Elasticity Theory and Finite Elements

Using a Porous Media Formulation

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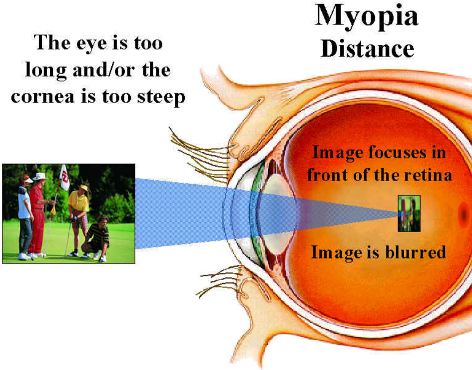
Table of Contents

A Brief Overview of Myopia

- Distant images focus anterior to the retinal plane (nearsightedness)
- Affects 33.1% of Americans over the age of 20

- Causes: Environmental and behavioral factors AND genetics
- Mechanism: Pressure causes soft tissue deformation and modeling of extracellular matrix



http://www.purevisionmethod.com/tag/high-myopia/

Classical Mechanics

- Statistical mechanics
  - Follows individual particles (small length scale)
  - A HUGE number of items to track
  - Uses probability theory to relate the microscale to macroscopic properties of the system
- Continuum mechanics
  - Follows material at a continuum level (large length scale)
  - Locally averaged properties
  - Fundamental laws of physics—conservation of mass, momentum, energy

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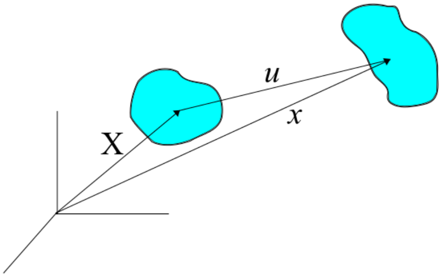
Preliminaries

- Elasticity
  - Reversible deformation
  - Ex. spring, rubber ball
  - Opposite of plasticity (ex. crumpled cars from a collision)
- Stress ("pressure")
  - Internal pressures that neighboring particles of a continuum exhibit upon one another
  - Dimension$=\dfrac{\text{force}}{\text{area}}$ $$\sigma = \frac{F}{A}$$
- Strain ("stretch")
  - Measurement of change in length relative to reference configuration length
  - Dimensionless $$\varepsilon = \frac{\Delta L}{L}$$

Linear Elastic Model: Kinematics



A point moves from undeformed configuration X to deformed configuration x over a period of time t

- Lagrangian formulation: $x(X, t)$ (we follow the mass as it moves)
- Displacement: $u = x - X$

Linear Elastic Model: Equilibrium equation

Conservation of momentum

- Newton's second law ($\sum F = ma$)
  - Per unit volume ($\sum \frac{F}{V} = \frac{ma}{V} = \rho a$)
  - Assume quasistatic, so $\rho a = 0$

Linear Elastic Model: Equilibrium equation

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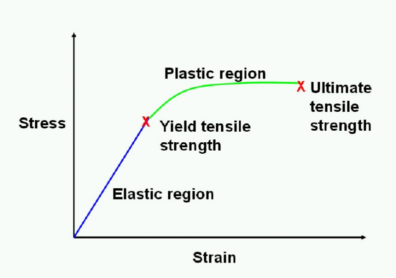
- Newton's second law ($\sum F = ma$)
  - Per unit volume ($\sum \frac{F}{V} = \frac{ma}{V} = \rho a$)
  - Assume quasistatic, so $\rho a = 0$
- Then for stress σ

$$\nabla \cdot \sigma = \frac{\partial \sigma}{\partial X} = 0$$

  - Recall $\sigma = \frac{\text{force}}{\text{area}}$, so then $\frac{\partial \sigma}{\partial X} = \frac{\text{force}}{\text{volume}}$

