A proof of the Thompson conjecture using Poisson geometry

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The Thompson conjecture posits a correspondence between self-adjoint matrices with prescribed sets of eigenvalues and complex matrices with prescribed sets of singular values. (Recall that the singular values of a matrix A are the eigenvalues of AA^*.) More precisely, given real numbers λ_j^k for $1 \leq j \leq n$ and $1 \leq k \leq r$, there exist self-adjoint matrices B_j, $j = 1, \ldots, n$ such that B_j has eigenvalues λ_j^k and $\sum_{j=1}^n B_j = 0$ if and only if there exist complex matrices A_j, $j = 1, \ldots, n$ such that A_j has singular values $\exp(\lambda_j^k)$ and $\prod_{j=1}^n A_j = I$.

This result was first proved by Klyachko in [Kly00]. The proof presented here, due to Alekseev, Meinrenken, and Woodward, uses relatively new techniques in Poisson geometry, some of which were motivated by problems in linear algebra such as the Thompson conjecture. The basic scheme involves realizing the above sets of matrices as certain moduli spaces. These moduli spaces are then shown to be homeomorphic as a consequence of a general theorem in Poisson geometry.

Given a Lie group G acting on a symplectic manifold (M, π) by symplectomorphisms, a moment map for this action is a map from M to the dual of the Lie algebra of G which, roughly speaking, singles out a quantity that is preserved by the action. If a moment map Φ exists for the action of G on M, we will call the triple (M, ω, Φ) a *Hamiltonian G-space.*

Suppose now that G is endowed with a Poisson structure, and M is thought of as a Poisson manifold. Assume the action of G on M is Poisson, meaning that if π_G and π_M are the Poisson tensors on G and M, respectively, then $\pi_G + \pi_M$ pushes forward under the action map to π_M. One of the primary technical ingredients in our proof will be a generalization of the notion of a moment map from symplectic actions to Poisson actions, as defined by J.H Lu and A. Weinstein in [LW90]. This is a generalization in the sense that if G has the zero Poisson structure, a moment map in this general sense is a moment map in the usual sense. A generalized moment map will be called a *Lu-Weinstein moment map.* A *Poisson G-space* is a triple (M, Ω, J), where J is a Lu-Weinstein moment map for the G-action on M with respect to Ω.

The categories of Hamiltonian G-spaces and Poisson G-spaces admit natural structures of tensor categories. That is, there are product, sum, and conjugation operations on each category satisfying the usual axioms. Of interest in the proof of the Thompson conjecture are the product operations: The product of two Hamiltonian G-spaces (M_2, ω_2, Φ_2) and

(M_2, ω_2, Φ_2) is a Hamiltonian G-space $(M_1 \times M_2, \omega_1 + \omega_2, \Phi_1 + \Phi_2)$ where the action of G on $M_1 \times M_2$ is the diagonal G-action; the product of two Poisson G-spaces (M_1, Ω_1, J_1) and (M_2, Ω_2, J_2) is a Poisson G-space $(M_1 \times M_2, \Omega_1 + \Omega_2, J_1 J_2)$ where the action of G on $M_1 \times M_2$ is a certain "twisted diagonal" action.

Under certain conditions on G, it is possible to convert a Poisson G-space to a Hamiltonian G-space by perturbing the symplectic form on M. This process is called *linearization*. In effect, linearization involves exchanging a Poisson action for a Hamiltonian action. From a more abstract point of view, linearization is a functor from the category of Poisson G-spaces to the category of Hamiltonian G-spaces.

The general result which leads to the proof of the Thompson conjecture is that linearization commutes with products. That is, given two Poisson G-spaces, the Hamiltonian G-space obtained by linearizing the product is symplectomorphic to the Hamiltonian G-space obtained by taking the product of the linearizations. This equivalence of Hamiltonian G-spaces descends to an equivalence of reduced symplectic spaces (the moduli spaces mentioned above), which in turn implies the Thompson conjecture.

The goal of this paper is to present the proof of the Thompson conjecture as given in [AMW01] for a general mathematical audience. Additional examples, explanations, and proofs are included here, as well as an introduction to Poisson Lie groups. Note that some of the notation and terminology used here differs from [AMW01]. In particular, we will use J instead of Ψ for Lu-Weinstein moment maps and the term *Poisson G-space* instead of *Hamiltonian G-space with G^*-valued moment map*.

1 Poisson Manifolds

In this section, we collect some basic facts and definitions regarding Poisson manifolds.

Definition 1.1. A *Poisson manifold* is a manifold P endowed with a bilinear, antisymmetric bracket $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ which satisfies the Jacobi identity and the following Leibniz rule:

$$\{\phi_1\phi_2, \phi_3\} = \phi_1\{\phi_2, \phi_3\} + \phi_2\{\phi_1, \phi_3\}.$$

Since $\{\cdot,\cdot\}$ is a derivation in each argument, and since the tangent space at each point $x \in P$ can be identified with the space of derivations at x, it follows that there exists a $\pi \in \Gamma(\wedge^2 TM)$ such that $\{f, g\} = \pi(df, dg)$. From this point of view, the Jacobi identity for the Poisson bracket is equivalent to $[\pi, \pi] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket, a generalization of the Lie bracket to multivector fields.

Example 1.2. If (P, ω) is a symplectic manifold, then the inverse of the matrix for ω defines a Poisson tensor on P. Thus, every symplectic manifold is a Poisson manifold. Conversely, if π is a nondegenerate Poisson tensor on P, then the inverse of the matrix representing π defines a symplectic form on P. In this sense, a Poisson manifold can be thought of as a generalization of a symplectic manifold in which degeneracy is allowed.

Contracting the Poisson bivector field π with a 1-form η yields a vector field on P. It seems that the convention is to contract in the second slot. In other words,

$$\eta \to \pi(\cdot, \eta)$$

is a map from T^*M to TM. This map will be denoted by $\pi^\#$. For $f \in C^\infty(P)$, $\pi^\#(df)$ is called the *Hamiltonian vector field* of f. The image of $\pi^\#$ defines an involutive distribution on P. The integral submanifolds of this distribution are called the *symplectic leaves* of the Poisson manifold P.

Example 1.3 (Poisson Structures on $\mathbb{R}^3$). Define a Poisson structure on $\mathbb{R}^3$ by

$$\pi = P\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + Q\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + R\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The condition $[\pi, \pi] = 0$ (where $[\cdot, \cdot]$ denotes the Schouten bracket) is equivalent to

$$P(R_y - Q_z) + Q(P_z - R_x) + R(Q_x - P_y) = 0.$$

If $\alpha = Adx + Bdy + Cdz$, then

$$\pi^\#(\alpha) = (QC - RB)\frac{\partial}{\partial x} + (RA - PC)\frac{\partial}{\partial y} + (PB - QA)\frac{\partial}{\partial z}.$$

For instance, consider the 2-tensor $\pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. It is easy to check that $[\pi, \pi] = 0$. Using the formula above, $\pi^\#(dx) = (zx)dy - (yx)dz$, which, for each point (x, y, z), can be identified with the vector $(0, zx, -yx)$ (based at the point (x, y, z)). Since $(0, zx, -yz) \cdot (x, y, z) = 0$, $\pi^\#(dx)$ can be thought of as a vector tangent to the sphere of radius $\sqrt{x^2 + y^2 + z^2}$. Similar computations show that $\pi^\#(dy)$ and $\pi^\#(dz)$ can also be identified with vectors tangent to a sphere. Thus, at each point p, $\pi^\#(\alpha)_p$ is tangent to a sphere for any 1-form α, and it is clear that any sphere centered at the origin is an integral submanifold of the involutive distribution defined by the image of $\pi^\#$. That is to say, spheres centered at the origin are symplectic leaves of the Poisson manifold $(\mathbb{R}^3, \pi)$. Furthermore, along with the single point $\{(0, 0, 0)\}$, these are all of the symplectic leaves of $\mathbb{R}^3$ with this Poisson structure.

Definition 1.4. A *Poisson map* (or *morphism*) from a Poisson manifold P to a Poisson manifold Q is a map $\phi : P \to Q$ such that

$$\phi^*\{f, g\}_Q = \{\phi^* f, \phi^* g, \}_P.$$

Viewing the Poisson structure as a bivector π, this is equivalent to $\phi_*(\pi_P) = \pi_Q$.

