Math 215 - Section 004
Test 1

You have 75 minutes to complete the test. ***Show all of your work clearly.*** *Unless otherwise stated, each answer must be explained and justified. Do not use decimal approximations.*

1. Find the solution set of the following linear system via Gauss-Jordan Elimination. Label all of your steps clearly using the notation from section 2.2 in the text (e.g., $R_1 + 2R_2$, $5R_3$, etc.).

$$\begin{aligned} x + y - z &= 0 \\ 2x - 4y + 2z &= 4 \\ x + 4y + -3z &= -4 \end{aligned}$$

Solution:
The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -4 & 2 & 4 \\ 1 & 4 & -3 & -4 \end{bmatrix}.$$

Apply elementary row operations to obtain

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the RREF of this matrix. The third row is equivalent to the equation 0=1, which cannot be satisfied by any values of x, y, and z. Therefore, the corresponding system of equations has no solutions, which implies that the original system has no solutions. Hence, the solution set of the original system is the empty set.

2. a. State the Cauchy-Schwarz Inequality (for vectors in $\mathbb{R}^n$). Use complete sentences.

Solution:
For any vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$.

b. Use the Cauchy-Schwarz Inequality to prove the Triangle Inequality (for vectors in $\mathbb{R}^n$).

Solution:
See page 20 in the text.

3. Let $\vec{u} = [1, 1, 0]$ and $\vec{v} = [-1, 1, 1]$. Is $\vec{w} = [2, -2, -1]$ a linear combination of $\vec{u}$ and $\vec{v}$? Justify your answer **algebraically**.

Solution:
By definition, $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$ if and only if the following linear system has a solution:

$$\begin{aligned} x - y &= 2 \\ x + y &= -2 \\ y &= -1 \end{aligned}$$

Substituting $y = -1$ into the second equation gives $x = -1$. Substituting $x = -1$, $y = -1$ into the first equation now gives $-1 - 1 = 2$, which is a contradiction. Therefore, the system has no solutions, which implies that $\vec{w}$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

4. Suppose $\vec{u}$ and $\vec{v}$ are vectors such that $||\vec{u}|| = 2$, $||\vec{v}|| = 3$, and $\vec{u} \cdot \vec{v} = \frac{3}{2}$. Find $||\vec{u} + \vec{v}||$.

   Solution:
   $||\vec{u} + \vec{v}||^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = ||\vec{u}||^2 + 2\vec{u} \cdot \vec{v} + ||\vec{v}||^2 = 2^2 + 2(\frac{3}{2}) + 3^2 = 16 \Rightarrow ||\vec{u} + \vec{v}|| = 4.$

5. Find the distance between the parallel planes $\mathcal{A}_1$ and $\mathcal{A}_2$ given by equations $2x+y-2z = 0$ and $2x + y - 2z = 5$, respectively.

   Solution:
   This problem can be done in several ways. The shortest is to use the formula distance=$\frac{|d_2-d_1|}{||\vec{n}||}$ from section 1.3 in the text. If you have spent a lot of time proving this formula, you might remember it.

   A second way, which requires only geometric intuition, is to do the following: First, find points P_1 and P_2 on $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. I chose $P_1 = (0, 0, 0)$ and $P_2 = (1, 1, -1)$. Then, using the fact that $\vec{n} = [2, 1, -2]$ is normal to both planes, the distance between $\mathcal{A}_1$ and $\mathcal{A}_2$ is

$$\begin{aligned}
\left\|\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}\right\| &= \left\|\frac{\vec{n} \cdot \overrightarrow{P_1P_2}}{\vec{n} \cdot \vec{n}}\vec{n}\right\| \\
&= \left\|\frac{5}{9}[2, 1, -2]\right\| \\
&= \frac{5}{9}\,\|[2, 1, -2]\| \\
&= \frac{5}{3}
\end{aligned}$$

6. Find an equation in **vector form** of the line in $\mathbb{R}^3$ that passes through $P = (1, 2, 3)$ and is perpendicular to the plane with general equation $x - y + 2z = 3$.

Solution:
$\vec{x} = [1, 2, 3] + t[1, -1, 2]$.

7. Show **algebraically** that the set of all points that are equidistant from the points $P = (1, 2, 3)$ and $Q = (-1, 4, 2)$ is a plane in $\mathbb{R}^3$.

Solution:
We are considering the set of points (vectors) $\vec{x}$ which satisfy the equation $d(\vec{x}, \vec{P}) = d(\vec{x}, \vec{Q})$. Since these quantities are nonnegative, this is equivalent to

$$\begin{aligned} & d(\vec{x}, \vec{P})^2 = d(\vec{x}, \vec{Q})^2 \\ \Leftrightarrow \quad & (x-1)^2 + (y-2)^2 + (z-3)^2 = (x+1)^2 + (y-4)^2 + (z-2)^2 \\ \Leftrightarrow \quad & x^2 - 2x + 1 + y^2 - 4x + 4 + z^2 - 6z + 9 = x^2 + 2x + 1 + y^2 - 8y + 16 + z^2 - 4z + 4 \\ \Leftrightarrow \quad & 4x - 4y + 2z = -7. \end{aligned}$$

This is the equation of a plane in $\mathbb{R}^3$.

8. Find formulas for two planes in $\mathbb{R}^3$ whose intersection is given by the parametric equations $x = 3t - 1$, $y = 3t - 2$, $z = t$. Write your formulas in **general form**.

Solution:
There are many possible solutions, but the simplest is probably

$$\begin{aligned} x - 3z &= -1 \\ y - 3z &= -2. \end{aligned}$$