Linear Elastic Model: Constitutive equations

- Stress (σ)-strain (ε) relationship



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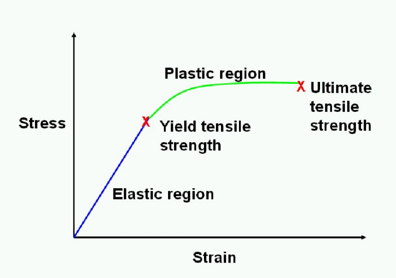

- The elastic region acts like a spring—
  - Compare to $F = -k\Delta x$
    The amount of force applied to the spring determines the amount of displacement

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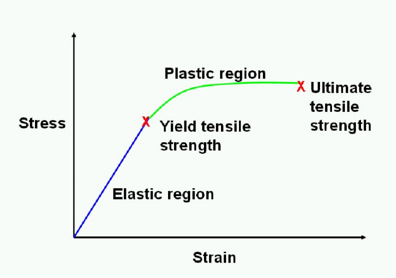

- The elastic region acts like a spring—
  - Compare to $F = -k\Delta x$
    The amount of force applied to the spring determines the amount of displacement
- Assume small deformation
- Assume linear with elastic modulus E

$$\sigma = E\varepsilon$$

Linear Elastic Model: Constitutive equations

$$\frac{\partial \sigma}{\partial X} = 0$$

$$\sigma = E\varepsilon$$

- Strain (ε)-displacement(u) relationship

$$\varepsilon = \frac{\partial u}{\partial X} \qquad \left(= \frac{\Delta L}{L} \right)$$

Linear Elastic Model: Constitutive equations

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- Strain (ε)-displacement(u) relationship

$$\varepsilon = \frac{\partial u}{\partial X} \qquad \left(= \frac{\Delta L}{L}\right)$$

- Combining the equilibrium equation and constitutive equations, we obtain

$$Eu_{XX} = 0$$

Linear Elastic Model: Constitutive equations

- Theory of porous media first developed for analysis of wet soil (1941)
- Porous media applied to biomechanics (1980s)
- More complicated models developed—mixed porohyperelastic model with transport and swelling (MPHETS)
- 1D model of MPHETS with growth and modeling—stress causes model to add mass (2012)

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- **Immediate goal:** Develop axisymmetric model of poroelasticity
- **Long-term goal:** Connect axisymmetric model with growth and modeling

Modeling Soft Tissue as Porous Media

A saturated sponge:

+

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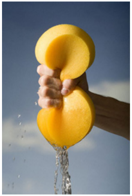


Sponge:
incompressible
elastic material

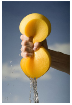
Fluid bath:
incompressible

Sponge + Fluid:
compressible
biphasic material
continuum approach

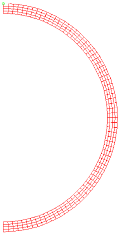
- The stress on the solid balances external stresses plus the fluid pore pressure
- Fluid pore pressure creates fluid flow via Darcy's Law

www.clay-king.com, www.purelightchiro.com, shinmike.wordpress.com

Modeling Soft Tissue as Porous Media

 +  = 

Idea: Add pressure, the material deforms

 $\Rightarrow$

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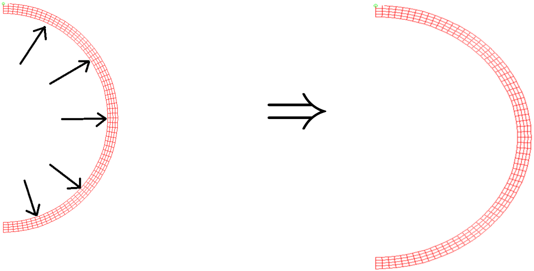

Linear Poroelastic Model: Governing equations

- Conservation of total momentum, quasi steady-state

$$\frac{\partial \sigma}{\partial X} = 0$$

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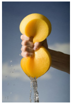
 $+$  $=$ 

- Conservation of total momentum, quasi steady-state

$$\frac{\partial \sigma}{\partial X} = 0$$

- Assume effective stress principle holds:

$$\sigma^{\text{eff}} = \sigma + p^f$$

$$\sigma = \sigma^{\text{eff}} - p^f$$

for solid stress σ^{eff}, total stress σ, and pore fluid pressure p^f

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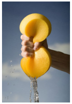
 $+$  $=$ 