Definition 1.5. Given two Poisson manifolds P and Q, the *product Poisson structure* on $P \times Q$ is defined by

$$\{f, g\} = \{f(\cdot, y), g(\cdot, y)\}_P(x) + \{f(x, \cdot), g(x, \cdot)\}_Q(y).$$

From the point of view of tensors, the product tensor $\pi_{P\times Q}$ is the sum $\pi_P + \pi_Q$.

1.1 Poisson Lie Groups

The proof of the Thompson conjecture will involve a compact Lie group acting on a symplectic manifold. The Lie group will be endowed with a Poisson structure that is compatible with the group multiplication. In this section, we will state some general facts about Poisson structures on Lie groups. Proofs can be found in [LW90] and in [Lu90].

Definition 1.6. A Lie group G is called a *Poisson Lie group* if G is a Poisson manifold and the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is endowed with the product Poisson structure. In this case, the Poisson tensor π is said to be *multiplicative.*

A bivector field π on a Lie group G is multiplicative if and only if

$$\pi(gh) = l_g \pi(h) + r_h \pi(g)$$

for all $g, h \in G$, where l_g denotes left translation by g and r_h denotes right translation by h.

Let G be a connected Lie group, and let $\rho : G \times V \to V$ be a representation of G on a vector space V. A *1-cocycle on G relative to ρ* is a map $\phi : G \to V$ such that

$$\phi(gh) = \phi(g) + g \cdot \phi(h) \; \forall g, h \in G,$$

where $\cdot$ denotes the action of G on V induced by ρ (that is, $g \cdot \phi(h) = \rho(g, \phi(h))$).

The differential at the identity of the representation ρ is a map from $\mathfrak{g} \times V$ to V. A *1-cocycle on $\mathfrak{g}$ relative to $d\rho$* is a map $\epsilon : \mathfrak{g} \to V$ such that

$$X \cdot \epsilon(Y) - Y \cdot \epsilon(X) = \epsilon([X, Y]) \; \forall X, Y \in \mathfrak{g},$$

where $\cdot$ represents the action of $\mathfrak{g}$ on V induced by $d\rho$.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra with dual $\mathfrak{g}^*$. Then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a *Lie bialgebra* if there is given a Lie algebra structure on $\mathfrak{g}^*$ such that the map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

Let G be a Lie group endowed with a bivector field π satisfying $\pi_e = 0$. Define a map $d_e\pi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ by

$$X \mapsto \left(L_{\bar{X}} \pi\right)_e,$$

where $\bar{X}$ is any vector field on G with $\bar{X}_e = X$. This is a linear map, and its dual, called the *linearization of π at e*, is a map

$$[\cdot, \cdot]_\pi : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}$$

given by

$$[\xi, \eta]_\pi = d_e(\pi(\overline{\xi}, \overline{\eta}),$$

where $\xi, \eta \in \mathfrak{g}$ and $\overline{\xi}$ and $\overline{\eta}$ are any 1-forms on G with $\overline{\xi}_e = \xi$ and $\overline{\eta}_e = \eta$. Lu and Weinstein have proved the following theorem [LW90], which gives criteria in terms of the maps $[\cdot, \cdot]_\pi$ and $d_e\pi$ for the bivector field π to be Poisson and/or multiplicative.

Theorem 1.8.

1. *If* π *is multiplicative, then* $d_e\pi$ *is a 1-cocycle relative to the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g} \wedge \mathfrak{g}$*. Conversely, if* G *is connected and simply-connected, then for any 1-cocycle* $\epsilon : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ *relative to the adjoint representation of* $\mathfrak{g}$ *on* $\mathfrak{g} \wedge \mathfrak{g}$*, there is a unique multiplicative bivector field* π *such that* $\epsilon = d_e\pi$*.*

2. *If* π *is a Poisson tensor, then the bracket* $[\cdot,\cdot]_\pi$ *on* $\mathfrak{g}^*$ *induced by* π *satisfies the Jacobi identity, i.e., it is a Lie bracket on* $\mathfrak{g}^*$*. Moreover, when* G *is connected, a multiplicative bivector field* π *is a Poisson tensor if and only if its derivative at* e *defines a Lie bracket* $[\cdot,\cdot]_\pi$ *on* $\mathfrak{g}^*$*.*

Therefore, we can associate to any Poisson Lie group G a particular Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ using $\delta = [\cdot,\cdot]_\pi$. This Lie bialgebra is called the *tangent Lie bialgebra* of the Poisson Lie group (G, π).

For connected and simply-connected Lie groups, there is a correspondence between Lie bialgebras and Poisson structures. More precisely,

Theorem 1.9. *If* (G, π) *is a Poisson Lie group, then the linearization of* π *at* e *defines a Lie algebra structure on* $\mathfrak{g}^*$ *such that* $(\mathfrak{g}, \mathfrak{g}^*)$ *form the tangent Lie bialgebra to* (G, π)*. Conversely, if* G *is connected and simply-connected, then for every Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *there is a unique multiplicative Poisson structure* π *on* G *such that* $(\mathfrak{g}, \mathfrak{g}^*)$ *is the tangent Lie bialgebra to the Poisson Lie group* (G, π)*.*

Let G^* be the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}^*$. Then by Theorem 1.9, there is a unique Poisson structure π' on G^* such that the tangent Lie bialgebra to the Poisson Lie group (G^*, π') is $(\mathfrak{g}^*, \mathfrak{g})$. The Poisson Lie group (G^*, π') is called the *dual* of (G, π).

We will need two more definitions.

Definition 1.10. A triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a *double Lie algebra* if $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces.

Lu has proved in [Lu90], Theorem 2.22, that given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, it is always possible to define a Lie bracket on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ so that $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ form a double Lie algebra.

There is a corresponding definition at the group level.

Definition 1.11. A triple of Lie groups (G, G_+, G_-) is a *double Lie group* if G_+ and G_- are both closed Lie subgroups of G such that the map $\alpha : G_+ \times G_- \to G$ defined by $(g_+, g_-) \mapsto g_+g_-$ is a diffeomorphism.

Theorem 3.7 in [LW90] gives conditions under which a double Lie algebra can be used to produce a double Lie group. More specifically,

Theorem 1.12. *Suppose* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *is a double Lie algebra and* G *is the connected, simply connected Lie group with Lie algebra* $\mathfrak{g}$*. Let* G_+ *and* G_- *be the connected Lie subgroups of* G *with Lie algebras* $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$*, respectively. If* G_+ *is compact and* G_- *is closed in* G*, then* (G, G_+, G_-) *is a double Lie group.*

1.2 The Lu-Weinstein Poisson Tensor

Lu and Weinstein showed in [LW90] that every connected, compact, semisimple Lie group admits a nontrivial, multiplicative Poisson tensor. The proof is constructive, and the resulting tensor is sometimes called the Lu-Weinstein Poisson tensor (though Lu and Weinstein used the term *Bruhat-Poisson* structure.)

The construction is as follows. Given a connected, simply-connected, compact Lie group K with Lie algebra $\mathfrak{k}$, let $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, $G = K^{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition for $\mathfrak{g}$, and let $G = KAN$ be the corresponding decomposition of G. Now let B denote an inner product on $\mathfrak{k}$ that is invariant under automorphisms of $\mathfrak{k}$ (such as the Killing form, for example). The complexification $B^{\mathbb{C}}$ of B defines a non-degenerate pairing

$$\langle \cdot, \cdot \rangle = 2\mathrm{Im}\, B^{\mathbb{C}}$$

between $\mathfrak{k}$ and $\mathfrak{a} + \mathfrak{n}$. The corresponding identification $\mathfrak{k}^* \cong \mathfrak{a} + \mathfrak{n}$ induces a Lie algebra structure on $\mathfrak{k}^*$. By Theorem 1.9, there is a unique Poisson structure π on K such that the pair $(\mathfrak{k}, \mathfrak{k}^*)$ is the tangent Lie bialgebra of (K, π). In this setting, the Poisson dual K^* is AN, and $G = K^{\mathbb{C}} = KAN$ is the double.

