

Directions: Read all questions carefully. Use a pencil and erase all unnecessary marks. Show all of your work in the space provided and display your answer on the line given if requested. You will lose points if you make an approximation and fail to indicate the approximation. Be careful to use proper notation to indicate vector versus scalar quantities as well.

1. Determine whether the following are true or false (T or F). You do not need to give your reasons.

- (a) **T** Level sets of a function $f(x, y, z)$ of different levels never intersect.
- (b) **T** If $f(x, y) = k$ for all points (x, y) in a region R then $\int \int_R f(x, y) dA = k \cdot \text{Area}(R)$.
- (c) **F** If f and g are two functions continuous on a region R then $\int \int_R fg dA = \int \int_R f dA \int \int_R g dA$.
- (d) **F** If W_1 and W_2 are solid regions of space with $\text{volume}(W_1) > \text{volume}(W_2)$ then

$$\int \int \int_{W_1} f dV > \int \int \int_{W_2} f dV.$$

- (e) **T** The function $z(u, v) = u \cos v$ satisfies the equation

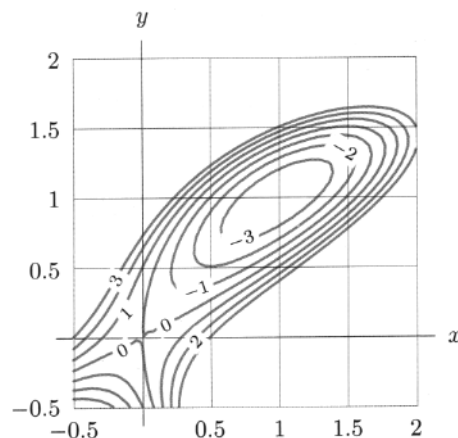
$$\cos v \frac{\partial z}{\partial u} - \frac{\sin v}{u} \frac{\partial z}{\partial v} = 1.$$

- (f) **F** The local linearization of $f(x, y) = x^2 + y^2$ at the point $(1, 1)$ gives an overestimate of the value of $f(x, y)$ at the point $(1.1, 1.2)$.
- (g) **T** If you know each of the directional derivatives of f at (a, b, c) in the directions of 3 perpendicular unit vectors in space then you can determine the gradient of f at the point (a, b, c) .

2. The figure shows the contours for a function $f(x, y)$.

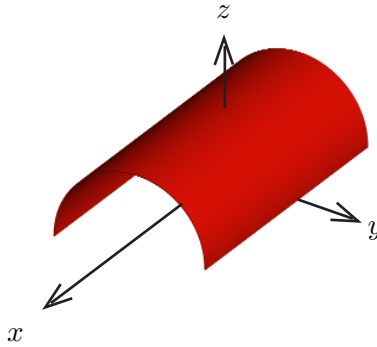
Using this contour diagram indicate with a check mark whether the given derivative is positive, negative, or zero. Let $\vec{u} = (1/\sqrt{2})(\vec{i} + \vec{j})$.

	positive	negative	zero
$f_x(0.5, 0)$	X		
$f_y(1.5, 1)$		X	
$f_{xx}(0.5, 0.5)$	X		
$f_{yy}(0.5, 0)$	X		
$f_{xy}(0.5, 1)$		X	
$f_{\vec{u}}(1, 0.5)$		X	



3. Let $g(x, y, z) = z - f(x, y)$ where $f(x, y) = \sqrt{1 - y^2}$.

(a) Sketch a graph of the level surface $g = 0$.



(b) Explain why the vectors $\vec{v} = \vec{i} + f_x(3, -1/\sqrt{2})\vec{k}$ and $\vec{w} = \vec{j} + f_y(3, -1/\sqrt{2})\vec{k}$ would be tangent to the level surface if their tails were attached at the point $P = (3, -1/\sqrt{2}, 1/\sqrt{2})$.