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- Linear stress-strain relationship, with elastic modulus E

$$\sigma^{\text{eff}} = E\varepsilon$$

- Strain-displacement $$\varepsilon = \frac{\partial u}{\partial X}$$

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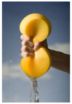
 +  = 

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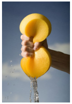
 $+$  $=$ 

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- Combining these, we obtain

$$Eu_{XX} - p^f_X = 0$$

Linear Poroelastic Model: Governing equations

- Incompressibility constraint for total material velocity v

$$\nabla \cdot v = 0$$

- Split into apparent relative fluid velocity v^{fr} and solid velocity v^s

$$\nabla \cdot (v^{fr} + v^s) = \nabla \cdot v^{fr} + \nabla \cdot v^s = 0.$$

  - Assume Darcy's law holds, with permeability k

$$v^{fr} = -k \frac{\partial p^f}{\partial X}.$$

  - Solid velocity is solid displacement derivative in time:

$$v^s = \frac{du}{dt}$$

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$$-kp^f_{XX} + u_{Xt} = 0,$$

Linear Poroelastic Model: Governing equations

- Conservation of total momentum

$$-kp^f_{XX} + u_{Xt} = 0$$

- Incompressibility constraint

$$Eu_{XX} - p^f_X = 0.$$

Linear Poroelastic Model: Special Solution

- We can simplify these to one equation

$$\begin{cases} -kp^f_{XX} + u_{Xt} = 0 \\ Eu_{XX} - p^f_X = 0 \end{cases}$$

$$\implies p^f_X = Eu_{XX}$$

Linear Poroelastic Model: Special Solution

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Linear Poroelastic Model: Special Solution

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$$\implies p^f_X = Eu_{XX}$$

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- So we obtain

$$-kEu_{XXX} + u_{Xt} = 0$$

- Integrating in X,

$$-kEu_{XX} + u_t = C,$$

for some constant of integration C
$\implies$ For the simplified model, displacement u is diffusive.

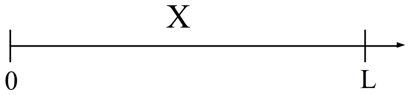
A Brief History of Finite Elements[1]

- The term "finite element" was first coined by Clough in 1960.
- In the early 1960s, engineers used the method for approximate solutions of problems in stress analysis, fluid flow, heat transfer, and other areas.
- The first book on the FEM by Zienkiewicz and Chung was published in 1967.
- In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems.
- Most commercial FEM software packages originated in the 1970s. (ABAQUS, Ansys, etc.)

[1] http://web.mit.edu/16.810/www/16.810_L4_CAE.pdf

Overview: Finite Element Method

- Variational/“weak” formulation of the problem



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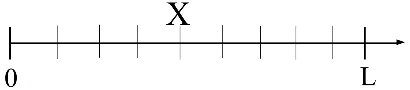

- Mesh system into elements; choose interpolation functions

Overview: Finite Element Method

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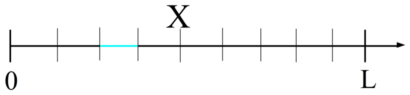

- Mesh system into elements; choose interpolation functions
- Set up element vectors and matrices (apply BCs)

Overview: Finite Element Method

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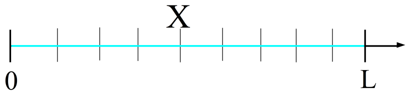

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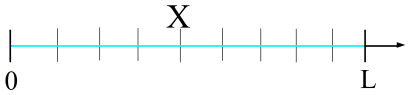

- Mesh system into elements; choose interpolation functions
- Set up element vectors and matrices (apply BCs)
- Assemble global vectors and matrices
- Solve a system of linear equations

$$[K]\mathbf{u} = \mathbf{f}$$

  - $[K]$ stiffness matrix
  - $\mathbf{u}$ displacement vector
  - $\mathbf{f}$ forcing vector