In fact, one can give an explicit formula for this tensor (see [LW90], p521-522). Let

$$\begin{aligned} \Pi_{\mathfrak{k}} &: \mathfrak{g} \to \mathfrak{k} \quad \text{and} \\ \Pi_{\mathfrak{k}^*} &: \mathfrak{g} \to \mathfrak{k}^* = \mathfrak{a} + \mathfrak{n} \end{aligned}$$

denote the projections. Then for α, $\beta \in \mathfrak{k}^*$, $k \in K$, the Lu-Weinstein tensor π is defined by

$$(r_{k^{-1}})_* \pi(\alpha, \beta) = \mathrm{Im}\, B(\Pi_{\mathfrak{k}^*}(\mathrm{Ad}_{k^{-1}}\alpha), \Pi_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}}\beta)). \tag{1.1}$$

Now suppose K is not simply-connected. Then there is a unique simply-connected covering group $\tilde{K}$ of K. Moreover, $K \cong \tilde{K}/N$, where N is a subgroup which is contained in the center of $Z(\tilde{K})$. Let $\tilde{\pi}$ be the Lu-Weinstein tensor on $\tilde{K}$, as constructed above. Observe that for $k \in Z(\tilde{K})$, $\Pi_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}\beta}) = 0$. Therefore, $\tilde{\pi}$ vanishes on $Z(\tilde{K})$. It follows that $\tilde{\pi}$ descends to a Poisson tensor π on the quotient $\tilde{K}/N \cong K$.

According to [FL04], this tensor can also be constructed in the following way. Since $\mathfrak{k}$ is a compact form of $\mathfrak{g}$, there is a corresponding Cartan involution, θ. Choose a Cartan subalgebra, $\mathfrak{t}$, of $\mathfrak{k}$ and let $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}$ be the complexification of $\mathfrak{t}$. Construct a root system, R, for $\mathfrak{h}$ via the usual eigenspace decomposition. Choose a set of positive roots, $R^+ \subset R$, and use the positive roots, $\alpha \in R^+$ to construct a Chevalley basis for $\mathfrak{h}$. This basis consists of elements E_α and $E_{-\alpha}$, $H_\alpha = [E_\alpha, E_{-\alpha}]$ for $\alpha \in R^+$. The Cartan subalgebra $\mathfrak{t}$ and set of positive roots, R^+, must satisfy $\theta(E_\alpha) = -E_{-\alpha}$ for all $\alpha \in R^+$. This guarantees that θ also fixes $X_\alpha = E_\alpha - E_{-\alpha}$, $Y_\alpha = i(E_\alpha + E_{-\alpha})$, and iH_α for all $\alpha \in R^+$, i.e., that X_α, Y_α, and iH_α are in $\mathfrak{k}$. Then $\Lambda = \sum_{\alpha \in R^+} X_\alpha \wedge Y_\alpha$ generates a Poisson tensor for K by $\pi = r_g\Lambda - l_g\Lambda$.

Note that some choices were made in the constructions above. In the first, we chose a particular Iwasawa decomposition. In the second, we chose a Cartan subalgabra $\mathfrak{t}$ of $\mathfrak{k}$ and then a set of positive root vectors, R^+. Allowing these choices to vary, one obtains a family of Lu-Weinstein tensors. However, most authors simply refer to “the” Lu-Weinstein tensor.

1.3 Dressing Actions

Let G be a compact, connected Poisson Lie group, and let G^* be the dual. Then G^* acts on G as follows: For $\xi \in \mathfrak{g}^*$, let ξ^l and ξ^r be respectively the left and right-invariant 1-forms on G with value ξ at e. Define maps

$$\lambda : \mathfrak{g}^* \to \chi(G) : \xi \mapsto \pi_G^{\#}(\xi^l),$$

and

$$\rho : \mathfrak{g}^* \to \chi(G) : \xi \mapsto -\pi_G^{\#}(\xi^r).$$

Then λ is a Lie algebra anti-homomorphism and ρ is a Lie algebra homomorphism.

For each $\xi \in \mathfrak{g}$, $\lambda(\xi)$ and $\rho(\xi)$ are called left and right *dressing vector fields*, respectively. Integrating λ gives rise to a local action of G^* on G as follows: Let $p \in G^*$ with $p = e^{t_0 X}$ for some $X \in \mathfrak{g}^*$, $t_0 \in \mathbb{R}$. For each $q \in G$, there is a 1-parameter group of diffeomorphisms, ϕ_t, generated by $\lambda(X)$ and defined on a neighborhood V of q. Define the action by

$$p \cdot q = \phi_{t_0}(q).$$

This (left) action is defined for p in some neighborhood U of the identity in G^* and for all $q \in G$. If exp maps $\mathfrak{g}^*$ onto G^* (if G^* is compact, for example), and if the dressing vector fields $\lambda(\xi)$ are complete, then we have a global action. Integrating ρ gives rise to a (right) action of G^* on G.

Similarly, if we define maps

$$\lambda' : \mathfrak{g} \to \chi(G^*) : \xi \mapsto \pi_{G^*}^{\#}(\xi^l),$$

and

$$\rho' : \mathfrak{g} \to \chi(G^*) : \xi \mapsto -\pi_{G^*}^{\#}(\xi^r),$$

integrating λ' and ρ' gives rise to left and right actions, respectively, of G on G^*. These actions of G and G^* on each other are called *dressing actions*.

In the above description of dressing actions, we started at the Lie algebra level (the infinitesimal version) and then integrated to get an action at the Lie group level. It is also possible to define the action at the Lie group level in the first place. Let d be the double Lie algebra of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, and let D be the corresponding connected, simply-connected Lie group. Then in some cases, D decomposes as the direct product of G and G^* in two ways:

$$D = GG^* = G^*G.$$

For $k \in G$ and $l \in G^*$, we have $kl = l'k'$ for some $l' \in G^*$ and some $k' \in G$. Define the action of k on l (denoted by l^k) by $l^k = l'$. Similarly, define the action of l on k by $k^l = k'$. The the two actions are related by the equation

$$kl = l^k k^l.$$

A short computation shows that the action of G on G^* is a left action, and the action of G^* on G is a right action.

Remark 1.13. By composing with the inverse, the former can be made into a left action, and the latter can be made into a right action. In [Lu90], the actions resulting from this composition are called the dressing actions. That is, the dressing actions defined here differ from the dressing actions of Lu and Weinstein (and also of Flaschka and Ratiu in [FR96]) by composition with the inverse.

2 Symplectic, Hamiltonian and Poisson Actions

For this section, let M be a symplectic manifold with symplectic form ω, and let G be a Lie group acting on M.

Definition 2.1. The action of G on M is called *symplectic* if it preserves ω, that is, if $g^*(\omega) = \omega$ for all $g \in G$.

Any $X \in \mathfrak{g}$ induces a vector field, X_M on M by

$$(X_M)_m := \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot m$$

for any $m \in M$. (Here, $\cdot$ denotes the action of G on M.) Then the action of G on M preserves ω if and only if $\mathcal{L}_{X_M}\omega = 0$ for every $X \in \mathfrak{g}$. by Cartan's formula, $\mathcal{L}_{X_M}\omega = d(\iota_{X_M}\omega) + \iota_{X_M}(d\omega)$. Since ω is closed in this case, we have $\mathcal{L}_{X_M}\omega = d(\iota_{X_M}\omega)$. Therefore, the action of G on M is symplectic if and only if $\mathcal{L}_{X_M}\omega = 0$ if and only if $\iota_{X_M}\omega$ is closed for all $X \in \mathfrak{g}$.

A *moment map* for the action of G on M is a map $\Phi : M \to \mathfrak{g}^*$ such that

$$d(\Phi^X) = \iota_{X_M}\omega = \omega(X_M, \cdot)$$

for all $X \in \mathfrak{g}$, where Φ^X denotes the function from M to $\mathbb{R}$ defined by

$$\Phi^X(m) = \langle \Phi(m), X \rangle.$$

Moment maps are usually also required to be equivariant with respect to the action of G:

$$g \cdot \Phi(m) = \Phi(g \cdot m) \text{ for all } g \in G,$$

where the $\cdot$ on the left-hand side of the equation represents the coadjoint action of G on $\mathfrak{g}^*$, and the $\cdot$ on the right-hand side of the equation represents the given action of G on M.

