The Stiff Limit of the Exponential Rosenbrock-Type Method

Michelle Hine

Advisor: Dr. Stepanov

Program in Applied Mathematics
University of Arizona

May 13, 2011
Second Semester Term Paper Conference

Outline

Stiff Decay

- An system is considered 'stiff' if the ODE contains multiple time scales, with both rapidly oscillating modes and slow dynamics.
- After enough time has passed, a larger step size is sufficient to accurately capture the behavior of the system.
- To determine whether a problem has stiff decay, we consider the eigenvalues of the Jacobian from the equation of motion.
- Problems exhibiting stiff decay can have
    - a mixture of eigenvalues with large, negative real parts and eigenvalues with small negative real parts, or
    - eigenvalues that are large and almost purely imaginary.

The Exponential Rosenbrock-type Method

- The first prototypes of exponential integrators have existed for nearly 50 years.
- Historically, most exponential-style integrators linearized the equation at the initial time step only.
- Recently, people began to linearize the problem with each time step, resulting in a much more accurate solution curve.

- The exponential Rosenbrock-type integrator is an explicit one-step method.
- With each step, the method linearizes the equation, using a Taylor expansion.

$$\begin{aligned}\mathbf{y}' &= f(\mathbf{y})\\ &\approx f(\mathbf{y}_{k-1}) + \frac{df}{d\mathbf{y}}\Big|_{\mathbf{y}=\mathbf{y}_{k-1}}(\mathbf{y}-\mathbf{y}_{k-1})\\ &= f(\mathbf{y}_{k-1}) + J(\mathbf{y}-\mathbf{y}_{k-1})\end{aligned}$$

- Using the matrix exponential, the exponential Rosenbrock-type integrator solves the linear equation exactly, then steps forward by the exact solution of the linear equation.

- The solution to the linear equation is

$$\mathbf{y}_k = e^{(t-t_{k-1})J}\mathbf{y}_{k-1} + \int_{t_{k-1}}^{t_k} e^{(t_k-\tau)J} d\tau \left(f(\mathbf{y}_{k-1}) - J\mathbf{y}_{k-1}\right)$$

- Assuming the inverse Jacobian exists, one can use J^{-1} to calculate the integral of the matrix exponential, and for $h = t - t_{k-1}$, the equation becomes

$$\begin{aligned}\mathbf{y}(t) &= e^{hJ}\mathbf{y}_{k-1} + (e^{hJ} - I)J^{-1}\left(f(\mathbf{y}_{k-1}) - J\mathbf{y}_{k-1}\right) \\ &= e^{hJ}\mathbf{y}_{k-1} + (e^{hJ} - I)J^{-1}f(\mathbf{y}_{k-1}) - (e^{hJ} - I)\mathbf{y}_{k-1}\end{aligned}$$

The Exponential Rosenbrock-Type Method

$$\mathbf{y}(t) = \mathbf{y}_{k-1} + (e^{hJ} - I)J^{-1}f(\mathbf{y}_{k-1})$$

Standard Methods to Test Stiff Decay

- Consider testing the exponential Rosenbrock-type method with the autonomous 'test' equation $\mathbf{y}' = \lambda \mathbf{y}$.
- For $z = h\lambda$, we obtain $\mathbf{y}_k = e^z \mathbf{y}_{k-1}$, so the method solves the linear autonomous equation exactly.
- Hence, Ascher and Petzold's approach to determining stiff decay is useless here, since the stiff decay property for the linear autonomous equation is trivial.
- Instead, we test the method on systems with stiff decay.
- To achieve stiff decay, we use test equations containing both fast and slow modes.

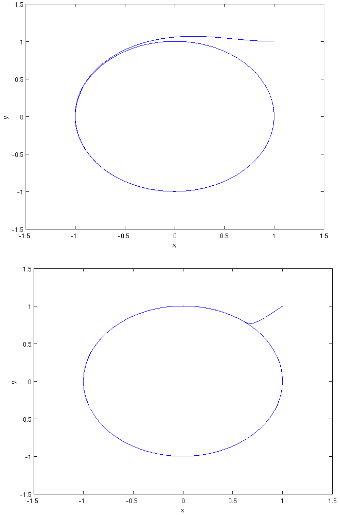
Standard Methods to Test Stiff Decay

- Consider testing the exponential Rosenbrock-type method with the autonomous ‘test’ equation $\mathbf{y}' = \lambda \mathbf{y}$.
- For $z = h\lambda$, we obtain $\mathbf{y}_k = e^z \mathbf{y}_{k-1}$, so the method solves the linear autonomous equation exactly.
- Hence, Ascher and Petzold’s approach to determining stiff decay is useless here, since the stiff decay property for the linear autonomous equation is trivial.
- Instead, we test the method on systems with stiff decay.
- To achieve stiff decay, we use test equations containing both fast and slow modes.

