Math 215 - Section 004
Review Questions for Test 3

This review is not comprehensive.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear.

   a. Explain why the image of T cannot be all of $\mathbb{R}^3$. (In fact, this is **not** the case if T is not assumed to be linear. It is possible to construct so-called "space-filling curves." These are maps from $\mathbb{R}$ onto $\mathbb{R}^n$.)

   b. Recall that a function $f : A \to B$ is called *one-to-one* if for any $x, y \in A$ with $x \neq y$, $f(x) \neq f(y)$. Give a condition on the matrix of T that determines whether T is one-to-one. (Hint: Number 54 on page 223 of the text may be helpful.)

2. Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that rotates a point (or a vector based at $\vec{0}$) by an θ degrees about the y-axis. The direction of rotation is counterclockwise if you are looking down the positive y-axis toward the xz-plane. Find $[R_\theta]$. You may assume that R_θ is linear.

3. (Challenge) Find the standard matrix for the (linear) transformation of $\mathbb{R}^3$ that rotates points by θ degrees about the line $x = z$, $y = 0$. (Hint: Choose a basis for $\mathbb{R}^3$ in which the matrix for this transformation is easy to write down. Then use Theorem 6.12 to convert to the standard basis.)

4. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map that projects points onto the plane $x = -2y - 3z$ along lines perpendicular to the plane. (This is the usual projection, as discussed in chapter 1.) Compute $[P]$.

5. Suppose T and S are two maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that $T([1, 2, -1]) = S([1, 2, -1])$, $T([2, 1, -3]) = S([3, 1, -3])$, and $T([-3, 1, 7]) = S([-3, 1, 7])$. Is it true that T and S must be the same? If so, explain why. If not, give a counterexample.

6. Let A be an $n \times n$ matrix with determinant 2. Let B be the matrix obtained by reversing the order of each column of A. Find $\det B$. (Hint: Your answer should depend on n, but don't spend a lot of time writing down a general formula. Just figure out the first 10 cases or so and find a pattern.)

7. A map $f : A \to B$ is called *onto* (or *surjective*) if the image of f is all of B. That is, f is onto if for every $b \in B$, there is an $a \in A$ (not necessarily unique) such that $f(a) = b$.

   a. Explain geometrically why a linear map $T : \mathbb{R} \to \mathbb{R}$ is one-to-one if and only if it is onto.

   b. Prove that a **linear** map $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto.

   c. Give a counterexample to show that this statement is not true for nonlinear maps. (Hint: Choose $n = 1$ to keep things simple.)

8. Give an example of a 3x3 matrix with eigenvalues $\lambda = 1$ and $\lambda = 2$ (and no other eigenvalues).

9. Let S be the set of points $p = (x, y, z)$ in $\mathbb{R}^3$ such that the distance between p and the z-axis is equal to the distance between p and the xy-plane. Is S a subspace of R^3? If so, prove it; if not, give a counterexample to show that S is not a subspace of $\mathbb{R}^3$. (Hint: Draw a picture of S.)

10. Define a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) := \begin{bmatrix} 7x - y + 2z \\ 2x + 3y + 4z \\ -x + y - 2z \end{bmatrix}.$$

    a. Is T linear? Prove or give a counterexample.
    b. Determine whether T is invertible.

11. Let A is an invertible, $n \times n$ matrix, and suppose λ is an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.