

INTERFACES ARISING IN REACTION-DIFFUSION EQUATIONS

EMILY P. MEISSEN
ADVISOR: KARL B. GLASNER

ABSTRACT. Diffuse interfaces are internal layers which arise in reaction-diffusion equations. We work through two canonical examples, the Allen-Cahn equation and Cahn-Hilliard equation, and find how the solutions and interfaces develop in these systems. We then work through an example in 1-D based on the Cahn-Hilliard model and again describe the solution and movement of the interface.

CONTENTS

1. REACTION-DIFFUSION EQUATIONS	2
2. ALLEN-CAHN EQUATION	2
2.1. GRADIENT FLOW	2
2.2. DESCRIPTION OF THE INTERFACE	3
3. CAHN-HILLIARD	7
3.1. GRADIENT FLOW	7
3.2. DESCRIPTION OF THE INTERFACE	8
4. MODIFIED CAHN-HILLIARD	10
4.1. GRADIENT FLOW	10
4.2. ASSUMPTIONS	11
4.3. MATCHING CONDITIONS	12
4.4. OUTER LAYER	13
4.5. INTERMEDIATE LAYER	14
4.6. INNER LAYER	14
5. EXTENSIONS	15
REFERENCES	15

1. REACTION-DIFFUSION EQUATIONS

Reaction-diffusion equations are commonly used to describe chemical and physical systems. For example, they can be used to represent the concentration of a chemical in a system. In such a situation, diffusion works to spread out the element while local chemical reactions build up the concentration. In these systems, multiple states are often possible, and so interfaces which separate these states often arise from the balance between diffusion and reactions [1]. Reaction-diffusion equations in \mathbb{R}^n are written in the form

$$u_t(x, t) = D\Delta u(x, t) + R(u(x, t))$$

where $u \in \mathbb{R}^n$ is a state variable, $x \in \Omega \subseteq \mathbb{R}^n$, $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix of diffusion coefficients, and R is a function which describes the local reactions. A simple example is the heat equation with an added source term,

$$u_t = u_{xx} + u.$$

This source term favors further build-up in areas where the chemical is already highly concentrated.

An example presented in depth in Fife [1] is flame theory. Consider a forest which is being burned. The moving flame front is an interface known as the combustion zone, which is thin because the reaction rate depends strongly on the temperature; this interface separates the preheat zone, which is untouched and has a low temperature, and the burned zone, which is in its final state. The diffusion of heat propagates the flame along with production of radicals. The model Fife uses is based on the simpler Allen-Cahn equation, which we will be presenting.

Many such systems can be written as a gradient flow in terms of an energy functional, i.e. $u_t = -\delta E/\delta u$ (in the weak sense with respect to an inner product). Here E describes the “free energy” in the system and $\delta E/\delta u$ denotes the first variation of E . Systems of these types develop in such a way as to minimize E .

2. ALLEN-CAHN EQUATION

As a first example, we look at the Allen-Cahn equation,

$$u_t = \Delta u + 2\epsilon^{-2}u(1 - u^2).$$

Here ϵ is a small parameter denoting that the reaction rate is very fast compared to diffusion. In this discussion we take $x \in \mathbb{R}^2$ and Neumann boundary conditions $\lim_{x \rightarrow \pm\infty} u_x(x, t; \epsilon) = 0$. This equation is usually used to model phase separation. Here u is known as an *order parameter* and is not necessarily conserved.