Here are two possible explanations: 1. Computation shows that $\nabla g(P) \cdot \vec{v} = 0$ and $\nabla g(P) \cdot \vec{w} = 0$. Since the gradient of g is perpendicular to the level set $g = 0$, and \vec{v} and \vec{w} are perpendicular to the gradient of g at P , it must be the case that \vec{v} and \vec{w} would be tangent to the set $g = 0$. 2. When its tail is attached at the point P , the vector \vec{v} is a line segment in the plane $y = -1/\sqrt{2}$ with slope equal to $f_x(P)$, which is the slope of the tangent line to the surface in the x -direction. Similarly, we can argue that \vec{w} is tangent to the surface.

(c) Determine the component of $\vec{v} \times \vec{w}$ perpendicular to $\text{grad } g(3, -1/\sqrt{2}, 1/\sqrt{2})$.

Since \vec{v} and \vec{w} are tangent to the surface at P , $\vec{v} \times \vec{w}$ is perpendicular to the surface at the point P and thus parallel to $\nabla g(P)$. So the component of $\vec{v} \times \vec{w}$ perpendicular to ∇g is $\vec{0}$.

(d) Calculate the directional derivative $g_{\vec{u}}(3, -1/\sqrt{2}, 1/\sqrt{2})$ where $\vec{u} = \frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$.

$$g_{\vec{u}}(P) = \nabla g(P) \cdot \vec{u} = (-\vec{j} + \vec{k}) \cdot \left(\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}\right) = -\frac{2}{\sqrt{5}}$$

(e) Find the equation of the plane tangent to the level surface $g = 0$ at the point $(3, -1/\sqrt{2}, 1/\sqrt{2})$.

Using $\nabla g(P)$ as a normal vector, and the fact that the plane passes through P , we find that one possible equation for the plane is $-(y + 1/\sqrt{2}) + (z - 1/\sqrt{2}) = 0$.

4. The level surfaces $f = -1$ and $g = 0$ of the functions $f(x, y, z) = z^2 - x^2 - y^2$ and $g(x, y, z) = y^2 + z^2 - 1$ intersect along a curve that passes through the point $P = (1, 1/\sqrt{2}, 1/\sqrt{2})$. Use the gradients of f and g to find a vector which would be tangent to this curve if its tail was attached at the point P .

The tangent line to the curve is the intersection of the tangent planes to $f = -1$ and $g = 0$ at P . The vectors $\nabla f(P)$ and $\nabla g(P)$ are normal to these planes, thus $\nabla f(P) \times \nabla g(P)$ is parallel to their intersection.

$$\nabla f(P) = -2\vec{i} - \sqrt{2}\vec{j} - \sqrt{2}\vec{k} \text{ and } \nabla g(P) = \sqrt{2}\vec{j} + \sqrt{2}\vec{k}$$

$$\nabla f(P) \times \nabla g(P) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{vmatrix} = -4\vec{i} + 2\sqrt{2}\vec{j} - 2\sqrt{2}\vec{k}$$

5. The voltage V (in volts) across a circuit is given by Ohm's Law: $V = IR$, where I is the current (in amps) flowing through the circuit and R is the resistance (in ohms). If we place two circuits, with resistance R_1 and R_2 in parallel, then their combined resistance, R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Suppose the current is 2 amps and increasing at a rate of 10^{-2} amps/sec and R_1 is 3 ohms and decreasing at a 0.5 ohms/sec while R_2 is 5 ohms and decreasing at 0.1 ohms/sec. Calculate the rate at which the voltage is changing.