Formulating Finite Element in 1D

- We use a "weak" formulation to create a functional
  - Others: energy formulation, principle of virtual work
- Begin with our main equation

$$\nabla_0 \cdot \sigma = \frac{\partial \sigma}{\partial X} = 0$$

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$$\nabla_0 \cdot \sigma = \frac{\partial \sigma}{\partial X} = 0$$

- Multiply this by a test function w (also called weighting function)

$$w \left(\frac{\partial \sigma}{\partial X} \right) = 0$$

- Integrate over the domain $\Omega = [0, L]$

$$\int_\Omega w \left(\frac{\partial \sigma}{\partial X} \right) d\Omega = 0$$

Formulating Finite Element in 1D

Integration by parts

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

- Integrate by parts

$$\int_\Omega w \left(\frac{\partial \sigma}{\partial X}\right) d\Omega = w\sigma\Big|_\Gamma - \int_\Omega \sigma \left(\frac{\partial w}{\partial X}\right) d\Omega = 0$$

- Rearranging,

$$\int_\Omega \sigma \left(\frac{\partial w}{\partial X}\right) d\Omega = w\sigma\Big|_\Gamma$$

- The left hand side will be the "stiffness matrix"

Derivation of Stiffness Matrix in 1D

$$\int_{\Omega} \left(\frac{\partial w}{\partial X} \right) \sigma d\Omega$$

- Now, recall $\sigma = E\varepsilon$

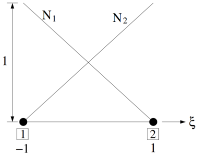
$$\int_{\Omega} \left(\frac{\partial w}{\partial X} \right) E\varepsilon d\Omega$$

- And $\varepsilon = \dfrac{\partial u}{\partial X}$

$$\int_{\Omega} \left(\frac{\partial w}{\partial X} \right) E \frac{\partial u}{\partial X} d\Omega$$

Interpolation Functions

- Isoparametric linear shape functions on coordinates [-1,1]



$$N_1(\xi) = -\frac{1}{2}(\xi - 1)$$
$$N_2(\xi) = \frac{1}{2}(1 - \xi)$$

- Interpolate by linear combination of function values on the nodes

$$x(\xi) = \sum_{i=1}^{2} N_i(\xi) x_{ei} = N^T x_e$$

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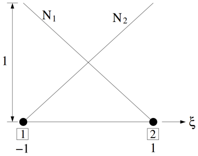

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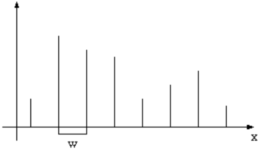

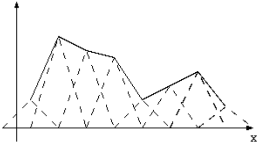

http://www.imechanica.org/files/femintro.pdf, http://idav.ucdavis.edu/ okreylos/

Interpolation Functions

- Represent displacement and test function by linear combination of shape functions
- On each element, we have

$$u(\xi) = \sum_{i=1}^{2} N_i(\xi) u_{ei} = N^T u_e$$

$$w(\xi) = \sum_{i=1}^{2} N_i(\xi) w_{ei} = N^T w_e$$

for

$$N^T = \begin{bmatrix} N_1(\xi) \\ N_2(\xi) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\xi - 1) \\ \frac{1}{2}(1 - \xi) \end{bmatrix}$$

$$u_e = \begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix}, \qquad w_e = \begin{bmatrix} w_{e1} \\ w_{e2} \end{bmatrix}$$

Derivation of Stiffness Matrix

$$\int_\Omega \left(\frac{\partial w}{\partial X}\right) E \frac{\partial u}{\partial X} d\Omega$$

$$u(\xi) = u_{ei} = N^T u_e, \qquad w(\xi) = N^T w_e$$

- Substituting in the shape functions yields

$$\int_\Omega \left(\frac{\partial N^T w_e}{\partial X}\right)^T E \frac{\partial N^T u_e}{\partial X} d\Omega$$