- For the first test equation, consider a particle with some initial position.

$$\begin{cases} \dot{x} = -y - \lambda x(x^2 + y^2 - 1) \\ \dot{y} = \quad x - \lambda y(x^2 + y^2 - 1) \end{cases}$$

- The decay to the circle provides the fast mode, then counter-clockwise rotation acts as the slow mode.
- Higher values of λ correspond with stiffer decay
    - Top figure: $\lambda = 1$, bottom figure: $\lambda = 20$

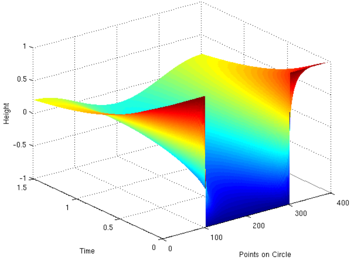


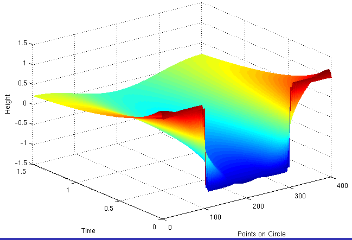
- For the second test equation, consider points a circle with diffusion or hyperviscosity and nonlinear forcing:

$$u_t = u_{xx} - u^3$$

$$u_t = u_{xxxx} - u^3$$

- Finite differences are used for the derivative in x.
- A larger number of points corresponds with greater stiffness.
- Figures show 400 points, $t = 1.5$.





Properties of a Method Exhibiting Stiff Decay

- If the method properly handles stiff decay, it will traverse the slow dynamics with a step size that is not small.
- The exponential Rosenbrock-type method allows the fast modes to continue to decay as the system travels through the slow dynamics.
- Once the system has passed through the region of rapid decay, the step size h should be prescribed by the slow dynamics of the system.
- Ideally, the step size depends on the tolerance level set for the solver, and is not based on the stiffness of the problem.

Analysis of Step Size Choice

- We apply the exponential Rosenbrock-type method to the equation $\mathbf{y}' = \lambda \mathbf{F} + \mathbf{f} = \mathscr{F}$
  - $\mathbf{F}$ denotes the fast decaying modes of the system.
  - $\mathbf{f}$ denotes the slow modes of the system.
  - By assumption, $\mathbf{f}$ is not large in λ.
- Then for the Jacobian we have $\lambda J + j = \mathscr{J}$

- For simplicity, consider the method beginning at time 0 with a step size of h.
- Apply one step of h to obtain $\mathbf{y}_k$ and two steps of $h/2$ to obtain $\bar{\mathbf{y}}_k$.

$$\mathbf{y}_k = \mathbf{y}_{k-1} + \int_0^h e^{\tau \mathscr{J}_{k-1}} \mathscr{F}_{k-1} d\tau$$

$$\begin{aligned} \mathbf{y}_{k-1/2} &= \mathbf{y}_{k-1} + \int_0^{h/2} e^{\tau \mathscr{J}_{k-1}} \mathscr{F}_{k-1} d\tau \\ \bar{\mathbf{y}}_k &= \mathbf{y}_{k-1/2} + \int_0^{h/2} e^{\tau \mathscr{J}_{k-1/2}} \mathscr{F}_{k-1/2} d\tau \end{aligned}$$

Let $\mathscr{J} \triangleq \mathscr{J}_{k-1}$, $\mathscr{F} \triangleq \mathscr{F}_{k-1}$, $\tilde{\mathscr{J}} \triangleq \mathscr{J}_{k-1/2}$, and $\tilde{\mathscr{F}} \triangleq \mathscr{F}_{k-1/2}$.

