Math 215 - Section 004
Test 2 - Solutions

You have 75 minutes to complete the test. ***Show all of your work clearly.*** *Unless otherwise specified, each answer must be explained and justified. Do not use decimal approximations.*

1. Determine whether the vectors $\vec{v}_1 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 & 1 & 8 \end{bmatrix}$ are linearly dependent or independent. If the vectors are linearly dependent, find an explicit linear dependency.

   Solution:
   The vectors $\vec{v}_1$, $\vec{v}_1$, and $\vec{v}_1$ are linearly independent if and only if the system $A\vec{x} = \vec{0}$ has a nontrivial solution, where
   $$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 8 \end{bmatrix}.$$
   The RREF of $[A|\vec{0}]$ is
   $$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
   which corresponds to the system $c_1 + 3c_3 = 0$, $c_2 - 2c_3 = 0$. Setting $c_3 = t$, the solution set for this system (and hence the original system) is $\{[-3t, 2t, t] \mid t \in \mathbb{R}\}$. Choosing, $t = 1$, we have the explicit linear dependency $-3\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}$. Thus, the vectors are linearly dependent.

2. Compute the matrix product $ABCD$,

   where $A = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$, $B = \begin{bmatrix} 313 & 538 & 400 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

   (Hint: The associativity of matrix multiplication means that there is more than one way to carry out this computation. Choose the way that makes the computation as simple as possible.)

   Solution:

$$A(BC)D = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} (939 - 538 - 400) \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 3 \\ 5 & -10 & 5 \\ 7 & -14 & 7 \end{bmatrix}$$

3. If A and B are symmetric, $n \times n$ matrices, which of the following (if any) is not necessarily symmetric? You do not need to justify your answer.

   a. $A + B$
   b. $A^T A$
   c. AB
   d. $2B^T$

   Solution:
   The matrix AB is not necessarily symmetric.

4. Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric. (Recall that a matrix A is skew-symmetric if $B^T = -B$.)

Solution:
Let $B = A - A^T$. Using the fact that the transpose operation commutes with matrix addition, $B^T = (A - A^T)^T = A^T - A = -(A - A^T) = -B$. $\square$

5. Prove that if A is an invertible matrix, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Solution:
We need to show that $\left(A^T\right)\left(A^{-1}\right)^T = I$ and $\left(A^{-1}\right)^T\left(A^T\right) = I$. In fact, by a theorem proved in class, it is enough to show that either one of these equations holds. Using a property of the transpose, $\left(A^T\right)\left(A^{-1}\right)^T = (A^{-1}A)^T = I^T = I$, since the identity matrix equal to its own transpose. $\square$

6. Complete the statement of the Fundamental Theorem for Invertible Matrices (as in section 3.3 of our textbook). You do not need to prove anything for this problem.

Let A be an $n \times n$ matrix. Then the following statements are equivalent:
a. *A is invertible.*

b. The system $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.

c. The system $A\vec{x} = \vec{0}$ has only the trivial solution.

d. *The RREF of A is I_n.*

e. A can be written as a product of elementary matrices.

7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Write A as a product of elementary matrices or explain why this is impossible.

Solution:
Since $\det(A) = -2 \neq 0$, A is invertible, which means that it is possible write A as a product of elementary matrices. The sequence of row operations $R_2 - 3R_1$, $R_1 + R_2$, $-\frac{1}{2}R_2$, transforms A into I_2. The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

The inverses of these matrices are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Therefore, since $E_3E_2E_1A = I_2$, we have

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

8. Let A, B, and X be invertible $n \times n$ matrices. Solve for X in the following equation. Simplify your answer as much as possible.

$$B^{-2}(X + A^T)^T(A^{-1})^T = (A^T B)^{-1}.$$

Solution:

$$\begin{aligned}
& B^{-2}(X + A^T)^T(A^{-1})^T = (A^T B)^{-1} \\
\Leftrightarrow\ & (X + A^T)^T = B^2 B^{-1}(A^T)^{-1}A^T \\
\Leftrightarrow\ & (X + A^T)^T = B \\
\Leftrightarrow\ & X + A^T = B^T \\
\Leftrightarrow\ & X = B^T - A^T = (B - A)^T.
\end{aligned}$$