- But w_e, u_e are constant, so can pull them out of derivative and integral

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- But w_e, u_e are constant, so can pull them out of derivative and integral $\Rightarrow$ stiffness matrix!

$$w_e^T \int_\Omega \left(\frac{\partial N}{\partial X}\right) E \frac{\partial N^T}{\partial X} d\Omega u_e = w_e^T [K_e] u_e$$

Weak Formulation in Higher Dimensions

- Begin with our main equation

$$\nabla \cdot \sigma = 0$$

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$$\int_{\Omega} \mathbf{w} \cdot (\nabla \cdot \sigma) d\Omega = 0$$

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$$\int_{\Omega} \mathbf{w} \cdot (\nabla \cdot \sigma) d\Omega = 0$$

- Note that

$$\mathbf{w} \cdot \nabla \cdot \sigma = (\nabla \cdot \sigma) \cdot \mathbf{w} + (\nabla \mathbf{w})^T : \sigma,$$

  for the dyadic product (or double dot product) of two tensors $\mathbf{A}$ and $\mathbf{B}$ defined as $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A} \cdot \mathbf{B}) = A_{ij}B_{ij}$

Weak Formulation in Higher Dimensions

- Then we have

$$\int_{\Omega} (\nabla \cdot (\sigma \cdot \mathbf{w}))\, d\Omega - \int_{\Omega} \left((\nabla \mathbf{w})^T : \sigma\right) d\Omega = 0$$

Weak Formulation in Higher Dimensions

- Then we have

$$\int_{\Omega} (\nabla \cdot (\sigma \cdot \mathbf{w}))\, d\Omega - \int_{\Omega} \left((\nabla \mathbf{w})^T : \sigma \right) d\Omega = 0$$

Theorem (Divergence)

$$\iiint (\nabla \cdot \mathbf{T}) dV = \iint (\mathbf{n} \cdot \mathbf{T}) dA$$

- Use the divergence theorem on first integral

$$\int_{\Gamma} \mathbf{w} \cdot (\sigma \cdot \mathbf{n}) d\Gamma - \int_{\Omega} \left((\nabla \mathbf{w})^T : \sigma \right) d\Omega = 0$$

Weak Formulation in Higher Dimensions

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- Use the divergence theorem on first integral

$$\int_\Gamma \mathbf{w} \cdot (\sigma \cdot \mathbf{n}) d\Gamma - \int_\Omega \left((\nabla \mathbf{w})^T : \sigma\right) d\Omega = 0$$

- By symmetry of Cauchy stress tensor, we can write

$$(\nabla \mathbf{w})^T : \sigma = \frac{1}{2}\left[(\nabla \mathbf{w}) + (\nabla \mathbf{w})^T\right] : \sigma \doteq (\varepsilon^w)^T \sigma$$

Weak Formulation in Higher Dimensions

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Weak Formulation in Higher Dimensions

- Then we have

$$\int_\Gamma \mathbf{w} \cdot (\sigma \cdot \mathbf{n}) d\Gamma - \int_\Omega (\varepsilon^w)^T \sigma d\Omega = 0$$

- Rearranging and using $\sigma = E\varepsilon$

$$\int_\Omega (\varepsilon^w)^T E \varepsilon d\Omega = \int_\Gamma \mathbf{w} \cdot (\sigma \cdot \mathbf{n}) d\Gamma$$

- The left hand side will be our stiffness matrix.