- For simplicity, consider the method beginning at time 0 with a step size of h.
- Apply one step of h to obtain $\mathbf{y}_k$ and two steps of $h/2$ to obtain $\bar{\mathbf{y}}_k$.

$$\mathbf{y}_k = \mathbf{y}_{k-1} + \int_0^h e^{\tau \mathscr{J}} \mathscr{F} d\tau$$

$$\begin{aligned} \mathbf{y}_{k-1/2} &= \mathbf{y}_{k-1} + \int_0^{h/2} e^{\tau \mathscr{J}} \mathscr{F} d\tau \\ \bar{\mathbf{y}}_k &= \mathbf{y}_{k-1/2} + \int_0^{h/2} e^{\tau \tilde{\mathscr{J}}} \tilde{\mathscr{F}} d\tau \end{aligned}$$

Let $\mathscr{J} \triangleq \mathscr{J}_{k-1}$, $\mathscr{F} \triangleq \mathscr{F}_{k-1}$, $\tilde{\mathscr{J}} \triangleq \mathscr{J}_{k-1/2}$, and $\tilde{\mathscr{F}} \triangleq \mathscr{F}_{k-1/2}$.

- Then subtracting the two methods yields equation (4).

$$\mathbf{y}_k - \bar{\mathbf{y}}_k = \left[\mathbf{y}_{k-1} + \left(\int_0^h e^{\tau\mathscr{J}}\,d\tau\right)\mathscr{F}\right] - \left[\mathbf{y}_{k-1} + \left(\int_0^{h/2} e^{\tau\mathscr{J}}\,d\tau\right)\mathscr{F} + \left(\int_0^{h/2} e^{\tau\tilde{\mathscr{J}}}\,d\tau\right)\tilde{\mathscr{F}}\right]$$

- One can split the first integral into two pieces, $\int_0^{h/2} e^{\tau\mathscr{J}}\,d\tau$ and $\int_{h/2}^{h} e^{\tau\mathscr{J}}\,d\tau$.

- Consider the Taylor expansion of $\tilde{\mathscr{F}}$, where the ellipses represent the nonlinear terms.

$$\begin{aligned}\tilde{\mathscr{F}} &= \mathscr{F} + \mathscr{J}\left(\int_0^{h/2} e^{\tau \mathscr{J}}\right)\mathscr{F} + \dots \\ &= \left[I + \mathscr{J}\left(\int_0^{h/2} e^{\tau \mathscr{J}}\right)\right]\mathscr{F} + \dots\end{aligned}$$

- One can show that that $I + \mathscr{J}\left(\int_0^{h/2} e^{\tau \mathscr{J}}\right) = e^{\frac{h}{2}\mathscr{J}}$.
- Using this identity and solving for the nonlinear terms allows the expression for $\tilde{\mathscr{F}}$ to be rewritten as

$$\tilde{\mathscr{F}} = e^{\frac{h}{2}\mathscr{J}}\mathscr{F} + \left(\tilde{\mathscr{F}} - e^{\frac{h}{2}\mathscr{J}}\mathscr{F}\right)$$

- Finally, the method bounds the difference $|\mathbf{y} - \bar{\mathbf{y}}|$ by the tolerance level, TOL.

$$|\mathbf{y} - \bar{\mathbf{y}}| = \left| \left[\int_0^{h/2} \left(e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}} \right) d\tau \right] e^{\frac{h}{2} \mathscr{J}} \mathscr{F} + \left[\int_0^{h/2} e^{\tau \tilde{\mathscr{J}}} d\tau \right] \left(e^{\frac{h}{2} \mathscr{J}} \mathscr{F} - \tilde{\mathscr{F}} \right) \right| \leq \text{TOL}$$

- There are two main sources of difference between the methods used to obtain $\mathbf{y}$ and $\bar{\mathbf{y}}$: the difference in the exponential terms due to rotation of the Jacobian matrix, and the nonlinear terms of the expansion for $\tilde{\mathscr{F}}$.

