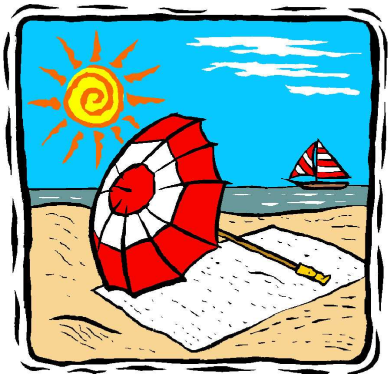

An Unusual Outbreak in Tropical Paradise: An Introduction to Compartmental Modeling of Diseases

Background

It's spring break, and you and your three best friends, Steph, Ashley, and Alex have finally convinced your parents to go on vacation *without* adult supervision. After an 18 hour flight on a private jet, you arrive on the tropical shore of Jacob Island. It is a beautiful and secluded island, with white sandy beaches and glistening coves. Before your arrival, each island only has _____ inhabitants.

The first day there, you and Alex decide to hang out at the beach to improve your tans. Steph and Ashley are feeling a bit more adventurous and decide to explore the neighboring retreat, Edward Island. While investigating their surroundings, Steph and Ashley find a wild, yet friendly monkey. They name him Ernesto, and try to teach him to screech on command. Ernesto loves bananas, and luckily, Steph packed a ready supply of fruit for lunch. After a full day of monkey-training, Steph and Ashley return to Jacob Island, and all four of you spend a lovely evening filled with truth or dare.

Your next day starts with breakfast on the beach. You notice that Steph and Ashley have slightly hoarse voices, but you assume that it is from all the singing the four of you did the previous night. After breakfast Steph and Alex return to the Edward Island to visit Ernesto. Ashley decides that though training a monkey was quite entertaining, you had the right idea of how to spend vacation. She decides to hang out on the beach with you. That afternoon, black clouds suddenly whip across the sky. A freak snow storm beats on both islands, with winds gusting up to 60 mph. The water is too dangerous for any transportation to or from either of the islands and the local meteorologist predicts that the storm will last for 10 days. The storm knocks out all modern communications.

Later that evening, a case of vespertiliovirum syndrome breaks out on both islands. It turns out that Steph and Ashley probably shouldn't have played with Ernesto, since they are the main carriers of vespertiliovirum syndrome.

This disease causes involuntarily speaking in a gruff voice and a sudden desire to fight crime. These symptoms last about two days, during which an infected individual is highly contagious. But, don't worry-- it's not fatal. Historical records on Jacob Island show that normally, once an individual has contracted vespertiliovirum syndrome, he or she cannot get it again. However, there has been no outbreak of vespertiliovirum syndrome since 1854, so none of the local residents are already immune.

Unfortunately, on the Edward Island, the strain has mutated. For this new strain, individuals no longer develop immunity to the disease. It looks like your vacation will be more exciting than you expected.

Developing the Models

Brainstorm: Write down facts from this story that may be useful in developing a model for the disease.

- Based on the story we see that we can separate the populations into three categories: **Susceptible, Infectious,** and **Recovered.**

- For each island, let **N** denote the total population and **S(t)**, **I(t)**, and **R(t)** denote the total number of susceptible, infectious, and recovered individuals at time **t**. Here we will let the unit of time be days.

- Let **ΔN(t)** be the change in the size of **N** from time **t-1** to time **t** and similarly for **S**, **I**, and **R**. Then

  ΔS(t)= # *of people who enter S at time t* - # *of people who leave S at time t*.

For each island, label the boxes on the next page and draw arrows to indicate which movements are allowed.

For example: a susceptible person cannot move to the recovered class without first entering the infectious class.

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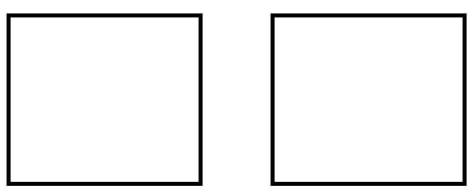
The Compartmental Models

Jacob Island (SIR)

Edward Island (SIS)



The Equations for the Models

Defining the parameters:

- It is reasonable to assume that the infection rate is proportional to the number of **Susceptible and Infected.** *Why?* Let **a=0.15** be the infection rate.

- We will let **r=0.5** be the recovery rate.