2.1. GRADIENT FLOW.

The Allen-Cahn equation can be written in variational form as a gradient flow with

$$E_\epsilon(u) = \int_{\Omega} \left[\frac{1}{\epsilon} V(u) + \frac{\epsilon}{2} |\nabla u|^2 \right] dx.$$

In this specific case, the potential $V(u) = -u^2 + u^4/2$. Here we calculate the first variation of E to show that $\langle \delta E_\epsilon/\delta u, \varphi \rangle = \langle -u_t, \varphi \rangle$ with respect to the usual L^2 inner product.

$$\begin{aligned} \left\langle \frac{\delta E_\epsilon}{\delta u}, \varphi \right\rangle &= \lim_{h \rightarrow 0} \frac{E_\epsilon(u + h\varphi) - E_\epsilon(u)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{\Omega} \left[\frac{1}{\epsilon} V(u + h\varphi) - \frac{1}{\epsilon} V(u) + \frac{\epsilon}{2} |\nabla(u + h\varphi)|^2 - \frac{\epsilon}{2} |\nabla u|^2 \right] dx \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{\Omega} \left[\frac{1}{\epsilon} V(u + h\varphi) - \frac{1}{\epsilon} V(u) + \frac{\epsilon h^2}{2} |\nabla \varphi|^2 + \epsilon h |\nabla u \nabla \varphi|^2 \right] dx \end{aligned}$$

We Taylor expand the first term around u :

$$V(u + h\varphi) = V(u) + h\varphi V_u(u) + h^2\varphi^2 V_{uu}(u) + \dots$$

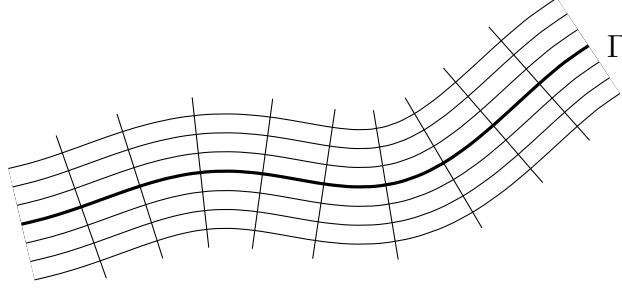


FIGURE 1. Example interface with local coordinate system. Plotted are Γ and the level curves of r and s .

For the third term,

$$|\nabla(u + h\varphi)|^2 = |\nabla u|^2 + 2h\nabla u \cdot \nabla\varphi + h^2|\nabla\varphi|^2.$$

Then by Green's First Identity,

$$\int_{\Omega} \nabla u \cdot \nabla\varphi dx = \int_{\partial\Omega} \varphi \nabla u \cdot n dS - \int_{\Omega} \varphi \Delta u dx.$$

With our $\nabla u \cdot n = 0$, this gives us $\int_{\Omega} \nabla u \cdot \nabla\varphi dx = - \int_{\Omega} \varphi \Delta u dx$.

So, after moving the limit inside the integral,

$$\begin{aligned} \left\langle \frac{\delta E_{\epsilon}}{\delta u}, \varphi \right\rangle &= \int_{\Omega} \left[\frac{1}{\epsilon} V_u(u)\varphi - \epsilon \Delta u \varphi + \lim_{h \rightarrow 0} \frac{\epsilon h^2}{2h} |\nabla\varphi|^2 \right] dx \\ &= \int_{\Omega} \left[\frac{1}{\epsilon} V_u(u)\varphi - \epsilon \Delta u \varphi \right] dx \\ &= \langle -u_t, \varphi \rangle. \end{aligned}$$

Hence $u_t = -\delta E_{\epsilon}/\delta u$ and so our system evolves in such a way as to minimize the free energy E_{ϵ} .

2.2. DESCRIPTION OF THE INTERFACE.

We would like to find an approximate solution to the Allen-Cahn equation and see how the interface moves.

We begin by assuming that we have a moving interface of width $\mathcal{O}(\epsilon)$, described by the curve $\Gamma(t; \epsilon)$. We define $\Gamma(t)$ to be the resulting interface in the limit as $\epsilon \rightarrow 0$. This interface separates two regions, which we will label D^+ and D^- .

