Math 223 – Vector Calculus Practice Exam 3

Instructor: Arlo Caine Name: Solutions

Directions: Read all questions carefully. Use a pencil and erase all unnecessary marks. Show all of your work in the space provided and display your answer on the line given if requested. You will lose points if you make an approximation and fail to indicate the approximation. Be careful to use proper notation to indicate vector versus scalar quantities as well.

1. Determine whether the following are true or false (T or F). You do not need to give your reasons.

   (a) T The parametric curve $x = t^2$, $y = 2$, $z = t$ is a parabola in space.

   (b) F If one particle moves with position $\vec{r}_1(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$ while another is moves with position $\vec{r}_2(t) = \vec{i} + (10 - t)\vec{k}$ then the two particles will collide at some time.

   (c) F If a particle is moving along a parameterized curve $\vec{r}(t)$ then the acceleration vector at any point is perpendicular to the velocity at that point.

   (d) T If a particle moves with constant velocity then the path must be a line.

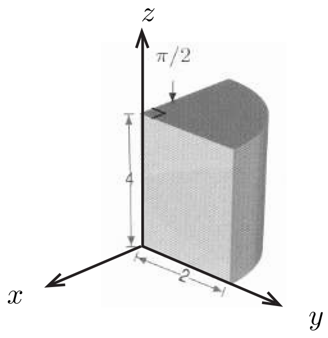
   (e) T There is a region R over which $\int_R f\, dA$ cannot be evaluated by a *single* iterated integral in the order $\int\int f\, dx\, dy$ or $\int\int f\, dy\, dx$.

   (f) F If the vector fields $\vec{F}$ and $\vec{G}$ have $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}$ for a particular path C then $\vec{F} = \vec{G}$.

   (g) T If $\vec{F} = \vec{i}$ is a vector field in 2-space, then $\int_C \vec{F} \cdot d\vec{r} > 0$, where C is the oriented line from $(0, 0)$ to $(1, 0)$.

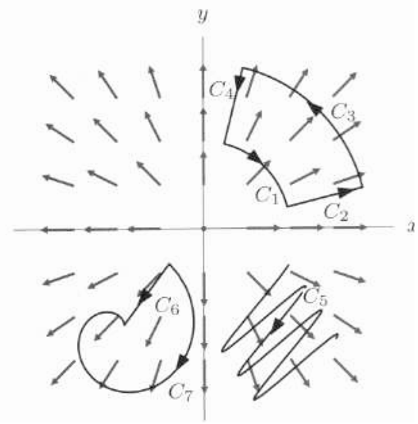
   (h) T If $\vec{F}$ is path-independent and defined on the whole plane, and C is any closed curve then $\int_C \vec{F} \cdot d\vec{r} = 0$.

2. Setup two integrals, one in rectangular coordinates, one in cylindrical coordinates to calculate the volume of the figure shown. Choose one of your integrals to evaluate and do so.



$$\begin{aligned}\text{Volume} &= \int_0^4 \int_0^2 \int_{-\sqrt{4-y^2}}^0 dx\, dy\, dz \\ &= \int_0^4 \int_{\pi/2}^{\pi} \int_0^2 r\, dr\, d\theta\, dz \\ &= 4\pi\end{aligned}$$

3. Let $\vec{F}(x,y)$ be the path-independent vector field shown in the figure. The vector field $\vec{F}$ associates to each point in the plane a unit vector pointing radially outward and thus has a singularity at the origin. The curves C_1, C_2, …, C_7 have the direction shown. Consider the line integrals $\int_{C_i} \vec{F} \cdot d\vec{r}$ for $i = 1, 2, \ldots, 7$.



Without calculating any integrals perform the following tasks and give a one or two sentence explanation of each answer.

(a) List all the line integrals which you expect to be zero.

$\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_3} \vec{F} \cdot d\vec{r}$ should be zero because the field is perpendicular to the curves.

(b) List all the line integrals which you expect to be negative.

$\int_{C_4} \vec{F} \cdot d\vec{r}$ should be negative because the orientation is opposite that of the field.

(c) Arrange the positive line integrals in ascending order.

This is somewhat of a trick question. The integrals

$$\int_{C_6} \vec{F} \cdot d\vec{r} = \int_{C_7} \vec{F} \cdot d\vec{r} = \int_{C_5} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

are all probably all equal. This field is actually the gradient of the function $\sqrt{x^2 + y^2}$ which measures the distance to the origin. The difference in the distances to the origin of the endpoints of path C_6 is the same as that for C_7, C_5, C_2.

4. A particle, following a straight line constant speed path through space, passes "downward" through a permeable barrier which is in the shape of the graph of $z = 10 - x^2 - y^2$, where x, y and z are measured in meters. It passes through the point $(2, 2, 2)$ in a direction perpendicular to the barrier with speed $\sqrt{33}$ m/s at time $t = 0$. Does it strike the barrier again? If your answer is "no", explain why. If your answer is "yes" find the time of intersection.