We can now write out the equations for our model:

Jacob Island (SIR)

N(t)=

ΔN(t)=

S(0)=

I(0)=

R(0)=

ΔS(t)=

ΔI(t)=

ΔR(t)=

Edward Island (SI)

N(t)=

ΔN(t)=

S(0)=

I(0)=

R(0)=

ΔS(t)=

ΔI(t)=

ΔR(t)=

We can now complete the picture of our compartmental models by labeling the arrows with the quantities in the equations.

Hypothesize: Who do you think will be more likely to contract the disease, you or Alex? Or do you think you will both be equally likely? Why?

Game Time!

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Interpreting the Behavior

What difference did we observe between the two models?

We will now examine our observations mathematically. To do this we will determine when compartments remain the same sizes.

1. Express **R** in terms of **S**, **I**, and **N**.

> Given this relation, we see that, given **S**, **I**, and **N**, we know the value of **R**. Therefore, it is sufficient to only consider the equations for **ΔS(t)** and **ΔI(t).**

2. Next, we will determine for what values of **S** and **I** there will be no change in the sizes of the compartments. To do this we set **ΔS(t)= ΔI(t)=0** and solve for **S** and **I**.

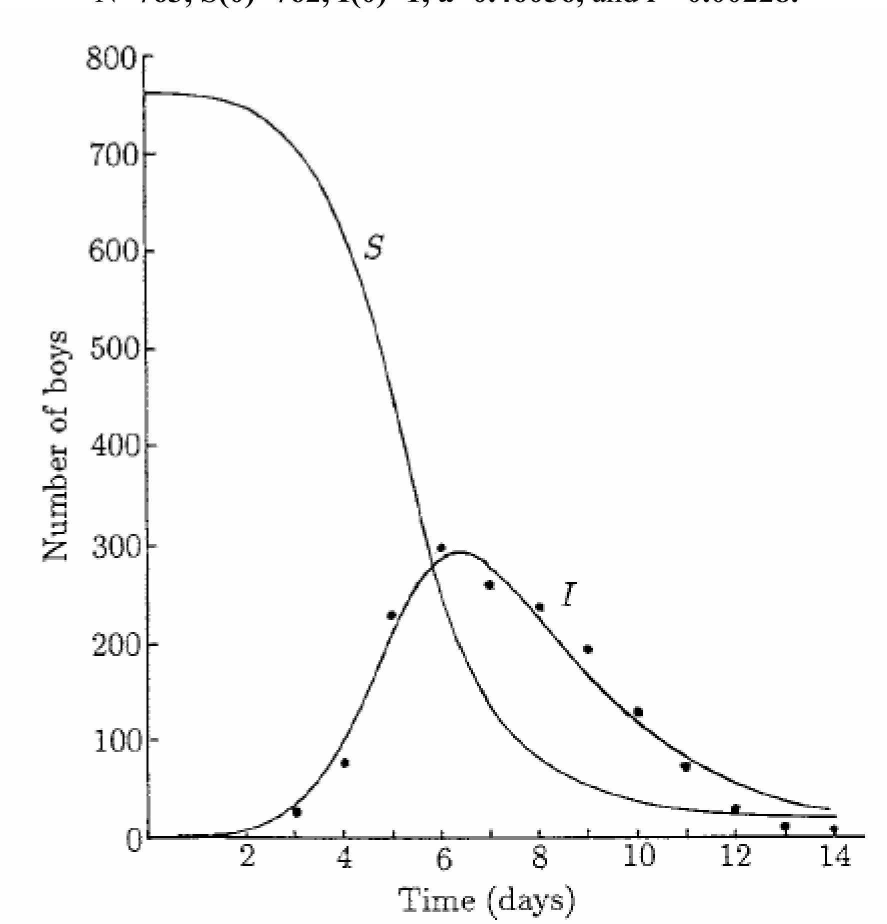
Graphing:

1. Plot the line **I=-S+N**. This line corresponds to different initial values **(S(0),I(0))**.
2. Plot the fixed points (the values **(S,I)** such that the sizes of the compartments do not change).
3. Plot the values **(S(t),I(t)).**

Applications of the Model

The model presented was highly simplified. But in certain cases it can provide a decent picture of the behavior a disease.
The graph below shows the data (dots) and the **SIR** model predictions (curves) for a 1978 outbreak of influenza at an English boarding school [1]. Here we have:

N=763, S(0)=762, I(0)=1, a=0.46056, and **r= 0.00228.**



Brainstorm: What aspects of a disease could be incorporated into our model to make it more realistic?

[1] J. D. Murray, Mathematical Biology, Springer-Verlag, Berlin, 1989, pp. 325-326