Now we expand $u(x, t; \epsilon)$ in power series away from Γ : $u(x, t; \epsilon) = \sum \epsilon^n u_n(x, t)$. In doing this, we assume that the parts $u_n(x, t)$ do not contain any components of $\mathcal{O}(\epsilon)$. The solution obtained in this region is called the *outer solution*. Then $u^N(x, t; \epsilon) = \sum_{n=1}^N \epsilon^n u_n(x, t)$ is called the *outer approximation* to $u(x, t; \epsilon)$. Each u_n is assumed to be smooth away from Γ .

In order to detect different behavior of u near Γ , we introduce a new local coordinate system near $\Gamma(t; \epsilon)$, denoted by $(r(x, t; \epsilon), s(x, t; \epsilon))$. Here s represents arclength along Γ and r represents signed distance from x to Γ . Arbitrarily, we choose $r > 0$ on D^+ and $r < 0$ on D^- . We know such a coordinate system exists [2]. See Figure 1. Note that Γ is the 0-level set $\{r(x, t; \epsilon) = 0\}$ in the limit as $\epsilon \rightarrow 0$.

In order to focus on u near Γ , we introduce a stretched variable $\rho(x, t; \epsilon) = r(x, t; \epsilon)/\epsilon$. As $\epsilon \rightarrow 0$, our ρ coordinate becomes more localized around Γ . Let $U(\rho, s, t; \epsilon) = u(x, t; \epsilon)$ represent $u(x, t; \epsilon)$ near Γ ; this is known as the *inner solution*. We also expand $U(\rho, s, t; \epsilon)$ in power series: $U(\rho, s, t; \epsilon) = \sum \epsilon^n U_n(\rho, s, t)$.

2.2.1. MATCHING CONDITIONS.

Once we derive the leading order approximations to $u(x, t; \epsilon)$ and $U(\rho, s, t; \epsilon)$ separately, we will match them with what are known as *matching conditions* to obtain a valid global approximation. Here we derive the matching conditions.

For r near 0,

$$\begin{aligned} u_i(r, s, t) &= u_i(0\pm, s, t) + r\partial_r u_i(0\pm, s, t) + \frac{r^2}{2}\partial_r^2 u_i(0\pm, s, t) + \mathcal{O}(r^3) \\ &= u_i(0\pm, s, t) + \epsilon\rho\partial_r u_i(0\pm, s, t) + \frac{\epsilon^2\rho^2}{2}\partial_r^2 u_i(0\pm, s, t) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Note u and thus u_i may be discontinuous at the interface, so $0+$ denotes the limit towards $r = 0$ from D^+ , and $0-$ denotes the limit towards $r = 0$ from D^- . Then to match our inner and outer solutions, we equate powers of ϵ as $\epsilon \rightarrow 0$ for small r in the equation $\sum \epsilon^n U_n(\rho, s, t) = \sum \epsilon^n u_n(r, s, t)$. This gives us the matching conditions as $\rho \rightarrow \pm\infty$:

$$\begin{aligned} U_0(\pm\infty, s, t) &= u_0(0\pm, s, t), \\ U_1(\rho, s, t) &\sim u_1(0\pm, s, t) + \rho\partial_r u_0(0\pm, s, t), \\ U_2(\rho, s, t) &\sim u_2(0\pm, s, t) + \rho\partial_r u_1(0\pm, s, t) + \frac{\rho^2}{2}\partial_r^2 u_0(0\pm, s, t), \end{aligned}$$

etc.

These will be used as boundary conditions when deriving the inner solution.

2.2.2. OUTER LAYER.

In the outer region, we look to solve the original differential equation

$$\epsilon^2 u_t = \epsilon^2 \Delta u + 2u(1 - u^2)$$

by substituting in $u(x, t; \epsilon) = \sum \epsilon^n u_n(x, t)$ and collecting in powers of ϵ . At first order,

$$\mathcal{O}(1) : 0 = 2u_0(1 - u_0^2),$$

or $u_0(1 - u_0)(1 + u_0) = 0$. Because ± 1 are the stable equilibria of $g'(u) = 2u(1 - u^2) = 0$, we assume our system has $u_0 = \pm 1$. Further, we assume our interface is separating two different states, so we choose $u_0 = 1$ on D^+ and $u_0 = -1$ on D^- . At next order,

$$\mathcal{O}(\epsilon) : u_1 - 3u_0^2 u_1 = 0,$$

so $u_1 = 0$. We stop here because we are simply looking for the leading order behavior of u .