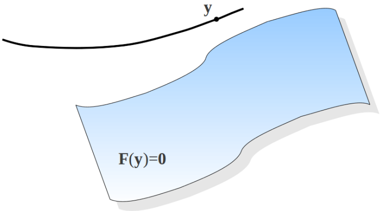
$$|\mathbf{y}-\bar{\mathbf{y}}| = \left| \left[\int_0^{h/2} \left(e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}} \right) d\tau \right] e^{\frac{h}{2}\mathscr{J}} \mathscr{F} + \left[\int_0^{h/2} e^{\tau \tilde{\mathscr{J}}} d\tau \right] \left(e^{\frac{h}{2}\mathscr{J}} \mathscr{F} - \tilde{\mathscr{F}} \right) \right| \leq \text{TOL}$$

- In the linear case, $\tilde{\mathscr{F}}$ exactly equals $e^{\frac{h}{2}\mathscr{J}} \mathscr{F}$.
- For the purposes of this analysis, we assume the nonlinear terms in the expansion are small, so the second term of the expression can be neglected.
- Then the difference becomes

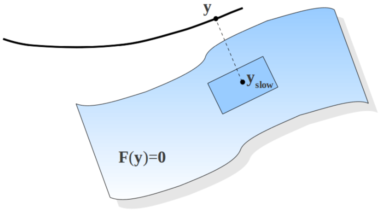
$$|\mathbf{y}-\bar{\mathbf{y}}| = \left| \int_0^{h/2} \left(e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}} \right) d\tau \; e^{\frac{h}{2}\mathscr{J}} \mathscr{F} \right| \leq \text{TOL}$$

Analysis of Decay to the Stable Manifold

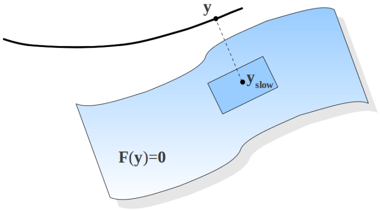
- Consider the equation $\mathbf{y}' = \lambda \mathbf{F}(\mathbf{y}) + \mathbf{f}(\mathbf{y}) \triangleq \mathscr{F}(\mathbf{y})$.
- We use estimation style analysis to evaluate behavior in the stiff limit ($\lambda \to \infty$).
- With a large value of λ, we expect fast decay to the stable manifold $\mathbf{F}(\mathbf{y}) = \mathbf{0}$.
- Let the dimension of this surface be m, while the dimension of the whole space $\mathbf{y}$ is M.



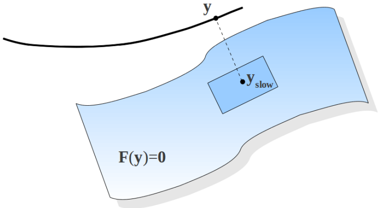
- Write $\mathbf{y}$ as $\mathbf{y} = \mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}}$, where $\mathbf{y}_{\text{fast}}$ corresponds to the fast decay part of the evolution, while $\mathbf{y}_{\text{slow}}$ represents the slow evolution of the system after the decay occurs.
- Of course, $\mathbf{y}' = (\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})' = \mathscr{F}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})$.
- But this yields only M equations for $2M$ variables.

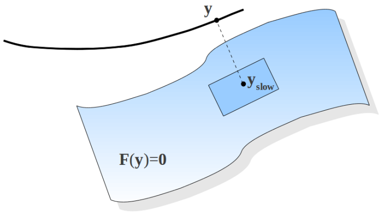


- Write $\mathbf{y}$ as $\mathbf{y} = \mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}}$, where $\mathbf{y}_{\text{fast}}$ corresponds to the fast decay part of the evolution, while $\mathbf{y}_{\text{slow}}$ represents the slow evolution of the system after the decay occurs.
- Of course, $\mathbf{y}' = (\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})' = \mathscr{F}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})$.
- But this yields only M equations for $2M$ variables.



- Choose the other M conditions according to the intuitive meaning of $\mathbf{y}_{\text{fast}}$ and $\mathbf{y}_{\text{slow}}$.
- Since the manifold contains the slow dynamics, $\mathbf{F}(\mathbf{y}_{\text{slow}}) = \mathbf{0}$. ($M - m$ conditions)
- Consider distributing $\mathbf{y}$ into $\mathbf{y}_{\text{fast}}$ and $\mathbf{y}_{\text{slow}}$.
- One way is to choose $\mathbf{y}_{\text{slow}}$ as the closest point on the surface $\mathbf{F}(\mathbf{y}) = \mathbf{0}$. This defines an "algorithm" of finding $\mathbf{y}_{\text{fast}}$ and $\mathbf{y}_{\text{slow}}$ given $\mathbf{y}$. (m conditions)





- Consider a point $\mathbf{y}_{slow}$ on the surface $\mathbf{F}(\mathbf{y}) = \mathbf{0}$.
- Let $\hat{P}_{\text{slow}}$ be the orthogonal projector matrix to the tangent plane at $\mathbf{y}_{slow}$.
- Denote the complimentary projector as $\hat{P}_{\text{fast}}$.
- The remaining conditions now are written as $\hat{P}_{\text{slow}}\mathbf{y}_{\text{fast}} = \mathbf{0}$.
- This yields m additional conditions because only m singular values of $\hat{P}_{\text{slow}}$ are non-zero.