The strategy is the following. Find parametric equations for the motion of the particle. Substitute those equations into the equation for the barrier. If the resulting equation in t has two solutions then the particle strikes the barrier again, otherwise it does not. The key is to set up the correct equations.

First, let $g = 10 - x^2 - y^2 - z$, then the barrier is given by the level set $g = 0$. In vector form, we can write the equations for the path as $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ where $\vec{r}_0 = 2\vec{i} + 2\vec{j} + 2\vec{k}$, so that $t = 0$ corresponds to the time of entry, and $\vec{v}$ is the velocity vector.

We must have that $\vec{v}$ is parallel to $\nabla g(2, 2, 2)$, has a negative $\vec{k}$-component and $\|\vec{v}\| = \sqrt{33}$. It turns out that $\nabla g(2, 2, 2) = -4\vec{i} - 4\vec{j} - \vec{k}$ works for $\vec{v}$. So, $\vec{r} = (2 - 4t)\vec{i} + (2 - 4t)\vec{j} + (2 - t)\vec{k}$.

$$\begin{aligned} z &= 10 - x^2 - y^2 \\ 2 - t &= 10 - (2 - 4t)^2 - (2 - 4t)^2 \\ 0 &= 33t - 32t^2 \end{aligned}$$

The particle does intersect the barrier again, at $t = 32/33$ seconds.

5. The table shows some of the values of a function $f(x, y)$.

$x\backslash y$	3	6	9	12
-4	10	12	15	19
-2	8	10	12	15
0	6	7	9	11
2	4	5	6	7

(a) Find parametric equations for the upper semi-circle of radius 3 traversed from $(2, 6)$ to $(-4, 6)$.

The position vector of the center of the semi-circle is $\vec{c} = -\vec{i} + 6\vec{j}$. If the path is traversed with constant speed as t ranges from 0 to 2π, then the displacement vector from the center to the point on the circle at time t is given by $\vec{d}(t) = 3\cos(t)\vec{i} + 3\sin(t)\vec{j}$. Thus, one set of parametric equations for the path is $\vec{r}(t) = \vec{c} + \vec{d}(t) = (3\cos(t) - 1)\vec{i} + (3\sin(t) + 6)\vec{j}$.

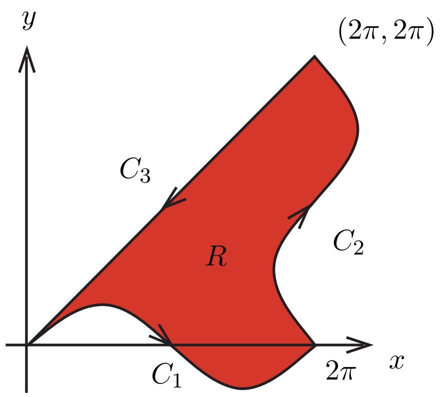
(b) Let C denote the curve in part (a). Estimate $\displaystyle\int_C \nabla f \cdot d\vec{r}$.

By the Fundamental Theorem of Line Integrals $\int_C \nabla f \cdot d\vec{r} = f(-4, 6) - f(2, 6) = 12 - 5 = 7$.

6. Sketch the region R in the plane whose boundary is the oriented curve $C = C_1 + C_2 + C_3$ where

$$C_1 : \begin{cases} x &= t \\ y &= \sin(t) \end{cases} \quad \text{and } C_2 : \begin{cases} x &= 2\pi - \sin(t) \\ y &= t \end{cases} \quad \text{and } C_3 : \begin{cases} x &= 2\pi - t \\ y &= 2\pi - t \end{cases}$$

where $0 \le t \le 2\pi$ in each parameterization. Use Green's Theorem to compute the area of the region R.



To compute the area we apply Green's Theorem with $\vec{F} = x\vec{j}$. For this field the scalar curl is $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$ and

$$\text{Area(R)} = \int\int_R 1\,dA = \int\int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$\begin{aligned}
\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} t\vec{j}\cdot(\vec{i} + \cos(t)\vec{j})dt = \int_0^{2\pi} t\cos(t)\,dt = t\sin(t)|_0^{2\pi} + \cos(t)|_0^{2\pi} = 0 \\
\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} (2\pi - \sin(t))\vec{j}\cdot(-\cos(t)\vec{i} + \vec{j})\,dt = \int_0^{2\pi} (2\pi - \sin(t))\,dt = (2\pi)^2 + [\cos(t)]_0^{2\pi} = 4\pi^2 \\
\int_{C_3} \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} (2\pi - t)\vec{j}\cdot(-\vec{i} - \vec{j})\,dt = \int_0^{2\pi} (t - 2\pi)dt = \left[\frac{1}{2}t^2 - 2\pi t\right]_0^{2\pi} = -2\pi^2
\end{aligned}$$

Thus the area is $4\pi^2 - 2\pi^2 = 2\pi^2$.