2.2.3. INNER LAYER.

In the inner region, we change from (x, t) to (ρ, s, t) coordinates and so must derive the Laplacian operator and time derivative in our new coordinates to obtain our inner equation. Doing so, we arrive at

$$\begin{aligned} \Delta_x u(r, s, t) &= u_{rr} + \Delta r u_r + u_s \Delta s + u_{ss} |\nabla s|^2, \\ u_t &= u_t + r_t u_r + s_t u_s, \end{aligned}$$

where $\Delta_x = \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

CHANGE OF COORDINATES.

LAPLACIAN.

Through chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x}.$$

Assuming smoothness of u throughout,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial}{\partial x} \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} \\ &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial s}{\partial x} \frac{\partial}{\partial s} \left(\frac{\partial u}{\partial x} \right) \\ &= \frac{\partial r}{\partial x} \left[\frac{\partial^2 u}{\partial r^2} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial r \partial x} + \frac{\partial^2 u}{\partial r \partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial^2 s}{\partial r \partial x} \right] + \frac{\partial s}{\partial x} \left[\frac{\partial^2 u}{\partial r \partial s} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial s \partial x} + \frac{\partial^2 u}{\partial s^2} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial^2 s}{\partial s \partial x} \right]. \end{aligned}$$

Because r and s are orthogonal coordinates, we have

$$\frac{\partial u}{\partial s} \frac{\partial^2 s}{\partial r \partial x} = 0, \quad \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial s \partial x} = 0.$$

Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \left(\frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x^2} + 2 \frac{\partial^2 u}{\partial r \partial s} \frac{\partial r}{\partial x} \frac{\partial s}{\partial x} + \frac{\partial^2 u}{\partial s^2} \left(\frac{\partial s}{\partial x} \right)^2 + \frac{\partial u}{\partial s} \frac{\partial^2 s}{\partial x^2}.$$

We have a similar expression for $\partial^2 u / \partial y^2$. Thus we write

$$\Delta u(r(x, t), s(x, t), t) = \frac{\partial^2 u}{\partial r^2} |\nabla r|^2 + \Delta r \frac{\partial u}{\partial r} + 2 (\nabla r \cdot \nabla s) \frac{\partial^2 u}{\partial r \partial s} + |\nabla s|^2 \frac{\partial^2 u}{\partial s^2} + \Delta s \frac{\partial u}{\partial s}.$$



Near Γ , note that $|\nabla r| = 1$. This is because the ∇r points in the direction of greatest rate of increase in r , and hence perpendicular to the level sets of r . In this direction, the rate of increase in r is 1. (See Figure blah.) Also, the level sets of r are orthogonal to the level sets of s (see Figure 1) and ∇r is orthogonal to the level sets of r and similarly for s , so ∇r is orthogonal to ∇s , i.e. $\nabla r \cdot \nabla s = 0$.

Thus, we have

$$\Delta u(r(x, t), s(x, t), t) = \frac{\partial^2 u}{\partial r^2} + \Delta r \frac{\partial u}{\partial r} + |\nabla s|^2 \frac{\partial^2 u}{\partial s^2} + \Delta s \frac{\partial u}{\partial s}.$$

TIME DERIVATIVE.