- Let $\hat{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}$.
- The structure of $\hat{J}$ determines the projectors $\hat{P}_{\text{slow}}$ and $\hat{P}_{\text{fast}} = \hat{I} - \hat{P}_{\text{slow}}$.
- The system of equations is the following:

$$(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})' = \lambda \mathbf{F}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}}) + \mathbf{f}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}});$$

$$\mathbf{F}(\mathbf{y}_{\text{slow}}) = \mathbf{0}, \quad \Rightarrow \quad \hat{J}(\mathbf{y}_{\text{slow}})\,\mathbf{y}'_{\text{slow}} = \mathbf{0} \quad \text{or} \quad \hat{P}_{\text{fast}}\mathbf{y}'_{\text{slow}} = \mathbf{0};$$

$$\hat{P}_{\text{slow}}\mathbf{y}_{\text{fast}} = \mathbf{0}.$$

- We can then use adiabatic approximation on the system. Here we just show the leading order.
- We expect $\mathbf{y}_{\text{fast}} \sim 1/\lambda$, so expanding about $\mathbf{y}_{\text{slow}}$, $\mathbf{F}(\mathbf{y}) = \left(\mathbf{F}(\mathbf{y}_{\text{slow}}) = \mathbf{0}\right) + \hat{J}(\mathbf{y}_{\text{slow}})\mathbf{y}_{\text{fast}} + \mathrm{O}(1/\lambda^2)$.
- Consider $(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})' = \lambda \mathbf{F}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}}) + \mathbf{f}(\mathbf{y}_{\text{fast}} + \mathbf{y}_{\text{slow}})$. The leading order is

$$\mathbf{y}'_{\text{slow}} = \lambda \hat{J}(\mathbf{y}_{\text{slow}})\mathbf{y}_{\text{fast}} + \mathbf{f}(\mathbf{y}_{\text{slow}})$$

- Acting on the system with the fast projector, $\hat{P}_{\text{fast}}$, projects the system onto the region of fast decay:

$$\lambda \hat{P}_{\text{fast}} \hat{J}(\mathbf{y}_{\text{slow}})\mathbf{y}_{\text{fast}} = -\hat{P}_{\text{fast}} \mathbf{f}(\mathbf{y}_{\text{slow}})$$

$$\lambda \hat{P}_{\text{fast}} \hat{J}(\mathbf{y}_{\text{slow}})\mathbf{y}_{\text{fast}} = -\hat{P}_{\text{fast}}\mathbf{f}(\mathbf{y}_{\text{slow}})$$

- We solve this equation for $\mathbf{y}_{\text{fast}}$.
- We can think of this system as a non-singular square system in the subspace related to $\hat{P}_{\text{fast}}$. It is a nondegenerate system because when considered in the $\hat{P}_{\text{fast}}$ subspace, the system decays in all directions.
- Note that in $1/\lambda$ order, the magnitude of $\mathbf{y}_{\text{fast}}$ is determined by $\hat{P}_{\text{fast}}\mathbf{f}$. Hence, $\mathbf{y}_{\text{fast}}$ is $O(1/\lambda)$ distance from the manifold.

Average Distance from $\mathbf{F}(\mathbf{y}) = \mathbf{0}$

After the system has decayed, the solution curve is at a distance of $O(1/\lambda)$.

$$\lambda \hat{P}_{\text{fast}} \hat{J}(\mathbf{y}_{\text{slow}})\mathbf{y}_{\text{fast}} = -\hat{P}_{\text{fast}}\mathbf{f}(\mathbf{y}_{\text{slow}})$$

- We solve this equation for $\mathbf{y}_{\text{fast}}$.
- We can think of this system as a non-singular square system in the subspace related to $\hat{P}_{\text{fast}}$. It is a nondegenerate system because when considered in the $\hat{P}_{\text{fast}}$ subspace, the system decays in all directions.
- Note that in $1/\lambda$ order, the magnitude of $\mathbf{y}_{\text{fast}}$ is determined by $\hat{P}_{\text{fast}}\mathbf{f}$. Hence, $\mathbf{y}_{\text{fast}}$ is $\mathrm{O}(1/\lambda)$ distance from the manifold.