Definition 2.2. The action of G on M is *Hamiltonian* if there exists an equivariant moment map Φ satisfying the properties listed above.

Example 2.3. Let $M = \mathbb{R}^2$, endowed with the symplectic form $\omega = rdr \wedge d\theta$, and let $G = S^1$ act on M by counterclockwise rotation. That is, for $\alpha \in S^1$, $\alpha \cdot (r, \theta) := (r, \theta + \alpha)$. Then for $X \in T_eS^1 \cong \mathbb{R}$, the vector field X_M is given by $(X_M)_{(r,\theta)} = X\frac{\partial}{\partial\theta}$, and $\iota_{X_M}\omega = -Xrdr$. We want a map $\Phi^X : M \to \mathbb{R}$ such that $d(\Phi^X) = -Xrdr$. Therefore, we must have

$\Phi^X(r,\theta) = X(-\frac{r^2}{2} + C)$ for some constant C. It is clear that the map $\Phi : M \to \mathfrak{g}^* \cong \mathbb{R}$ defined by

$$\Phi(r,\theta) = -\frac{r^2}{2} + C$$

satisfies the moment map equation $d\langle \Phi, X\rangle = \iota_{X_M}\omega$. Furthermore, S^1 is an abelian group, so in this case, equivariance of Φ is just invariance, i.e., $\Phi(\alpha \cdot (r,\theta)) = \Phi((r,\theta))$. Since the action of α affects only θ, it is clear that this condition is also satisfied. Therefore, Φ is a moment map for this action.

As described above, the action of G on M is symplectic if and only if $\iota_{X_M}\omega$ is closed for all $X \in \mathfrak{g}$. If it is also the case that $\iota_{X_M}\omega$ is exact for all $X \in \mathfrak{g}$, then one can construct a moment map Φ for this action as follows: Let $X \in \mathfrak{g}$. If $\iota_{X_M}\omega$ is exact, then there exists a function, which we will call Φ^X, such that $d(\Phi^X) = \iota_{X_M}\omega$. If every $\iota_{X_M}\omega$ is exact, then for each m in M, we have a function $\Phi^{\cdot}(m) : \mathfrak{g} \to \mathbb{R}$. It only remains to check that $\Phi^{\cdot}(m)$ is linear. This follows easily from the bilinearity of ω:

$$\begin{aligned} &d\Phi^{aX+bY} = \iota_p(aX+bY)\omega = \omega(aX+bY,\cdot) = a\omega(X,\cdot) + b\omega(Y,\cdot) \\ &= a\iota_{X_M}\omega + b\iota_{p(Y)} = ad(\Phi^X) + bd(\Phi^Y). \end{aligned}$$

Thus, Φ^{aX+bY} and $a(\Phi^X)+b(\Phi^Y)$ are equal up to a constant. If we choose all constants for the generating functions Φ^X to be zero in the first place, then Φ^{aX+bY} and $a(\Phi^X)+b(\Phi^Y)$ are equal.

Now suppose G is a Poisson Lie group with Poisson structure π_G, and think of M as a Poisson manifold with Poisson structure π_M.

Definition 2.4. An action $\mathcal{A} : G \times M \to M$ is *Poisson* if $\mathcal{A}$ is a Poisson map when $G \times M$ is endowed with the product Poisson structure, i.e., if $\mathcal{A}_*(\pi_G + \pi_M) = \pi_M$.

Example 2.5. Let $(G, \pi_G) = (\mathbb{R}^3, x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z})$, and let $(M, \pi_M) = (\mathbb{R}^3, x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. Suppose G acts on M by

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) := (x_2+x_1, y_2+x_1, z_2+x_1).$$

The Jacobian for the action map $\mathcal{A} : G \times M \to M$ is given by

$$D\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $\mathcal{A}_*(\pi_G + \pi_M) = 0 + \pi_M = \pi_M$. Therefore, this action is Poisson.

Note that if $\pi_G = 0$, then the action of G on M preserves π_M. If $\pi_G \neq 0$, however, then π_M may not be preserved by the G action.

Let G^* be the dual Poisson Lie group of G. For any $X \in \mathfrak{g}$, let X^l be the left-invariant 1-form on G^* with value X at $e \in G^*$. A moment map in the Lu-Weinstein sense for a left action $G \times M \to M$ is a map $J : M \to G^*$ such that

$$X_M = \pi_M^{\#}(J^*(X^l)) = \pi_M(\cdot, J^*(X^l)) \text{ for all } X \in \mathfrak{g}. \tag{2.1}$$

Similarly, a moment map for a right action $M \times G \to M$ must satisfy

$$X_M = -\pi_M^{\#}(J^*(X^r)) = -\pi_M(\cdot, J^*(X^r)) \text{ for all } X \in \mathfrak{g}, \tag{2.2}$$

where X^r is the right-invariant 1-form on G^* with value X at $e \in G^*$.

If π_M is the inverse of a symplectic structure $\Omega \in \Omega^2(M)$, then (2.1) is equivalent to

$$\iota_{X_M}\Omega = (J^*(X^l)) \text{ for all } X \in \mathfrak{g}, \tag{2.3}$$

and (2.2) is equivalent to

$$\iota_{X_M}\Omega = (J^*(X^l)) \text{ for all } X \in \mathfrak{g}. \tag{2.4}$$

In this case, the moment map is equivariant if for every $g \in G$,

$$J(m)^g = J(g \cdot m), \tag{2.5}$$

where the $\cdot$ on the right-hand side of the equation represents the given action of G on M. This is an equivalence in G^*.

Proposition 2.6. *The map J is Poisson if and only if it is equivariant. Furthermore, if the Poisson Lie group G acts on the Poisson manifold (M, π_M) on the left (resp. right), and if there is a Poisson map $J : M \to G^*$ satisfying (2.1) (resp. (2.1)), then the G-action is Poisson, and J is its moment map.*

If G has the trivial Poisson structure and M is a symplectic manifold (with Poisson structure induced by the symplectic form), then G^* can be identified with $\mathfrak{g}^*$, and a moment map in the Lu-Weinstein sense is actually a moment map in the usual sense. In other words, a Poisson action can be thought of as a generalization of a Hamiltonian action.

Example 2.7. Let K be a Poisson Lie group with dual group K^*, and suppose there is a Lie group G such that (G, K, K^*) is a double Lie group. Let $\mathcal{A} : K \times K^* \to K^*$ be the right dressing action of K on K^*. We will show that the identity on G^* is a moment map for this action. By [LW90], Theorem 2.7, $\mathcal{A}$ is Poisson. Therefore, by Theorem 2.6, it is sufficient to check that the moment map condition (2.2) is satisfied. For $X \in \mathfrak{g}$, the left-hand side of (2.2) is the vector field X_M generated by X and the dressing action. According to the infinitesimal version of the dressing action given in Section 1.3, $X_M = -\pi_{G^*}^{\#}(X^r)$, which is exactly the right-hand side of (2.2).

2.1 Symplectic and Poisson Reduction

Let $G \times M \to M$ be a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω) with moment map $\Phi : M \to \mathfrak{g}^*$. For $\mu \in \mathfrak{g}^*$, let $G_\mu \subset G$ be the isotropy subgroup of μ. If μ is a regular value of Φ and G_μ acts freely on $\Phi^{-1}(\mu)$, then there is a symplectic form ω_μ on $M_\mu := \Phi^{-1}(\mu)/G_\mu$.

Example 2.8. The following example is given in [dS01], Chapter 22. Suppose the Lie group $G = S^1$ acts on the symplectic manifold $(\mathbb{C}^n, \sum r_i dr_i \wedge d\theta_i)$ by rotation. That is, for $\alpha \in S^1$, $(z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\alpha \cdot (z_1, \ldots, z_n) := (e^{i\alpha} z_1, \ldots, e^{i\alpha} z_n).$$

(This is a higher-dimensional version of Example 2.3.) Then

$$\Phi : \mathbb{C}^n \to \mathfrak{g}^* \cong \mathbb{R} : \mathbf{z} \mapsto -\frac{|z|^2}{2} + C$$

is a moment map for this action for any constant C. If we choose $C = \frac{1}{2}$, then $\Phi^{-1}(0) = S^{2n-1} \subset \mathbb{C}^n$, and $\Phi^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$.

