Math 215 - Section 004

Test 3

You have 75 minutes to complete the test. ***Show all of your work clearly.*** *Unless otherwise specified, each answer must be explained and justified. Do not use decimal approximations.*

1. Let S be the set of vectors $[x, y, z]$ in $\mathbb{R}^3$ such that $xy + z = 0$. Is S a subspace of $\mathbb{R}^3$? If so, prove it; if not, give a counterexample to show that S is not a subspace of $\mathbb{R}^3$.

   Solution: The set S is not closed under addition and it is not closed under scaling. For example, $[1, 1, -1]$ is in S, but $2[1, 1, -1]$ is not. Therefore, S is not a subspace of $\mathbb{R}^3$.

2. Let T denote the linear transformation of $\mathbb{R}^2$ which corresponds to reflection over the line $y = x$ followed by projection onto the x-axis. Find $[T]$. (Recall that $[T]$ is the standard matrix representation of T.)

Solution: It is easy to see that $T(\vec{e_1}) = [0, 0]$ and $T(\vec{e_2}) = [1, 0]$. Therefore, $[T] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

3. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) := \begin{bmatrix} x + y + 2z \\ 2x + y + 4z \\ -x - 2z \end{bmatrix}.$$

Determine whether T is invertible.

Solution: The standard matrix of T is $[T] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$. Since $\det[T] = 0$, $[T]$ is not invertible, which implies that T is not invertible.

4. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}$. Compute the eigenvalues of A.

Solution: Setting $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{vmatrix} = (\lambda - 1)((1-\lambda)(-2-\lambda)+2) = \lambda(\lambda - 1)(\lambda + 1) = 0$, we find that the eigenvalues of A are 0, 1, and -1.

5. Let A be an $n \times n$ matrix. Which **one** of the following statements is not equivalent to the rest? (No explanation is required.)

   a. dim(row(A))=dim(col(A)).
   b. The rows of A are linearly independent.
   c. The nullity of A is 0.
   d. The associated linear map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible.
   e. The span of the columns of A is $\mathbb{R}^n$.
   f. $\det(A) \neq 0$.
   g. 0 is not an eigenvalue of A.

Solution: Statement (a) is true for every matrix, whereas the other statements are equivalent to the statement that A is invertible. Therefore, (a) is not equivalent to the rest.

6. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$.

a. Show that $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A and find the associated eigenvalue, λ.

Solution: Since $A\vec{v} = [0, 0, 0] = 0 \cdot \vec{v}$, $\vec{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

b. Find E_λ, where λ is the eigenvalue associated to the eigenvector $\vec{v}$.

Solution: $E_0 = (A - 0 \cdot I) = (A) = \text{span}[1, 0, 1], [0, 1, 0]$.

7. Suppose A is a 12×12 matrix with determinant 4. What is $\det(-A)$? Explain your answer.

   Solution: If $A = [\vec{a}_1|\dots|\vec{a}_{12}]$, then $-A = [-\vec{a}_1|\dots| - \vec{a}_{12}]$. Since the determinant is multilinear with respect to the columns of A, we can factor a -1 out of each column. Thus, $\det(-A) = (-1)^{12}\det(A) = \det(A) = 4$.

8. Let A be an invertible, $n \times n$ matrix, and suppose λ is an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

   Solution: If λ is an eigenvalue of A, then there exists a **nonzero** vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Multiplying this equation by A^{-1} on the left gives $\vec{v} = A^{-1}\lambda\vec{v}$. Since A is invertible, $\lambda \neq 0$ (by the F.T.I.M), so we can divide this equation by λ, which yields $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$. This implies (by definition) that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.