Average Distance from $\mathbf{F}(\mathbf{y}) = \mathbf{0}$

After the system has decayed, the solution curve is at a distance of $\mathrm{O}(1/\lambda)$.

- Consider the system $\mathbf{y}' = \mathscr{F} = \lambda F + f$, and project $\mathbf{y}$ onto the closest point in the manifold.
- Let J_* be the Jacobian at the projected point $\mathbf{y}_*$.
- $\mathscr{J} = \lambda J + j = \lambda(J - J_*) + \lambda J_* + j$.
  - By assumption, j represents the slow dynamics, so not much change is possible in a single time step.
  - Then $\tau j = \mathrm{O}(h)$, since $\tau = \mathrm{O}(h)$.
  - As before, the solution has essentially decayed to the manifold, so $\mathbf{y} - \mathbf{y}_* = \mathrm{O}(\frac{1}{\lambda})$
  - Then $J - J_* = \mathrm{O}(\frac{1}{\lambda})$.

- Consider the system $\mathbf{y}' = \mathscr{F} = \lambda F + f$, and project $\mathbf{y}$ onto the closest point in the manifold.
- Let J_* be the Jacobian at the projected point $\mathbf{y}_*$.
- $\mathscr{J} = \lambda J + j = \lambda(J - J_*) + \lambda J_* + j$.
  - By assumption, j represents the slow dynamics, so not much change is possible in a single time step.
  - Then $\tau j = \mathrm{O}(h)$, since $\tau = \mathrm{O}(h)$.
  - As before, the solution has essentially decayed to the manifold, so $\mathbf{y} - \mathbf{y}_* = \mathrm{O}\left(\frac{1}{\lambda}\right)$
  - Then $J - J_* = \mathrm{O}\left(\frac{1}{\lambda}\right)$.

- Consider the system $\mathbf{y}' = \mathscr{F} = \lambda F + f$, and project $\mathbf{y}$ onto the closest point in the manifold.
- Let J_* be the Jacobian at the projected point $\mathbf{y}_*$.
- $\mathscr{J} = \lambda J + j = \lambda(J - J_*) + \lambda J_* + j$.
    - By assumption, j represents the slow dynamics, so not much change is possible in a single time step.
    - Then $\tau j = \mathrm{O}(h)$, since $\tau = \mathrm{O}(h)$.
    - As before, the solution has essentially decayed to the manifold, so $\mathbf{y} - \mathbf{y}_* = \mathrm{O}\left(\frac{1}{\lambda}\right)$
    - Then $J - J_* = \mathrm{O}\left(\frac{1}{\lambda}\right)$.

- Then we have

$$\begin{aligned} e^{\tau \mathscr{J}} &= e^{\tau(\lambda J + j)} \\ &= e^{\tau[\lambda(J - J_*) + \lambda J_* + j]} \\ &= e^{\tau \lambda J_* + O(h)}, \end{aligned}$$

where $O(h)$ is from $\lambda(J - J_*)$ and from j.

Projectors

Then $e^{\tau \mathscr{J}}$ is a projector into the slow dynamics up to exponentially small terms, which are neglected. Similarly for $e^{\tau \tilde{\mathscr{J}}}$.

- Then we have

$$\begin{aligned} e^{\tau \mathscr{J}} &= e^{\tau(\lambda J + j)} \\ &= e^{\tau[\lambda(J - J_*) + \lambda J_* + j]} \\ &= e^{\tau \lambda J_* + \mathrm{O}(h)}, \end{aligned}$$

where $\mathrm{O}(h)$ is from $\lambda(J - J_*)$ and from j.

Projectors

Then $e^{\tau \mathscr{J}}$ is a projector into the slow dynamics up to exponentially small terms, which are neglected. Similarly for $e^{\tau \tilde{\mathscr{J}}}$.