A similar result is true in the Poisson setting. Let (M, Ω, J) be a Poisson G-space. For any $l \in G^*$, define

$$G_l := \{k \in G \mid l^k = l\},$$

and

$$M_l := J^{-1}(l)/G_l.$$

Assume that l is a regular value of J and that G_l acts freely and properly on $J^{-1}(l)$. Then by [Lu], Theorem 4.12, M_l is a symplectic manifold.

3 Poisson and Hamiltonian G-spaces

Let G be a connected Lie group, let (M, ω) be a symplectic manifold, and suppose G acts on M by symplectomorphisms. If the action is Hamiltonian, i.e., if there exists a moment map $\Phi : M \to \mathfrak{g}^*$, then we will call the triple (M, ω, Φ) a *Hamiltonian G-space.* Two Hamiltonian G-spaces, (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2), are equivalent if there is a bijective map $\tau : M_1 \to M_2$ such that $\tau^*\Phi_2 = \Phi_1$, $\tau^*(\omega_2) = \omega_1$, and τ is equivariant with respect to the given K-actions on M_1 and M_2.

Let (G, π_G) be a connected Poisson Lie group, let (M, π) be a Poisson manifold, and let $G \times M \to M$ be a Poisson action. Suppose there exists a Lu-Weinstein moment map $J : M \to G^*$. The triple (M, π, J) will be called a *Poisson G-space.* If π is the inverse of a symplectic form Ω, we will use the notation (M, Ω, J). Two Poisson G-spaces, (M_1, π_1, J_1) and (M_2, π_2, J_2), are equivalent if there is a bijective map $\tau : M_1 \to M_2$ such that $\tau^* J_2 = J_1$, $\tau_*(\pi_1) = \pi_2$, and τ is equivariant with respect to the given K-actions on M_1 and M_2.

We will now define product operations for Poisson and Hamiltonian G-spaces. The product of two Hamiltonian G-spaces, (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2), is the triple $(M_1 \times M_2, \omega_1 + \omega_2, \Phi_1 + \Phi_2)$. In this case, the G-action on $M_1 \times M_2$ is the diagonal action (i.e., G acts independently on the components). It is clear that this triple is a Hamiltonian G-space and that the product operation on Hamiltonian G-spaces is associative.

Given two Poisson G-spaces, (M_1, π_1, J_1) and (M_2, π_2, J_2), define an action of G on $M_1 \times M_2$ by

$$g \times (m_1, m_2) = (g \cdot m_1, g^{J(m_1)} \cdot m_2).$$

Associativity of the action is proved using the equivariance of J_2 and the equation, $kl = l^k k^l$, from the definition of the dressing action.

Proposition 3.1. *If $M_1 \times M_2$ is endowed with the product Poisson structure $\pi_1 \times \pi_2$, the map $J = J_1 J_2 : M_1 \times M_2 \to K^* : (m_1, m_2) \mapsto J_1(m_1) J_2(m_2)$ is a moment map for this action.*

Proof. First, since multiplication on $M_1 \times M_2$ is Poisson by definition, and since J_1 and J_2 are Poisson maps, it is clear that J is a Poisson map. Now by Proposition 2.6, it is enough to show that J satisfies (2.5). This is proved in [FR96], Lemma 2.23. $\square$

We can therefore define the product of (M_1, π_1, J_1) and (M_2, π_2, J_2) to be the Poisson G-space $(M_1 \times M_2, \pi_1 + \pi_2, J_1 J_2)$. It is easy to check that this product operation is associative.

It is also possible to define sum and conjugation operations which give the collection of Poisson (resp. Hamiltonian) G-spaces the structure of a tensor category. This additional structure can be used to extend the results presented here.

4 Linearization

For a compact group K acting on a symplectic manifold M, Alekseev has proved in [Ale97] that, under certain conditions, it is possible to convert a Poisson K-space to a Hamiltonian K-space by perturbing the symplectic form on M. This process is called *linearization.* In effect, linearization involves exchanging a Poisson action for a Hamiltonian action. From a category-theoretic point of view, linearization can be thought of as a functor from the category of Poisson G-spaces to the category of Hamiltonian G-spaces. This paper will be concerned with a slightly restricted version of Alekseev's result.

4.1 Linearization Theorem

From now on, K denotes a compact, connected Lie group with the Lu-Weinstein Poisson structure, $K^* = AN$ is the dual, and $G = K^{\mathbb{C}} = KAN$ is the double. Given a Poisson action of K on a symplectic manifold (M, Ω) with K^*-valued moment map J, we will construct a symplectic form ω on M such that the action is Hamiltonian, i.e., such that there exists a $\mathfrak{k}^*$-valued moment map Φ for the action with respect to ω. In order to construct the form ω and the map Φ, we will first need a map $E : \mathfrak{k}^* \to K^*$ which is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^*$ and the left dressing action on K^*, that is,

$$E(k \cdot \mu) = E(\mu)^k \text{ for } k \in K.$$

Let $\kappa : \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution given by complex conjugation relative to $\mathfrak{k}$, and let

$$\dagger : \mathfrak{g} \to \mathfrak{g} : \xi \mapsto \xi^\dagger$$

be the anti-involution given by

$$\xi^\dagger = -\kappa(\xi).$$

The induced anti-involution on G will also be denoted by $\dagger$. In the case $K = SU(n)$, $G = Sl_n(\mathbb{C})$, the $\dagger$ map is the conjugate transpose. Let $\mathfrak{p} \subset \mathfrak{g}$ denote the fixed point set of $\dagger$. That is, $\mathfrak{p} = i\mathfrak{k}$, so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition corresponding to κ. Let $G = KP$ be the corresponding group decomposition.

Lemma 4.1. *Any $p \in P$ admits a unique decomposition $p = ll^\dagger$ for some $l \in K^*$.*

Proof. It follows from the fact that $\exp : \mathfrak{p} \to P$ is a diffeomorphism onto P that the square root is an invertible operation defined on all of P. The uniqueness of the Iwasawa decomposition implies that there exist unique $k \in K$, $l \in K^*$ such that $\sqrt{p} = lk$. Then

$$\sqrt{p}k^{-1} = l. \tag{4.1}$$

Applying the dagger map to this equation yields

$$k\sqrt{p} = l^\dagger. \tag{4.2}$$

Multiplying (4.1) on the right by (4.2), we obtain $p = ll^\dagger$.

To show that this decomposition is unique, consider the map $P \to K^*$ given by $p \mapsto l$ where $\sqrt{p} = lk$. By the uniqueness of the Iwasawa decomposition, this map is well-defined and injective. We have shown that if $\sqrt{p} = lk$, then $p = ll^\dagger$, so $l \mapsto ll^\dagger$ is the inverse map. We wish to show that

$$K^* \to P : l \mapsto ll^\dagger$$

is injective. It only remains to show that the map $p \mapsto l$ is surjective. By the Cartan decomposition, for each $l \in K^*$, there exist $p \in P$, $k \in K$ such that $k\sqrt{p}^{-1} = l^{-1}$. Taking inverses, this is equivalent to $\sqrt{p} = lk$. □

Let $B^\# : \mathfrak{k}^* \to \mathfrak{k}$ denote the isomorphism induced by the Killing form, B, on $\mathfrak{k}$. By Lemma 4.1, for each $\mu \in \mathfrak{k}^*$, the element $p = \exp(iB^\#(\mu)) \in P$ admits a unique decomposition $p = ll^\dagger$.

Proposition 4.2. *The map*

$$E : \mathfrak{k}^* \to K^* : \mu \mapsto l$$

is a diffeomorphism. Furthermore, E is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^*$ and the dressing action on K^*.