Interpolation functions

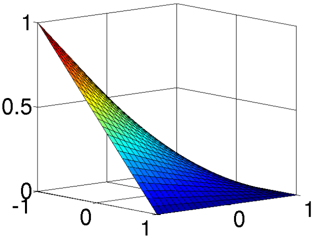
- Bilinear shape functions with respect to isoparametric coordinates $\xi, \eta \in [-1, 1]$

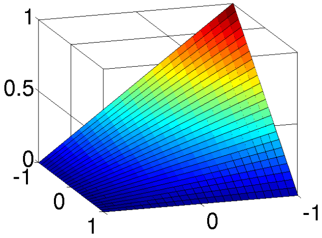
$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$N_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$

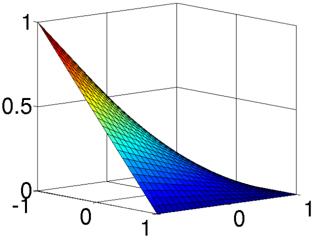
$$N_4(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

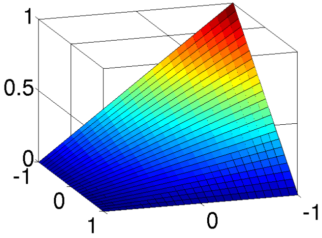




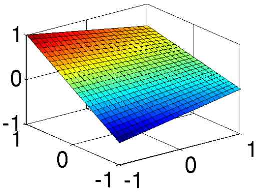
Interpolation functions

- Use isoparametric bilinear shape functions to interpolate





- Interpolate using weighted sum of four shape functions, e.g.



Derivation of Stiffness Matrix

$$\int_\Omega (\varepsilon^w)^T E \varepsilon d\Omega$$

- Recall the shape functions interpolate displacement over the nodes of the element

$$u|_{\Omega_e} = \sum_{j=1}^{\text{dof}} \sum_{i=1}^{\text{nodes}} N_{ij}(r,z) u_{eij} = N^T \mathbf{u_e}$$

- For an individual node in axisymmetry, the displacements are

$$\mathbf{u_{ei}} = (u_{ei}, w_{ei})^T$$

- In axisymmetry, the strain-displacement relationship is written

$$\varepsilon = \begin{pmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_{zr} \\ \varepsilon_\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \frac{u}{r} \end{pmatrix} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{bmatrix} \begin{pmatrix} N_i u_{ei} \\ N_i w_{ei} \end{pmatrix}$$

Derivation of Stiffness Matrix

- Define operator $[\hat{B}]$

$$\varepsilon_i = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{bmatrix} \begin{pmatrix} N_i u_{ei} \\ N_i w_{ei} \end{pmatrix} = [\hat{B}] N_i \begin{pmatrix} u_{ei} \\ w_{ei} \end{pmatrix}$$

- Write the nodal dof for the element in a vector

$$\mathbf{u_e} = [u_{e1} \quad w_{e1} \quad u_{e2} \quad w_{e2} \quad \cdots \quad u_{en} \quad w_{en}]^T$$

- Define operator $[B]$ for strain-displacement relationship on an entire element (note derivatives are of interpolation functions)

$$[B] = [\hat{B} N_1 \quad \cdots \quad \hat{B} N_n]$$

$$\varepsilon = [B]\mathbf{u_e}$$

- Finally, we obtain the stiffness matrix

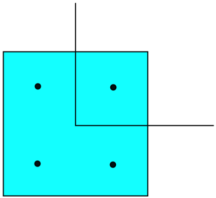
$$\mathbf{w_e}^T \int_\Omega [B]^T E [B] d\Omega \mathbf{u_e} = \mathbf{w_e}^T [K_e] \mathbf{u_e}$$

Numerical Integration

- Use Gaussian quadrature for numerical integration:

$$\iint f(x,y)dxdy \approx \sum_{i=1}^{n} w_i f(x_i, y_i)$$

Numerical Integration



- Use Gaussian quadrature for numerical integration:

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- Here, we use 2×2 Gaussian quadrature
  - Four integration points located at $(\xi, \eta) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$
  - Weights $w_i = 1$ for $i = 1, \ldots, 4$
- Map to and from isoparametric coordinate system; this involves the Jacobian in the integral.

$$\iint f(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i) j(\xi_i, \eta_i)$$