This is a simple application of the chain rule:

$$\frac{du(x, t)}{dt} = \frac{du(r(x, t), s(x, t), t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t}.$$

Here, Δr represents the curvature of the level sets of r . Specifically, as given by [2],

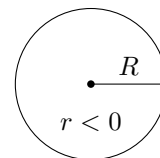
$$\Delta r = \frac{\kappa}{1 + \kappa r},$$

where κ is the curvature of the $r = 0$ level set, or of Γ . For a simple example to illustrate this equality, consider a circle of radius R and let r be the signed distance from the boundary of the circle with $r < 0$ inside and $r > 0$ outside. Then $r = |x| - R$, so

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}, \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

and so

$$\Delta r = \frac{1}{\sqrt{x^2 + y^2}}.$$



$r < 0$

FIGURE 2. A simple example of $\Delta r = \frac{\kappa}{1 + \kappa r}$.

The circle, and hence the $r = 0$ level set, has curvature $1/R$. Thus

$$\frac{\kappa}{1 + \kappa r} = \frac{1/R}{1 + (|x| - R)/R} = \frac{1}{R + |x| - R} = \frac{1}{\sqrt{x^2 + y^2}} = \Delta r.$$

Because the coordinate $r(x, t; \epsilon)$ depends on Γ , we also expand $r(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n r_n(x, t)$. Hence, when we solve for the motion of Γ , we will be solving for r_{0t} , which is only a leading order approximation. Similarly, $s(x, t; \epsilon)$ must be expanded, though we neglect this for notational reasons since doing so will not affect any of our calculations. Also, note that in this inner region $\Delta r = \kappa \sum_{n=0}^{\infty} (-\kappa r)^n = \kappa + \mathcal{O}(\epsilon)$ because $r = \mathcal{O}(\epsilon)$.

Substituting in the form for the Laplacian and time derivative in our new coordinates, we obtain the inner equation for the inner solution $U(\rho, s, t; \epsilon)$:

$$\epsilon r_t U_\rho + \epsilon^2 U_t + \epsilon^2 s_t U_s = U_{\rho\rho} + \epsilon \Delta r U_\rho + \epsilon^2 U_{ss} |\nabla s|^2 + \epsilon^2 U_s \Delta s + 2U(1 - U^2).$$

Substituting in our expansions $U(\rho, s, t; \epsilon) = \sum \epsilon^n U_n(\rho, s, t)$ and $r(x, t; \epsilon) = \sum \epsilon^n r_n(x, t)$ and collecting in powers of ϵ , at first order we get:

$$\mathcal{O}(1) : 0 = U_{0\rho\rho} + 2(U_0 - U_0^3).$$

This ODE is in potential form, i.e. $U_{0\rho\rho} = -V'(U_0)$ where $V'(U) = U^2 - U^4/2$. Multiplying through by $U_{0\rho}$, we can integrate to get

$$\frac{1}{2}(U_{0\rho})^2 = \frac{1}{2}U_0^4 - U_0^2 + C_1.$$

From our matching conditions, we have $U_0(\pm\infty, s, t) = u_0(0\pm, s, t) = \pm 1$, and so $U_{0\rho}(\pm\infty, s, t) = 0$. Thus $C_1 = \frac{1}{2}$ and

$$(U_{0\rho})^2 = U_0^4 - 2U_0^2 + 1 = (U_0^2 - 1)^2.$$

Then by separation of variables,

$$U_{0\rho} = \pm(U_0^2 - 1) \quad \Rightarrow \quad \mp \tanh^{-1}(U_0) = \rho + C_2(s, t)$$

and so

$$U_0 = \tanh(\mp x + C_2(s, t))$$

where C_2 is an arbitrary constant. Through our boundary conditions, we see that the solution is

$$U_0(\rho, s, t) = \tanh(x + C_2(s, t)).$$

At the next order, we have

$$\mathcal{O}(\epsilon) : r_{0t} U_{0\rho} = U_{1\rho\rho} + \kappa U_{0\rho} + 2(U_1 - 3U_0^2 U_1)$$