Proof. It only remains to prove the K-equivariance of E. We want to show that for $\mu \in \mathfrak{k}^*$, $E(\mathrm{Ad}_k^*\mu) = E(\mu)^k$ for all $k \in K$ (where Ad_k^* represents the coadjoint action of K on $\mathfrak{k}^*$). Using the equivariance of $B^\#$ with respect to the adjoint and coadjoint K-actions and setting $\xi = B^\#(\mu)$, we have

$$\exp(i\mathrm{Ad}_k^* B^\#(\mu)) = \exp(\mathrm{Ad}_k i\xi) = k\exp(i\xi)k^{-1}.$$

Decomposing $\exp(i\xi) = ll^\dagger$ for $l \in K^*$, we have $\exp(i\mathrm{Ad}_k^* B^\#(\mu)) = kll^\dagger k^{-1}$. Using the fact that $\dagger$ acts as the inverse on K, one can show that

$$kll^\dagger k^{-1} = (l^k)(l^k)^\dagger.$$

This implies that $E(k \cdot \mu) = l^k$, as desired. □

Now we will construct a 1-form on $\mathfrak{k}^*$. First, let θ^L be the left-invariant Maurer-Cartan 1-form on K^*, and let $\theta^{L^\dagger}$ be its image under the map $\dagger : \mathfrak{k}^* \subset \mathfrak{g} \to \mathfrak{g}$, i.e., $\theta^{L^\dagger} = \dagger \circ \theta^L$. Then $B^{\mathbb{C}}(\theta^L, \theta^{L^\dagger})$ (where B is the invariant pairing used in the construction of the Lu-Weinstein Poisson structure on K) is an imaginary-valued 2-form on K^*, and we can define a real-valued 1-form β on $\mathfrak{k}^*$ by

$$\beta = \frac{1}{2i}\mathcal{H}\left(E^* B^{\mathbb{C}}\left(\theta^L, \theta^{L^\dagger}\right)\right),$$

where $\mathcal{H} : \Omega^*(\mathfrak{k}^*) \to \Omega^{*-1}(\mathfrak{k}^*)$ is the standard homotopy operator for the de Rham differential.

The following proposition is proved in the appendix of [AMW01].

Proposition 4.3. *The 1-form β satisfies*

$$\iota_{\xi_{k^*}} d\beta = E^*\langle \theta^R, \xi\rangle - d\langle \xi \cdot\rangle \textit{ for all } \xi \in \mathfrak{k},$$

where $\iota_{\xi_{k^*}}$ is the vector field on K^* generated by ξ and the coadjoint action of K on $\mathfrak{k}^*$.

Now let

$$\Phi = E^{-1} \circ J \text{ and } \omega = \Omega - d\Phi^*\beta,$$

Using Proposition 4.3, we will show that the moment map condition

$$\Omega(\xi_M, \cdot) = J^*\langle \theta^R, \xi\rangle$$

for J, is equivalent to the corresponding moment map condition

$$d\langle \Phi, \xi\rangle = \omega(\xi_M, \cdot)$$

for Φ. We have

$$\begin{aligned}
\omega(\xi_M, \cdot) &= \Omega(\xi_M, \cdot) - d(\Phi^*\beta)(\xi_M, \cdot) \\
&= J^*\langle \theta^R, \xi\rangle - d(\Phi^*\beta)(\xi_M, \cdot) \\
&= J^*\langle \theta^R, \xi\rangle - (\Phi^* d\beta)(\xi_M, \cdot).
\end{aligned}$$

The equivariance of E implies the following equivariance condition for E^{-1}: For $k \in K$, $l \in K^*$,

$$k \cdot E^{-1}(l) = E^{-1}(l^k),$$

where the $\cdot$ on the left-hand side represents the coadjoint action of K on $\mathfrak{k}^*$. Using this equation and the equivariance of J, we have

$$\begin{aligned}
\Phi_*(\xi_M)_m &= E_*^{-1} \circ J_*(\xi_M)_m \\
&= \frac{d}{dt}\left(E^{-1}\left(J(\exp(t\xi)\cdot m)\right)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(E^{-1}\left(J(m)^{\exp(t\xi)}\right)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\exp(t\xi)\cdot E^{-1}\left(J(m)\right)\right)\Big|_{t=0} \\
&= (\xi_{\mathfrak{k}^*})_{\Phi(m)}\,.
\end{aligned}$$

Thus, $(\Phi^* d\beta)(\xi_M, \cdot) = \Phi^*(d\beta(\xi_{\mathfrak{k}^*}, \cdot))$. Using Proposition 4.3,

$$\begin{aligned}\omega(\xi_M, \cdot) &= J^*\langle\theta^R, \xi\rangle - \Phi^*(E^*\langle\theta^R, \xi\rangle - d\langle\xi, \cdot\rangle) \\ &= J^*\langle\theta^R, \xi\rangle - \Phi^*(\langle\theta^R, \xi\rangle - d\langle\Phi, \xi\rangle) \\ &= d\langle\Phi, \xi\rangle,\end{aligned}$$

as desired.

The Hamiltonian K-space (M, ω, Φ) is called the *linearization* of the Poisson K-space (M, Ω, J).

4.2 Linearization Commutes with Products

In this section, we will show that linearization commutes with the product operations defined in Section 3. We will need the following Moser isotopy lemma.

Lemma 4.4. *Let* (M, ω^s, Φ^s) *be a family of compact Hamiltonian* K*-manifolds,* $s \in [0, 1]$*. For* $\xi \in \mathfrak{k}$*, let* ξ_M^s *denote the Hamiltonian vector field generated by* ξ *and the* K*-action associated to* (M, ω^s, Φ^s)*. Suppose* ω^s *and* Φ^s *depend smoothly on* s *and that there exists a smooth family of 1-forms* α^s *such that*

$$\dot{\omega^s} = d\alpha^s, \tag{4.3}$$

where the dot stands for $\frac{d}{ds}$*. Suppose, also, that for each* $\xi \in \mathfrak{k}^K$
(where $\mathfrak{k}^K = \{\xi \in \mathfrak{k} \mid Ad_k\xi = \xi\, \forall k \in K\}$*),*

$$\langle\dot{\Phi^s}, \xi\rangle + \iota_{\xi_M^s}\alpha^s = 0. \tag{4.4}$$

Then there is a smooth isotopy $\phi^s : M \to M$ *which intertwines the* K*-actions for the parameters 0,* s *and which satisfies*

$$(\phi^s)^*\,\omega^s = \omega^0, \;\; (\phi^s)^*\,\Phi^s = \Phi^0.$$

Proof. For each $s \in [0, 1]$, let $j^s : M \to \tilde{M} := [0, 1] \times M$ be the inclusion $j^s(m) = (s, m)$. Equip $\tilde{M}$ with the K-action such that the maps j^s are equivariant with respect to the K-action on M defined by ω^s, Φ^s. Define $\Phi : \tilde{M} \to \mathfrak{k}^*$ by $(j^s)^*(\Phi^s) = \Phi$, and let ω, α be forms on $\tilde{M}$ which pull back to ω^s, α^s, respectively, under j^s and which vanish on $\frac{\partial}{\partial s}$. Now let

$$\tilde{\omega} = \omega + ds \wedge \alpha \in \Omega^2(\tilde{M}) \tag{4.5}$$

Then (4.3) is equivalent to

$$d\tilde{\omega} = 0, \tag{4.6}$$

and (4.4) is equivalent to the moment map condition

$$d\langle\Phi, \xi\rangle = \iota_{\xi_{\tilde{M}}}\tilde{\omega}. \tag{4.7}$$

According to the proof given in [AMW01], we may assume that $\tilde{\omega}$ is K-invariant.