$$\begin{aligned} \frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = \frac{R_1 R_2}{R_1 + R_2} & R &= \frac{15}{8} \\ \frac{\partial R}{\partial R_1} &= \frac{R_2^2}{(R_1 + R_2)^2} = \frac{25}{64} & \frac{\partial V}{\partial I} &= R = \frac{15}{8} \\ \frac{\partial R}{\partial R_2} &= \frac{R_1^2}{(R_1 + R_2)^2} = \frac{9}{64} & \frac{\partial V}{\partial R} &= I = 2 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{\partial R}{\partial R_1} \frac{dR_1}{dt} + \frac{\partial V}{\partial R} \frac{\partial R}{\partial R_2} \frac{dR_2}{dt} \\ &= \frac{15}{8} \cdot 10^{-2} + 2 \cdot \frac{25}{64} \cdot (-0.5) + 2 \cdot \frac{9}{64} \cdot (-0.1) \approx -0.4 \frac{\text{volts}}{\text{sec}} \end{aligned}$$

6. The hull of a boat has width $w(x, y)$ feet at a point x feet from the front and y feet below the water line. Values of w are in given in the table.

$y \backslash x$	0	10	20	30	40	50	60
0	2	8	13	16	17	16	10
2	1	4	8	10	11	10	8
4	0	3	4	6	7	6	4
6	0	1	2	3	4	3	2
8	0	0	1	1	1	1	1

- (a) Set up a definite integral that calculates the volume of the hull below the water line. Explain each step.

A slice of the hull of length dx and depth dy , x feet from the front of the ship and y feet below the water line has volume $w(x, y) dx dy$ cubic feet. Thus, the total volume of the hull below the water line is given by the double integral $\int_0^8 \int_0^{60} w(x, y) dx dy$ cubic feet.

- (b) Estimate your integral using the table.

The natural grid for the table has $\Delta x = 10$ and $\Delta y = 2$.

$$\begin{aligned} \text{Lower Sum} &= (1 + 4 + 8 + 10 + 10 + 8 + 0 + 3 + 4 + 6 + 6 + 4 + 0 + 1 + 2 + 3 + 3 \\ &\quad + 2 + 0 + 0 + 1 + 1 + 1 + 1)20 = 1580 \end{aligned}$$

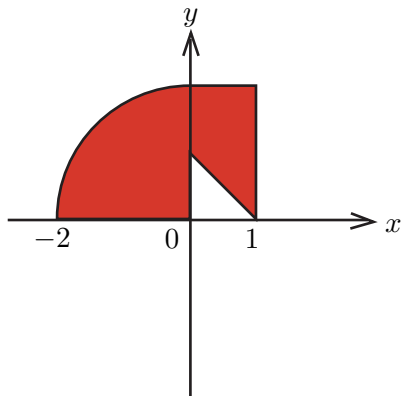
$$\begin{aligned} \text{Upper Sum} &= (8 + 13 + 16 + 17 + 17 + 16 + 4 + 8 + 10 + 11 + 11 + 10 + 3 + 4 + 6 + 7 + 7 \\ &\quad + 6 + 1 + 2 + 3 + 4 + 4 + 3)20 = 3820 \end{aligned}$$

$$\int_0^8 \int_0^{60} w(x, y) dx dy \approx \frac{1}{2}(\text{Lower} + \text{Upper}) = \frac{1}{2}(1580 + 3820) = 2700 \text{ cubic feet.}$$

7. Sketch the region of integration and evaluate the iterated integral

$$I = \int_{-2}^0 \int_0^{\sqrt{4-x^2}} x dy dx + \int_0^1 \int_{1-x}^2 x dy dx$$

Show each step of the calculation.



$$\begin{aligned} I &= \int_{\pi/2}^{\pi} \int_0^2 r \cos \theta r dr d\theta + \int_0^1 x(2 - (1 - x)) dx \\ &= \left(\int_{\pi/2}^{\pi} \cos \theta d\theta \right) \left(\int_0^2 r^2 dr \right) + \int_0^1 (x^2 - x) dx \\ &= [\sin \theta]_0^{\pi/2} \left[\frac{r^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \\ &= -1 \cdot \frac{8}{3} + \frac{1}{3} - \frac{1}{2} = -\frac{15}{6}. \end{aligned}$$