Boundary Conditions

$$[K_e]\mathbf{u_e} = \mathbf{f_e}$$

- Essential boundary conditions
- Natural boundary conditions

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Boundary Conditions

$$[K_e]\mathbf{u_e} = \mathbf{f_e}$$

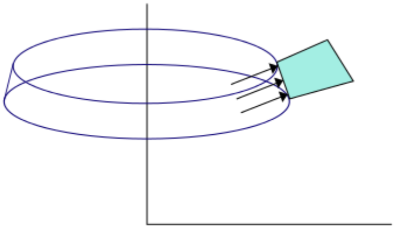
- Essential boundary conditions
  - Dirichelet/"displacement"/$u(x) = \alpha$
  - Appear in the u vector
- Natural boundary conditions
  - Neumann/"nodal forcing"
  - Appear in the f vector/$u_X(x) = \alpha$

Neumann Boundary Conditions in Axisymmetry

- Coordinate system (r, z, θ); integration along a surface of revolution
- Pressure enters as nodal forcing. For uniform pressure Φ and $\bar{N}$ indicating shape functions evaluated along the boundary

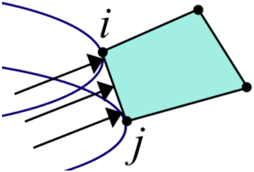
$$f_s = \int_0^{2\pi} \int_\Gamma [\bar{N}]^T \Phi r d\Gamma d\theta$$

$$f_s = 2\pi \int_\Gamma [\bar{N}]^T \Phi r d\Gamma$$



Neumann Boundary Conditions in Axisymmetry

$$f_s = 2\pi \int_\Gamma [\bar{N}]^T \Phi r d\Gamma,$$



- $\bar{N}$ is the shape function along the boundary between $l_i = (r_i, z_i)$ and $l_j = (r_j, z_j)$
- The boundary of a bilinear shape function is just a linear shape function:

$$\bar{N}_1 = \frac{l_j - l}{l_j - l_i} \qquad \bar{N}_2 = \frac{l - l_i}{l_j - l_i}$$

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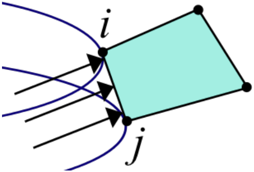

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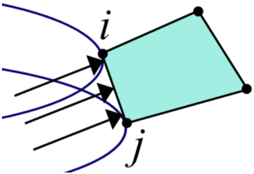
- Express r as a sum of shape functions:

$$r = \bar{N}_1 r_i + \bar{N}_2 r_j$$

Neumann Boundary Conditions in Axisymmetry

$$
\begin{aligned}
f_s &= 2\pi\Phi \int_\Gamma [\bar{N}]^T (r) d\Gamma \\
&= 2\pi\Phi \int_\Gamma [\bar{N}]^T (\bar{N}_1 r_i + \bar{N}_2 r_j) d\Gamma
\end{aligned}
$$



- Parameterizing the curve along the boundary Γ between points $l_i = (r_i, z_i)$ and $l_j = (r_j, z_j)$ gives the contribution along the line

$$
f_s = 2\pi\Phi \int_{l_i}^{l_j} \begin{bmatrix} \dfrac{l_j - l}{l_j - l_i} \\ \dfrac{l - l_i}{l_j - l_i} \end{bmatrix} \left(\frac{l_j - l}{l_j - l_i} r_i + \frac{l - l_i}{l_j - l_i} r_j \right) dl
$$

Neumann Boundary Conditions in Axisymmetry

$$f_s = 2\pi\Phi \int_{l_i}^{l_j} \begin{bmatrix} \dfrac{l_j - l}{l_j - l_i}\left(\dfrac{l_j - l}{l_j - l_i} r_i + \dfrac{l - l_i}{l_j - l_i} r_j\right) \\ \dfrac{l - l_i}{l_j - l_i}\left(\dfrac{l_j - l}{l_j - l_i} r_i + \dfrac{l - l_i}{l_j - l_i} r_j\right) \end{bmatrix} dl$$