Let $\tilde{X}$ be the unique vector field on $\tilde{M}$ such that $\iota_{\tilde{X}}\tilde{\omega} = 0$ and $\iota_{\tilde{X}}ds = 1$. It is K-invariant, preserves $\tilde{\omega}$, and its flow $\tilde{\phi}^s$ takes the slice at 0 to the slice at s. Let ϕ^s be the isotopy of M defined by $\tilde{\phi}^s \circ j^0 = j^s \circ \phi^s$. Then $(\tilde{\phi}^s) = \tilde{\omega}$ implies $(\phi^s)^* \omega^s = \omega^0$. Similarly, for $\xi \in \mathfrak{k}^K$, we have

$$L_{\tilde{X}} \langle \Phi, \xi \rangle = \iota_{\tilde{X}} d \langle \Phi, \xi \rangle = \iota_{\tilde{X}} \iota_{\xi_{\tilde{M}}} \tilde{\omega}.$$

This shows that $(\tilde{\phi}^s)^* \langle \Phi, \xi \rangle = \langle \Phi, \xi \rangle$, or equivalently, $(\phi^s)^* \langle \Phi^s, \xi \rangle = \langle \Phi^0, \xi \rangle$. Since the flow $\tilde{\phi}^s$ is K-equivariant, ϕ^s intertwines the K-actions on M for the parameters 0,s. Finally, since moment maps are determined up to a constant in $(\mathfrak{k}^*)^K$, this proves that $(\phi^s)^* \Phi^s = \Phi^0$. □

Theorem 4.5 (Linearization commutes with products)**.**
Let (M_i, Ω_i, J_i), $i = 1, 2$, be two Poisson K-spaces with linearizations (M_i, ω_i, Φ_i). Consider the products

$$\begin{aligned} (M, \Omega, J) &:= (M_1 \times M_2, \Omega_1 + \Omega_2, J_1 J_2) \\ (M, \omega, \Phi) &:= (M_1 \times M_2, \omega_1 + \omega_2, \Phi_1 + \Phi_2). \end{aligned}$$

The linearization of (M, Ω, Φ) is equivalent to the product (M, ω, Φ). That is, there exists a diffeomorphism $\phi : M \to M$ which takes the diagonal K-action to the twisted K-action and satisfies

$$\phi^* \Omega = \omega + d\Phi^* \beta, \quad \phi^* J = E \circ \Phi.$$

Proof. Recall that the inner product B on $\mathfrak{k}$ is used in the construction of the Lu-Weinstein tensor, in the definition of a K^*-valued moment map, and in the linearization process. For any $s > 0$, consider the rescaled inner product $B^s = s^{-1} B$, and define $\zeta^s : \mathfrak{k}^* \to \mathfrak{k}^*$ by $\zeta^s(\mu) = s\mu$. Replacing B by B^s replaces the map E in the linearization process by $E^s = (\zeta^s)^* E$, and the form β by $\beta^s = s^{-1} (\zeta^s)^* \beta$. For each $s > 0$ and for $j = 1, 2$, setting

$$\begin{aligned} \Omega_j^s &= \omega_j + d\Phi_j^* \beta^s, \\ J_j^s &= E^s \circ \Phi_j, \end{aligned}$$

we obtain two families of Poisson K-spaces, (M_j, Ω_j^s, J_j^s). Let (M, Ω^s, J^s) denote the product $(M_1 \times M_2, \Omega_1^s + \Omega_2^s, J_1^s J_2^s)$, and let (M, ω^s, Φ^s) denote the corresponding linearization, with

$$E^s \circ \Phi^s = (E^s \circ \Phi_1)(E^s \circ \Phi_2), \tag{4.8}$$

$$\omega^s = \omega + d(\Phi - \Phi^s)^* \beta^s. \tag{4.9}$$

Consider the limit as $s \searrow 0$. Since the family of maps $\{E^s\}$ extends smoothly to $s = 0$ by $E^0 = E$, it follows from (4.8) that the family of moment maps Φ^s extends smoothly to $s = 0$ by $\Phi^0 = \Phi$. On the other hand, since β is the image of a form under the de Rham homotopy operator, it vanishes at the identity (see Appendix A). It follows that the family of 1-forms $\{\beta^s\}$ extends smoothly to $s = 0$ by $\beta^0 = 0$. Therefore, by (4.9), the forms ω^s extend smoothly to $s = 0$ by $\omega^0 = \omega$. Thus, for $s \in [0, 1]$, we have a family of compact Hamiltonian K-spaces, (M, ω^s, Φ^s) connecting (M, ω, Φ) with the linearization of (M, Ω, J).

The proof is completed by an application of Lemma 4.4 with

$$\alpha^s = \frac{d}{ds}(\Phi - \Phi^s)^* \beta^s.$$

$\square$

Since the diffeomorphism ϕ satisfies $\phi^* J = E \circ \Phi$, we have the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "J"] \\
\mathfrak{k}^* \arrow[r, "E"'] & K^*
\end{tikzcd}$$

Therefore, for $l \in K^*$ and $\mu = E^{-1}(l) \in \mathfrak{k}^*$, $\phi(\Phi^{-1}(\mu)) = J^{-1}(l)$. Furthermore, since ϕ takes the diagonal K-action to the twisted K-action on $M = M_1 \times M_2$, we have the following corollary to Theorem 4.5.

Corollary 4.6. *With notation as in Theorem 4.5, the reduced spaces $(M_1 \times M_2)_l$ at $l \in K^*$ and $(M_1 \times M_2)_\mu$ at $\mu = E^{-1}(l)$ are diffeomorphic.*

5 Thompson Conjecture

We will now restrict our attention to the case $M = (K^*)^n$, where $(K^*)^n$ is the n-fold Cartesian product and $(K^*)^n \times K \to (K^*)^n$ is the twisted right dressing action of K on $(K^*)^n$. In this setting, the moment map $J : K^* \to K^*$ is the product of the projections onto each coordinate.

Let $\mathcal{O}_1 \ldots, \mathcal{O}_n$ be coadjoint orbits in $\mathfrak{k}^*$, and let $\mathcal{D}_i = E(\mathcal{O}_i)$ be the corresponding dressing orbits in K^*. Consider the following moduli spaces:

$$\mathcal{M}_{\mathcal{O}} = \{(\xi_1, \ldots, \xi_n) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \,|\, \xi_1 + \cdots + \xi_n = 0\}/K \tag{5.1}$$

$$\mathcal{M}_{\mathcal{D}} = \{(g_1, \ldots, g_n) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \,|\, g_1 \ldots g_n = e\}/K \tag{5.2}$$

where the K-action in (5.1) is the diagonal coadjoint action, and the K-action in (5.2) is the twisted dressing action.

Using the fact that the isotropy subgroups of $0 \in \mathfrak{k}^*$ and $e \in K^*$ for the coadjoint and dressing actions, respectively, are both equal to K, since $\mathcal{M}_{\mathcal{O}} = \Phi^{-1}(0)/K$ and $\mathcal{M}_{\mathcal{D}} = J^{-1}(0)/K$, it follows from Corollary 4.6 that $\mathcal{M}_{\mathcal{O}}$ and $\mathcal{M}_{\mathcal{D}}$ are diffeomorphic.

Now consider the special case $K = U(r)$, $G = K^{\mathbb{C}} = GL(r, \mathbb{C})$. The Lie algebra $\mathfrak{k}$ consists of anti-Hermitian matrices, and the dual $\mathfrak{k}^*$ can be identified with the set of Hermitian matrices via the pairing

$$\langle \mu, \xi \rangle = \frac{1}{i} tr(\mu\xi), \text{ for } \mu \in \mathfrak{k}^*, \xi \in \mathfrak{k}.$$

It is clear that the coadjoint action preserves eigenvalues, so the orbits $\mathcal{O}_j$ consist of Hermitian matrices with prescribed eigenvalues. As noted in the proof of Proposition 4.2, for $l \in K^*$, we have

$$l^k (l^k)^\dagger = k l l^\dagger k^{-1}.$$

This implies that the eigenvalues of $ll^\dagger$ are preserved by the dressing action. Therefore, the orbits $\mathcal{D}_j$ consist of matrices with prescribed singular values. (Recall that the singular values of a matrix A are the eigenvalues of $AA^\dagger = AA^*$.) As observed above, Corollary 4.6, applied to this special case, proves the following theorem.

Theorem 5.1. *(The Thompson Conjecture)*
Let λ_j^k for $1 \leq j \leq n$ and $1 \leq k \leq r$ be given real numbers. The following conditions are equivalent:

1. *There exist complex matrices A_j, $j = 1, \ldots, n$ such that A_j has singular values $\exp(\lambda_j^k)$ and $\prod_{j=1}^n Aj = I$.*
2. *There exist self-adjoint matrices B_j, $j = 1, \ldots, n$ such that B_j has eigenvalues λ_j^k and $\sum_{j=1}^n B_j = 0$.*

By considering Poisson anti-involutions, this result is extended in [AMW01] to real matrices. In particular, the following are also equivalent to (1) and (2) in Theorem 5.1:

3. *There exist real matrices A_j, $j = 1, \ldots, n$ such that A_j has singular values $\exp(\lambda_j^k)$ and $\prod_{j=1}^n Aj = I$.*
4. *There exist real, symmetric matrices B_j, $j = 1, \ldots, n$ such that B_j has eigenvalues λ_j^k and $\sum_{j=1}^n B_j = 0$.*

APPENDICES

A Algebraic Topology

The *standard homotopy operator* on the De Rham complex, as described in [BT95], is a particular set of linear operators H_k such that each H_k takes k-forms to $k-1$-forms and which map closed forms to exact forms. As with the exterior derivative, when there is no potential for confusion, it is common to drop the k and denote the operator simply by H.

