Math 223 - Section 010
Test 3a with Solutions

You have 50 minutes to complete the test. ***Show all of your work clearly.*** Unless otherwise stated, each answer must be explained and justified.

1. For each of the following statements, circle TRUE or FALSE. You do not need to explain your answers for this problem. Assume that all functions are smooth.

   a. TRUE or FALSE: The parametric equation $\vec{r}(t) = (t^3+1)\vec{i} - 2(t^3+1)\vec{j} - 5\vec{k}$ describes a cubic curve in the plane $z = -5$.

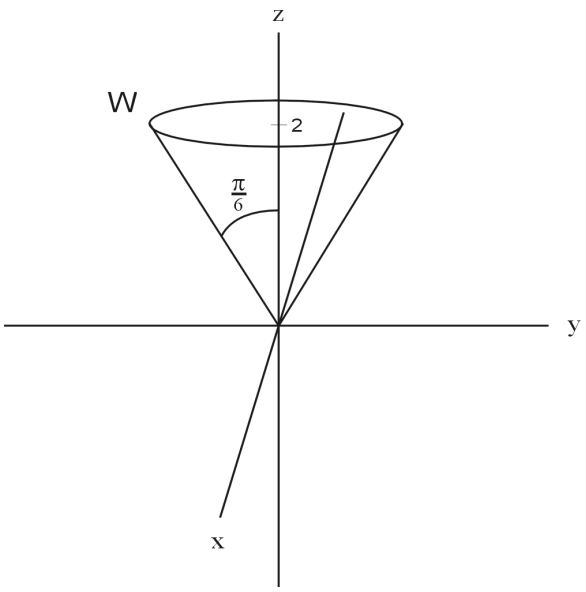
   b. TRUE or FALSE: Let $\vec{F} = -y\vec{i} + x\vec{j} + z\vec{k}$ and let C be the curve given by $\vec{r}(t) = \cos t\vec{i} + \sin(t)\vec{j} + e^t\vec{k}$ with $0 \leq t \leq 4\pi$. Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} > 0$.

   c. TRUE or FALSE: Let $f(x, y, z)$ be a smooth function, and suppose the level set $f = -1$ is a smooth curve, C. Then $\displaystyle\int_C \nabla f \cdot d\vec{r} = 0$.

   d. TRUE or FALSE: If the parametric curve $\vec{r}(t)$ describes the motion of a particle which is traveling at constant speed, then at any point on the curve, the velocity vector and the acceleration vector of the particle are perpendicular.

   e. TRUE or FALSE: If a particle moves along a straight line and never stops, then whenever the particle's acceleration vector is non-zero, it points in the same direction as the velocity vector.

2. Let W be the flat-topped cone of height 2 centered on the z-axis with its point at the origin and vertex angle $\frac{\pi}{6}$. (See the figure below.)



Find (but do not compute) iterated integrals that compute the volume of W in

a. Cartesion coordinates with order of integration $dz\,dy\,dx$,
Answer:

$$\int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} \int_{-\sqrt{\frac{4}{3}-x^2}}^{\sqrt{\frac{4}{3}-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{2} 1\,dz\,dy\,dx$$

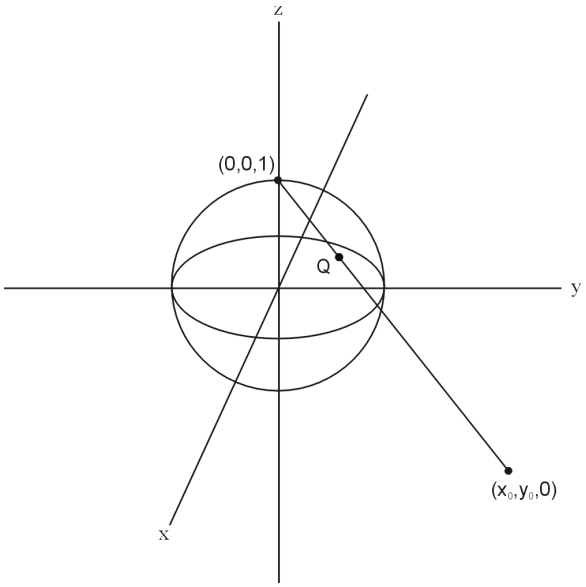
b. cylindrical coordinates with order of integration $dz\,dr\,d\theta$, and
Answer:

$$\int_0^{2\pi}\int_0^{\frac{2}{\sqrt{3}}}\int_{\sqrt{3}r}^{2} r\,dz\,dr\,d\theta.$$

c. spherical coordinates with order of integration $d\rho\,d\phi\,d\theta$.
Answer:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\frac{2}{\cos\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

3. Let S be the unit sphere centered at the origin in space. Given any point $(x_0, y_0, 0)$ in the xy-plane, the straight line from $(x_0, y_0, 0)$ to the point $(0, 0, 1)$ intersects the sphere at exactly one other point, Q. (See the figure below.) Find the coordinates of Q in terms of x_0 and y_0.



Answer:

$$Q = \left(\frac{2x_0}{x_0^2 + y_0^2 + 1}, \frac{2y_0}{x_0^2 + y_0^2 + 1}, \frac{x_0^2 + y_0^2 - 1}{x_0^2 + y_0^2 + 1} \right).$$

4. Let $\vec{F}(x,y) = (x-y)\vec{i}+(y-x)\vec{j}$. Calculate the work done by the force field $\vec{F}$ on a particle which travels from the point $(-1,-1)$ to the point $(2,8)$ along the graph of $y = x^3$.
Answer: Work done=18 (whatever units).

5. For each of the following, determine whether the given vector field is conservative. Explain your answers.

   a. $\vec{H}(x,y) = y\vec{i} - x\vec{j}$
   Answer: $\vec{H}$ is not conservative because curl$(\vec{H}) \neq 0$.

   b. $\vec{F}(x,y) = \frac{x\vec{i}+y\vec{j}}{x^2+y^2}$
   Answer: $\vec{H}$ is conservative because it is a gradient field.

6. Consider the vector field $\vec{F} = x\sin(x)\vec{i} + (xy + \ln(1+y^2))\vec{j}$ in the plane. Let C_1 be the curve given by $\vec{r}_1(t) = (t^2+1)\vec{i} + t\vec{j}$, $-1 \leq t \leq 1$, and let C_2 be the curve given by $\vec{r}_2(t) = 2\vec{i} + (1-t)\vec{j}$, $0 \leq t \leq 2$. Let $C = C_1 + C_2$. Use Green's Theorem to calculate the circulation of $\vec{F}$ around C.

Answer:

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$