Math 223 – Vector Calculus　　　　Practice Exam 1

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Directions: Read all questions carefully. Use a pencil and erase all unnecessary marks. Show all of your work in the space provided and display your answer on the line given if requested. You will lose points if you make an approximation and fail to indicate the approximation. Be careful to use proper notation to indicate vector versus scalar quantities as well.

1. Determine whether the following are true or false (T or F). You do not need to give your reasons.

   (a) T Two contours of $f(x,y)$ with different heights never intersect.

   (b) T There is only one point in the yz-plane that is a distance 3 from the point $(3,0,0)$.

   (c) T The graphs of $f(x,y) = \sin(xy)$ and $f(x,y) = \sin(xy) + 2$ do not intersect.

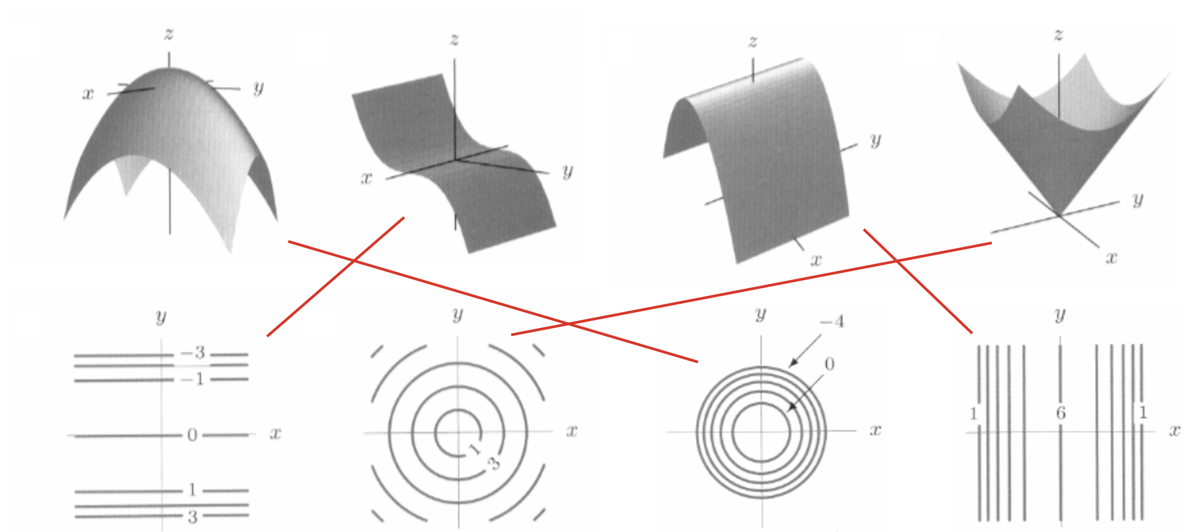
   (d) T There is no linear function whose graph is a plane parallel to the xz-plane.

   (e) T The vectors $3\vec{i} - 2\vec{j} - 4\vec{k}$ and $-39\vec{i} + 26\vec{j} + 52\vec{k}$ are parallel.

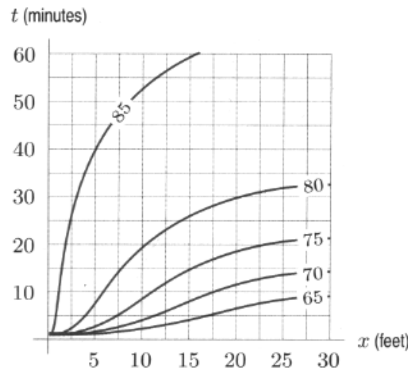
   (f) T The quantity $((\vec{b} \times \vec{a}) \cdot \vec{c})\vec{a}$ is a vector.

   (g) F If $\vec{v}$ and $\vec{w}$ are any two vectors then $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.

2. Match the graph with the contour diagram.



3. You are in a room 30 feet long with a heater at one end. In the morning the room is 65°F. You turn on the heater, which quickly warms the room up to 85°F. Let $H(x,t)$ be the temperature x feet from the heater, t minutes after the heater is turned on. The figure shows the contour diagram for H.



(a) How warm is it 10 feet from the heater 5 minutes after it was turned on?

Approximately 72° F

(b) How far from the heater was the temperature changing at a approximately a rate of 1°F/min, 5 minutes after the heater was turned on.

We are looking for a point where the contours are vertically separated by 5 minutes since the contours are in steps of 5° F. Along the line $t = 5$ minutes, at 17.5 feet we find this time rate of change.

4. Sketch a picture AND describe in words the following sets of points.

(a) The set of points in space whose distance from $(1, -2, 3)$ is 5.

Sphere of radius 5 centered at $(1, -2, 3)$.

(b) The set of points in space whose distance from the line $y = x$ in the xy-plane is the same as its distance to the plane $x + y = 0$.

This is a double cone centered on the line $y = x$ in the plane $z = 0$ with vertex at the origin and slope 1.

(c) The set of points in space which lie in the plane $z = 4$ whose distance from the z-axis is 3.

This is a circle of radius 3 centered at $(0, 0, 4)$ in the plane $z = 4$.

(d) The set of points in the plane for which the product of the distances from the point to the points $(1, 0)$ and $(-1, 0)$ is one.

