flux of E through S. (*Hint: Your answer will depend on whether the sphere surrounds the origin or not.*)