- Consider the tolerance equation:

$$|\mathbf{y} - \bar{\mathbf{y}}| = \left| \int_0^{h/2} \left(e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}} \right) d\tau \; e^{\frac{h}{2} \mathscr{J}} \mathscr{F} \right| \leq \text{TOL}$$

- One copy of h is obtained from the $d\tau$ parameter; and another $e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}}$.
- So then the local error is $\mathrm{O}(h^2)$, which corresponds with a first order method.

Order Reduction

In the stiff limit, the method suffers from an order reduction due to the rapid change allowed between the two projector matrices.

- Consider the tolerance equation:

$$|\mathbf{y} - \bar{\mathbf{y}}| = \left| \int_0^{h/2} \left(e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}}\right) d\tau \; e^{\frac{h}{2}\mathscr{J}} \mathscr{F} \right| \leq \text{TOL}$$

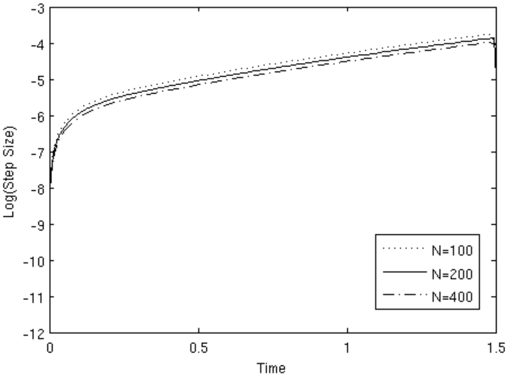
- One copy of h is obtained from the $d\tau$ parameter; and another $e^{\tau \mathscr{J}} - e^{\tau \tilde{\mathscr{J}}}$.
- So then the local error is $\mathrm{O}(h^2)$, which corresponds with a first order method.

Order Reduction

In the stiff limit, the method suffers from an order reduction due to the rapid change allowed between the two projector matrices.

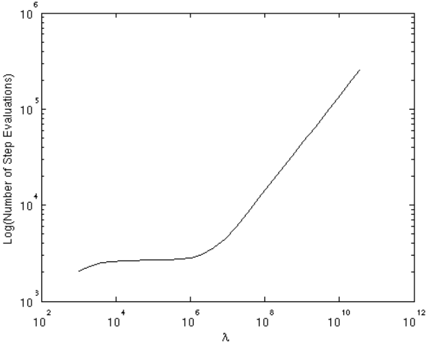
Stiff Decay Characteristics

- Figure shows Log(Step Size) vs. Time for the equation $u_t = u_{xx} - u^3$ with N=100, N=200, and N=400; $t = 1.5$.
- Step size does grow slightly with a larger number of points, however, the growth rate is linear for the period after decay of fast modes.



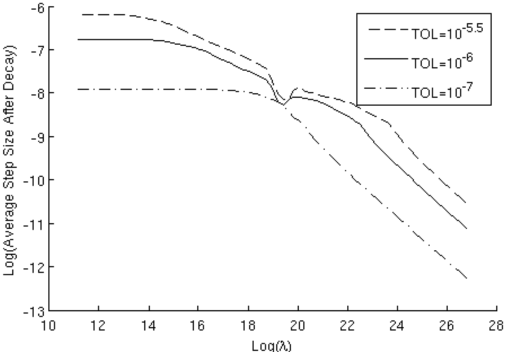
Stiff Decay Characteristics

- Figure shows decay to a circle, with Log(Number of Step Evaluations) vs. λ, $t = 3$.
- Note that there is a period of non-changing step size for λ values between 10^4 and 10^6.
- This period of constant step size indicates that the method exhibits stiff decay characteristics, but the increase after $\lambda = 10^6$ is troubling.



Dependence of Step Size on Tolerance Level

- Figure shows average step size after decay using different tolerance levels.
- Note that the plateau of non-increasing step size proceeds to the right for lower tolerance thresholds.

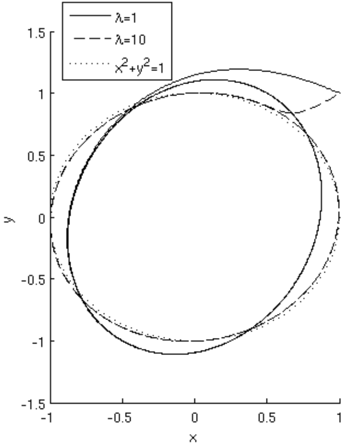


Average Distance from $\mathbf{F}(\mathbf{y}) = \mathbf{0}$

- We expect the average distance from the stable manifold to be $O(1/\lambda)$.