Neumann Boundary Conditions in Axisymmetry

$$f_s = 2\pi\Phi \int_{l_i}^{l_j} \begin{bmatrix} \frac{l_j - l}{l_j - l_i}\left(\frac{l_j - l}{l_j - l_i} r_i + \frac{l - l_i}{l_j - l_i} r_j\right) \\ \frac{l - l_i}{l_j - l_i}\left(\frac{l_j - l}{l_j - l_i} r_i + \frac{l - l_i}{l_j - l_i} r_j\right) \end{bmatrix} dl$$

- Use Mathematica and...

$$f_s = 2\pi\Phi \begin{bmatrix} \frac{1}{3}(l_j - l_i) r_i + \frac{1}{6}(l_j - l_i) r_j \\ \frac{1}{6}(l_j - l_i) r_i + \frac{1}{3}(l_j - l_i) r_j \end{bmatrix}$$

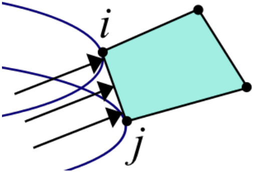
- Note that $(l_j - l_i)$ is the length of the line, denote l_{ij}

$$f_s = \frac{2\pi\Phi l_{ij}}{6} \begin{bmatrix} 2r_i + r_j \\ r_i + 2r_j \end{bmatrix}$$

- Interpret this as the contribution of force at point i, j

Neumann Boundary Conditions in Axisymmetry

$$f_s = 2\pi \int_{\Gamma} [\bar{N}] \Phi r d\Gamma,$$



- The length of the element edge is given by

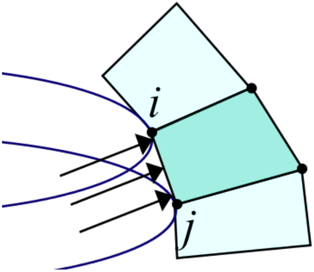
$$l_{ij} = \sqrt{(r_i - r_j)^2 + (z_i - z_j)^2}$$

- Then the contribution at point i is

$$f_{si} = \pi l_{ij} \left(\frac{2r_i + r_j}{3} \right)$$

Neumann Boundary Conditions in Axisymmetry

$$f_{si} = \pi l_{ij} \left(\frac{2r_i + r_j}{3} \right)$$

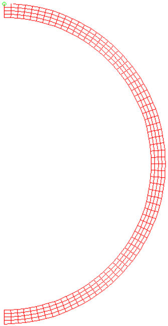


- Use the fact that pressure is normal to the surface to decompose f_{si} into radial f_{sir} and axial f_{siz} components
- Sum up components over nodes
- This becomes our forcing vector

$$[\mathbf{K}]\mathbf{u} = \mathbf{f}$$

Numerical Results: Linear Elasticity, Small Strain

- Results compared to ABAQUS (commercial software) to validate
- Boundary conditions
  - Pin top node (prevent rigid body motion)
  - Pressure $\Phi = 3333.1 Pa$ (normal human eye pressure)
  - Young's modulus $E = 3 \times 10^6 Pa$
  - Compressibility (Poisson's ratio) $\nu = .4$



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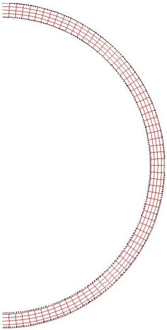

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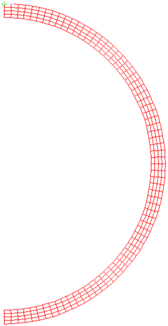

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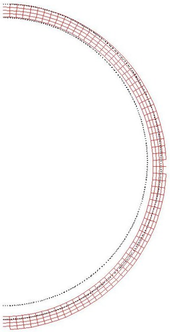

Summary: Finite Elements

- Pros:
  - Can deal with complex geometries
  - Axisymmetry allows much smaller computational time
  - Smaller mesh size, higher order elements can yield more accurate solution
- Cons:
  - Bookkeeping of FE is complicated
  - Axisymmetry means problem needs to be formulated in cylindrical coordinates; this is less intuitive than Cartesian coordinates

Thank You!

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