where $\Delta r = \kappa + \mathcal{O}(\epsilon)$. We rewrite this as $\mathcal{L}U_1 = r_t U_{0\rho} - \kappa U_{0\rho} = f$ where $\mathcal{L} = \partial_{\rho\rho} + 2 - 6U_0^2$. Note that \mathcal{L} is self-adjoint on $L^2(\mathbb{R}) \cap C^2(\mathbb{R})$:

$$\begin{aligned} \langle \mathcal{L}u, v \rangle &= \int_{-\infty}^{\infty} (u_{\rho\rho} v + 2uv - 6U_0^2 uv) d\rho \\ &= u_\rho v \Big|_{-\infty}^{\infty} - uv_\rho \Big|_{-\infty}^{\infty} + \langle u, \mathcal{L}v \rangle \\ &= \langle u, \mathcal{L}v \rangle. \end{aligned}$$

Then by the Fredholm Alternative, $\mathcal{L}U_1 = f$ has a solution when f is orthogonal to the nullspace of \mathcal{L}^* , or in the case of self-adjointness, of \mathcal{L} . Looking back at our $\mathcal{O}(1)$ equation, if we differentiate once we get $0 = U_{0\rho\rho\rho} + 2U_{0\rho} - 6U_0^2 U_{0\rho}$ and so $U_{0\rho}$ is in the null-space of L uniquely (though we do not prove uniqueness here). Then

$$\langle \mathcal{L}U_1, U_{0\rho} \rangle = \int_{-\infty}^{\infty} (r_{0t} U_{0\rho}^2 - \kappa U_{0\rho}^2) d\rho = 0$$

implies $r_{0t} = \kappa$. Thus, to leading order, we have motion by curvature (see Figure 3).

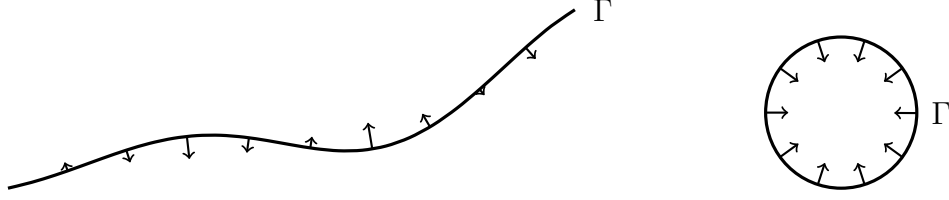


FIGURE 3. Illustrations of motion by curvature. When κ is large, Γ moves faster into the open area.

3. CAHN-HILLIARD

We now move on to study the Cahn-Hilliard equation,

$$\epsilon u_t = -\Delta (\epsilon^2 \Delta u + 2u(1 - u^2)).$$

For simplicity, we write this in coupled form as

$$\begin{aligned} \epsilon u_t &= \Delta \mu, \\ \mu &= -\epsilon^2 \Delta u - 2u(1 - u^2). \end{aligned}$$

The boundary conditions are

$$\nabla u \cdot n = 0, \quad \nabla \mu \cdot n = 0.$$

The first condition is known as the natural boundary condition and will be used when showing that this equation represents a gradient flow. The second restricts flux into the system. Our equation is written in conservation form, $u_t + \nabla J = 0$ where J represents the flux of the system, with $J = -\nabla \mu$. Here, μ is also known as the chemical potential. We again study the case where $x \in \Omega \subseteq \mathbb{R}^2$.

Although this PDE looks similar to the Allen-Cahn equation, it has an extra conserved quantity. We show that $\frac{d}{dt} \int u \, dx = 0$ by the Divergence Theorem, which implies conservation of mass.