The homotopy operator on $\mathbb{R}^n$ is constructed as follows. Let $\pi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} : (x,t) \mapsto x$ be the projection onto the first $n-1$ coordinates. Every form on $\mathbb{R}^n$ is a linear combination of the following two types of forms:

1. $(\pi^*\eta)f(x,t)$,
2. $(\pi^*\eta)f(x,t)dt$,

where ϕ is a form on $\mathbb{R}^{n-1}$. Define $H_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ by

$$(\pi^*\eta)f(x,t) \mapsto 0 \tag{A.1}$$

$$(\pi^*\eta)f(x,t)dt \mapsto (\pi^*\eta)\left(\int_0^t f(x,t)dt\right). \tag{A.2}$$

Note that if $t = 0$, $\left(\int_0^t f(x,t)dt\right) = 0$, which implies that $H_k(\pi^*\eta)f(x,t)dt) = 0$ whenever $t = 0$. In particular, the form $H_k(\eta)$ vanishes at the origin for any k-form.

For an arbitrary manifold, simply use the construction above in each coordinate chart.

B Lie Groups and Lie Algebras

B.1 Representations

Let G be a connected Lie group. For each $g \in G$, define a map $\Psi_g : G \to G$ given by

$$\Psi_g(h) = ghg^{-1}.$$

Then

$$(d\Psi_g)_e : T_eG \to T_eG$$

is an automorphism of $\mathfrak{g}$. The map $(d\Psi_g)_e$ is sometimes denoted by Ad_g. Thus, $g \mapsto Ad_g$ is a representation of the Lie group G on its Lie algebra, $\mathfrak{g}$, called the *adjoint representation.*

If a group G has a representation ρ on a vector space V, then G can also be represented on V^* as follows: Define $\sigma : G \to V^*$ by $\sigma(g) = \rho(g^{-1})^T$. The map σ is called the *dual* or *contragredient* representation. The coadjoint representation of a Lie group G on the dual of its Lie algebra, $\mathfrak{g}^*$, is the dual of the adjoint representation. The image of an element $g \in G$ under this representation will be denoted by Ad^*_g.

Given a representation $\rho : G \times V :\to V$ of a connected Lie group G on a vector space V, there is an induced representation of $\mathfrak{g}$ on V. The differential of ρ is a map from $\mathfrak{g} \times T_0V$ to T_0V. But T_0V can be identified with V, so $d\rho$ can be thought of as a map from $\mathfrak{g} \times V$ to V.

B.2 Maurer-Cartan Forms

For $g \in G$, $V \in T_gG$, define a $\mathfrak{g}$-valued 1-form ω on G by

$$\omega(V) = g^{-1}V,$$

where $g^{-1}V$ is the left-translation of V by g^{-1} (actually, g^{-1}_*, but the $*$ is often omitted). The form ω is sometimes called the *left-invariant Maurer-Cartan 1-form.* The *right-invariant Maurer-Cartan 1-form* is defined similarly.

B.3 Compact Real Forms, Cartan Involutions, and Cartan Decompositions

Given a complex Lie algebra $\mathfrak{g}$, a *real form* of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{k}$ such that $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{g}$ are isomorphic as Lie algebras. If there exists a real form $\mathfrak{u}$ of $\mathfrak{g}$ such that the Killing form of $\mathfrak{u}_0$ is negative definite, the connected, simply-connected Lie group corresponding to $\mathfrak{u}_0$ is then a compact (real) Lie group, and $\mathfrak{u}_0$ is called a *compact real form* of $\mathfrak{g}$.

Definition B.1. A *Cartan involution* of a real, semisimple Lie algebra $\mathfrak{g}_0$ is an involution of $\mathfrak{g}_0$ (an automorphism that squares to 1) such that the symmetric, bilinear form

$$B_\theta(X, Y) = -B(X, \theta(Y))$$

is positive definite. (Here, B denotes the Killing form of $\mathfrak{g}_0$.)

A complex, semisimple Lie algebra $\mathfrak{g}$ can be thought of as a real Lie algebra, which we will denote by $\mathfrak{g}^{\mathbb{R}}$. There is a correspondence between Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ and compact real forms of $\mathfrak{g}$: Given a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$, conjugation with respect to $\mathfrak{u}_0$ (thinking of $\mathfrak{g}$ as $\mathfrak{u}_0 \otimes \mathbb{C} = \mathfrak{u}_0 + i\mathfrak{u}_0$, this is literally just complex conjugation) is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$. Note that this involution, typically denoted by θ, satisfies $\theta(\xi) = \xi$ for all $\xi \in \mathfrak{u}_0$. It turns out that every Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ is given by conjugation with respect to a compact real form of $\mathfrak{g}$. That is to say, given a Cartan involution θ of a complex, semisimple Lie algebra $\mathfrak{g}$, the set

$$\mathfrak{u} = \{\xi \in \mathfrak{g} \,|\, \theta(\xi) = \xi\}$$

is a compact real form of $\mathfrak{g}$.

Any Cartan involution θ of $\mathfrak{g}^{\mathbb{R}}$ necessarily has eigenvalues ± 1. Therefore, θ induces an eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces with eigenvalues +1 and -1, respectively. This is called a *Cartan decomposition.* There is a corresponding decomposition of the Lie group G: Let K be the connected, simply-connected Lie group with Lie algebra $\mathfrak{k}$. Then the map from $K \times \mathfrak{p}$ to G given by

$$(k, X) \mapsto k \cdot \exp X$$

is a diffeomorphism onto. Furthermore, we have ([Kna02], p360)

1. K is closed,
2. K contains the center of G,
3. K is compact iff the center of G is finite,
4. when the center of G is finite, K is a maximal, compact subgroup of G.

B.4 Iwasawa Decompositions

The following description is summarized from [Kna02].

Given a matrix $X \in SL(n, \mathbb{C})$, applying the Gram-Schmidt orthogonalization process yields a unique decomposition $X = kan$, where $k \in SU(N)$, a is diagonal with real, positive entries, and n is upper-triangular with 1's on the diagonal. This decomposition generalizes to arbitrary semisimple Lie groups as follows.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. Since $\mathfrak{p}$ is finite-dimensional, there exists a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. The set $\{\mathrm{ad}_H \,|\, H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. It follows that $\mathfrak{g}$ is an orthogonal direct sum of simultaneous eigenspaces for this family of transformations. The corresponding eigenvalues can be identified with elements of $\mathfrak{a}^*$. For each $\lambda \in \mathfrak{a}^*$, let

$$g_\lambda = \{X \in \mathfrak{g} \,|\, (\mathrm{ad}_H)X \text{ for all } H \in \mathfrak{a}\}.$$

The set

$$\Sigma := \{\lambda \in \mathfrak{a} \,|\, \lambda \neq 0,\ \mathfrak{g}_\lambda \neq 0\}$$

constitutes a root system for the vector space $\mathfrak{g}$. It can be shown that $\mathfrak{g}$ decomposes as the orthogonal direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

Choose a set of positive roots $\Sigma^+ \subset \Sigma$, and let

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda.$$

Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$.

Proposition B.2 (Iwasawa Decomposition)**.** *The Lie algebra* $\mathfrak{g}$ *decomposes as the direct sum* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Note that this is a vector space direct sum, not a Lie algebra direct sum.

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then exponentiating an Iwasawa decomposition of $\mathfrak{g}$ yields a corresponding Lie group decomposition,

$$G = KAN,$$

where K is compact, A is abelian, and N is unipotent. This decomposition is unique, so that $G \cong K \times A \times N$.

B.5 Invariant Measures

Let G be an m-dimensional Lie group. There exists a nonvanishing, left-invariant m-form ω on G. Using ω, one can construct a nonzero Borel measure $d\mu_l$, called a *left Haar measure*, that is invariant under left-translation (for any Borel set E, $d\mu_l(L_g E) = d\mu_l(E)$ for all $g \in G$). Any two left Haar measures are proportional [Kna02].

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