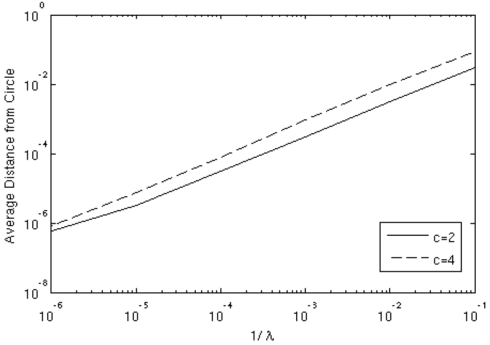
$$\begin{cases} \dot{x} = -y - \lambda x(x^2 + y^2 - 1) \\ \dot{y} = \ \ cx - \lambda y(x^2 + y^2 - 1) \end{cases}$$

- Now the dynamics push the solution curve toward an elliptical shape.
- Here, $c = 2$.



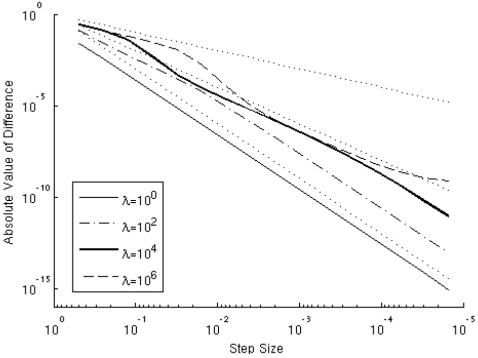
Average Distance from $\mathbf{F}(\mathbf{y}) = \mathbf{0}$

Figure: Average distance from unit circle vs. $1/\lambda$, for multiple values of c



Reduction in Order

Figure: $|\mathbf{y} - \bar{\mathbf{y}}|$, obtained by one step of size h, and two steps of size $h/2$, respectively. Dotted lines indicate slopes of h, h^2, h^3.



Concluding Remarks

- The method exhibits stiffness characteristics, but more numerical experimentation of the stiff limit is recommended.
- Ideally, this experimentation would be as independent of the choice of step size as possible.
- The code used for this research chooses step size based on the tolerance threshold; hence, step size is implicitly determined by the tolerance.
- Using estimation style analysis and numerical testing, we have shown that in the stiff limit, the method suffers an order reduction to first order.

Concluding Remarks

- The method exhibits stiffness characteristics, but more numerical experimentation of the stiff limit is recommended.
- Ideally, this experimentation would be as independent of the choice of step size as possible.
- The code used for this research chooses step size based on the tolerance threshold; hence, step size is implicitly determined by the tolerance.
- Using estimation style analysis and numerical testing, we have shown that in the stiff limit, the method suffers an order reduction to first order.

Concluding Remarks

- The method exhibits stiffness characteristics, but more numerical experimentation of the stiff limit is recommended.
- Ideally, this experimentation would be as independent of the choice of step size as possible.
- The code used for this research chooses step size based on the tolerance threshold; hence, step size is implicitly determined by the tolerance.
- Using estimation style analysis and numerical testing, we have shown that in the stiff limit, the method suffers an order reduction to first order.

Acknowledgements

I would like to thank Dr. Stepanov for providing support and encouragement throughout the semester.

[AP98] Uri M. Ascher and Linda R. Petzold. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1st edition, 1998.

[BHW85] Peter N. Brown, Alan C. Hindmarsh, and Homer F. Walker. Experiments with quasi-newton methods in solving stiff ode systems. *SIAM J. Sci. Stat. Comput.*, 6(2):297–313, April 1985.

[EHL75] W. H. Enright, T. E. Hull, and B. Lindberg. Comparing numerical methods for stiff systems of o.d.e:s. *BIT Numerical Mathematics*, 15:10–48, 1975. 10.1007/BF01932994.

[HOS09] M. Hochbruck, A. Ostermann, and J. Schweitzer. Exponential rosenbrock-type methods. *J. Numer. Anal.*, 47:786–803, 2009.