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u \, dx &= \int_{\Omega} u_t \, dx \\ &= \epsilon^{-1} \int_{\Omega} \Delta \mu \, dx \\ &= \epsilon^{-1} \int_{\Omega} \nabla \cdot (\nabla \mu) \, dx \\ &= \epsilon^{-1} \int_{\partial \Omega} -n \cdot \nabla \mu \, dS \quad \text{by the Divergence Theorem} \\ &= 0 \quad \text{by the second boundary condition.} \end{aligned}$$

3.1. GRADIENT FLOW.

The Cahn-Hilliard equation be written as a gradient flow with respect to the H^{-1} inner product, where H^{-1} is the dual space of H^1 , with

$$E_{\epsilon}(u) = \int_{\Omega} \left[\frac{\epsilon^2}{2} |\nabla u|^2 + g(u) \right] dx.$$

In this case, $\langle u_t, \varphi \rangle_{H^{-1}} = \langle -\delta E_{\epsilon} / \delta u, \varphi \rangle_{H^{-1}}$, which is equivalent to $\langle u_t, \varphi \rangle = \langle \Delta (\delta E_{\epsilon} / \delta u), \varphi \rangle$. In this, we use the form

$$u_t = \Delta (-\epsilon^2 \Delta u + g'(u)).$$

Using our previous calculations,

$$\begin{aligned}
\left\langle \frac{\delta E_\epsilon}{\delta u}, \varphi \right\rangle &= \lim_{h \rightarrow 0} \frac{E_\epsilon(u + h\varphi) - E_\epsilon(u)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \int_\Omega \left[\frac{\epsilon^2}{2} |\nabla(u + h\varphi)|^2 - \frac{\epsilon^2}{2} |\nabla u|^2 + g(u + h\varphi) - g(u) \right] dx \\
&= \int_\Omega \left[-\epsilon^2 \Delta u \varphi + g_u(u) \varphi \right] dx \\
&= \langle \Delta^{-1} u_t, \phi \rangle.
\end{aligned}$$

Hence the system develops in such a way as to minimize the free energy with respect to the H^{-1} inner product.

3.2. DESCRIPTION OF THE INTERFACE.

We again assume there exists some moving interface Γ and try to analyze the leading order behavior of the system and of the development of Γ . Here we have the same matching conditions for $u(x, t; \epsilon)$ as before. Our matching conditions for $\mu(x, t; \epsilon)$ are analogous to those for $u(x, t; \epsilon)$. We also use the same scalings for the inner layer as in the Allen-Cahn case.

3.2.1. OUTER LAYER.

To find the leading order behavior in the outer region, we again start by expanding the outer solution in asymptotic series:

$$u(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n u_n(x, t), \quad \mu(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \mu_n(x, t).$$

Plugging these into our equations and boundary conditions, at first order we get

$$\mathcal{O}(1) : \begin{cases} \Delta \mu_0 = 0, \\ \mu_0 = -2u_0(1 - u_0^2), \\ \nabla \mu_0 \cdot n = 0, \\ \nabla u_0 \cdot n = 0. \end{cases}$$

We also assume that μ_0 is continuously differentiable on Γ , which will be proven in our inner region analysis. This, along with the first and third conditions, require that μ_0 be constant with respect to x . We see this by Green's identity:

$$\int_\Omega \mu_0 \Delta \mu_0 dx = - \int_\Omega (\nabla \mu_0)^2 dx + \int_{\partial\Omega} \mu_0 (\nabla \mu_0 \cdot n) dS = - \int_\Omega (\nabla \mu_0)^2 dx.$$

Thus we have $\int_\Omega (\nabla \mu_0)^2 dx = 0$, which implies that $\nabla \mu_0 = 0$ and gives us $\mu_0(x, t) = \mu_0(t)$ and $\mu_0 = u_0(1 - u_0^2)$. We again choose $u_0 = \pm 1$, corresponding to the stable equilibria of our potential function. Then $\mu_0 = 0$.

At next order we have

$$\mathcal{O}(\epsilon) : \begin{cases} \Delta \mu_1 = u_{0t} = 0, \\ \mu_1 = -2u_1(1 - 3u_0^2) = 4u_1, \\ \nabla \mu_1 \cdot n = 0, \\ \nabla u_1 \cdot n = 0, \end{cases}$$

which gives us $u_1 = \mu_1(x, t)/4$.