This is Bernoulli's Lemniscate. The problem correlates to the calculus review problems. Start with the equation for the product of the distances to be one, $\sqrt{(x+1)^2 + y^2}\sqrt{(x-1)^2 + y^2} = 1$, then do some algebra to find the equation from the calculus review problems,

$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0.$$

5. Let $P_0 = (1, 0, 0)$, $\vec{n}_1 = \vec{i} + \vec{j} + \vec{k}$, and $\vec{n}_2 = \vec{i} - 2\vec{j} + \vec{k}$. Let $P = (x, y, z)$ be a variable point. Write $\vec{P} = x\vec{i} + y\vec{j} + z\vec{k}$ for the position vector of that point and $\overrightarrow{P_0P}$ for the displacement vector from P_0 to P. The following equations describe a line or a plane or a point in 3-space. Explain which is which.

(a) $\overrightarrow{P_0P} \times \vec{n}_1 = \vec{0}$

(b) $\overrightarrow{P_0P} \cdot \vec{n}_1 = 0$

(c) $\vec{P} \cdot \vec{n}_1 = 3$

(d) $(\vec{P} \times \vec{i}) \cdot \vec{j} = 0$

(e) $\begin{cases} \overrightarrow{P_0P} \cdot \vec{n}_1 &= 0 \\ \overrightarrow{P_0P} \cdot \vec{n}_2 &= 0 \end{cases}$

(f) $\begin{cases} \overrightarrow{P_0P} \cdot \vec{n}_1 &= 0 \\ \overrightarrow{P_0P} \cdot \vec{n}_1 &= 0 \end{cases}$

a) is line parallel to $\vec{n}$ through the point P_0. b) is a plane normal to $\vec{n}$ through P_0. c) is a plane with normal $\vec{n}$ through $(1, 1, 1)$. d) is a plane through the origin with normal vector $\vec{k}$ since $(\vec{P} \times \vec{i}) \cdot \vec{j} = -(\vec{j} \times \vec{i}) \cdot \vec{P} = \vec{k} \cdot \vec{P}$. e) is the intersection of the planes passing through P_0 with normal vectors $\vec{n}_1$ and $\vec{n}_2$. f) is the same as a).

6. Suppose that you are standing straight upright with your feet at the point $(-1, 1, 1)$. Your spine is aligned in the $\vec{j}$ direction and you are facing the $-\vec{i}$ direction. From this perspective is the point $(1, 2, 3)$ in front of or behind you? to your left or to your right? above or below you? Explain.

Behind, above, and to your left.

7. Find a vector parallel to the intersection of the planes described by $x + y - z = 0$ and $2x - y + z = 0$. A vector is parallel to the first plane if it is perpendicular to $\vec{n}_1 = \vec{i} + \vec{j} - \vec{k}$. A vector is parallel to the second plane if it is perpendicular to $\vec{n}_2 = 2\vec{i} - \vec{j} + \vec{k}$. The cross product, $\vec{n}_1 \times \vec{n}_2$ is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$ and therefore is parallel to the intersection.

$$\vec{n}_1 \times \vec{n}_2 = -3(\vec{j} + \vec{k})$$

Any non-zero multiple of this would also work.

8. The table below gives a partially completed table of a **linear** function $f(x, y)$.

x \ y	1	2	3	4
1	2	5	8	11
2	0	3	6	9
3	-2	1	4	7
4	-4	-1	2	5

(a) Complete the table using the fact that f is linear.

(b) Find c, m, and n such that the data is modelled by $f(x, y) = c + mx + ny$. The slope in the x-direction is $m = -2$. The slope in the y-direction is $n = 3$. Using the formula for a linear function, $f(x, y) = c + mx + ny$ we can fill in for m and n and use the data in the table to determine c.

$$f(x, y) = 1 - 2x + 3y$$

9. A pilot is flying a small airplane with an airspeed of 100 nm/hr. The vertical speed indicator reads that she is climbing at a rate of 500 ft/min. She is flying a heading chosen so that with the wind blowing southeast at 25 nm/hr she will follow a course due east. To do the following problem you may find it useful to know that there are 6080 feet in a nautical mile (nm).

(a) Compute the direction of her heading and express it in terms of degrees north/south of east.

First, we need to compute all figures into a common system of units. Converting ft/min to nm/hr, we see that the vertical component of velocity is 4.9 nm/hr. With the direction and magnitude, we can resolve the wind vector into components $\vec{W} \approx 17.6\vec{i} - 17.6\vec{j}$. Since $\vec{H} + \vec{W}$ must have no $\vec{j}$-component, we must have $\vec{H} \approx a\vec{i} + 17.6\vec{j} + 4.9\vec{k}$. With $\|\vec{H}\| = 100$, it must be the case that $a \approx 98.3$. The direction should be $\theta \approx \arctan(17.6/98.3) \approx 10°$ north of east.

(b) Compute her ground speed.

Her groundspeed is the magnitude of the $\vec{i}$,$\vec{j}$ part of $\vec{C} = \vec{H} + \vec{W} \approx 115.9\vec{i} + 4.9\vec{k}$. So her groundspeed is approximately 115.9 nm/hr.