3.2.2. INNER LAYER.

We let U and M represent the inner solution variables corresponding to u and μ . We again start by expanding

$$U(r, s, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n U_n(r, s, t), \quad M(r, s, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n M_n(r, s, t), \quad r(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n r_n(x, t).$$

In the new local coordinates, our inner equations are

$$\begin{aligned} \epsilon^2 r_t U_\rho + \epsilon^3 U_t + \epsilon^3 s_t U_s &= M_{\rho\rho} + \epsilon \Delta r M_\rho + \epsilon^2 \Delta s M_s + \epsilon^2 M_{ss} |\nabla s|^2, \\ M &= -U_{\rho\rho} - \epsilon \Delta r U_\rho - \epsilon^2 \Delta s U_s - \epsilon^2 U_{ss} |\nabla s|^2 - 2U(1 - U^2). \end{aligned}$$

Again, for boundary conditions in the inner region we use the matching conditions.

At first order, we have

$$\mathcal{O}(1) : \begin{cases} M_{0\rho\rho} = 0, \\ M_0 = -U_{0\rho\rho} - 2U_0(1 - U_0)^2, \\ U_0(\pm\infty, s, t) = u_0(0\pm, s, t) = \pm 1, \\ M_0(\pm\infty, s, t) = \mu_0(0\pm, s, t). \end{cases}$$

The first condition implies $M_0(\rho, s, t) = a_0(s, t)\rho + b_0(s, t)$. From the fourth condition, assuming $\mu_0(0\pm, s, t)$ is bounded, we see $a(s, t) = 0$ and $b(s, t) = \mu_0(0\pm, s, t)$. Thus μ_0 is continuous along Γ , as claimed previously. Also, $M_0 = 0$ and $U_0 = \tanh(\rho + C(s, t))$ as before.

At the next order, we have

$$\mathcal{O}(\epsilon) : \begin{cases} M_{1\rho\rho} + \kappa M_{0\rho} = 0, \\ M_1 = -U_{1\rho\rho} - \kappa U_{0\rho} - 2U_1(1 - 3U_0^2), \\ U_1(\rho, s, t) = u_1(0\pm, s, t) + \rho u_{0r}(0\pm, s, t), \\ M_1(\rho, s, t) = \mu_1(0\pm, s, t) + \rho \mu_{0r}(0\pm, s, t), \end{cases}$$

where we have used that $\Delta r = \kappa + \mathcal{O}(\epsilon)$. Using $M_0 = 0$, we have $M_1 = a_1(s, t)\rho + b_1(s, t)$. Through our matching conditions, we have $a_1(s, t) = \mu_{0r}(0\pm, s, t)$, which also gives us that μ_0 is continuously differentiable along Γ . Because $\mu_0 = 0$, we can determine through matching that $b_1(s, t) = \mu_1(0\pm, s, t)$, and so μ_1 is continuous on Γ and $M_1(\rho, s, t) = \mu_1(0\pm, s, t)$.

To solve for U_1 , we again use the Fredholm Alternative. Rewriting as $\mathcal{L}U_1 = -M_1 - \kappa U_{0\rho} = f$ where $\mathcal{L} = \partial_{\rho\rho} + 2 - 6U_0^2$ as before, we again have that \mathcal{L} is self-adjoint. Hence, $\mathcal{L}U_1 = f$ has a solution when f is orthogonal to the nullspace of \mathcal{L} . We again have $U_{0\rho}$ in the null-space of \mathcal{L} uniquely, so

$$\begin{aligned} \langle \mathcal{L}U_1, U_{0\rho} \rangle &= \int_{-\infty}^{\infty} (-M_1 U_{0\rho} - \kappa U_{0\rho}^2) d\rho \\ &= \int_{-\infty}^{\infty} (-\mu_1(0\pm, s, t) \operatorname{sech}^2(\rho + C(s, t)) - \kappa \operatorname{sech}^4(\rho + C(s, t))) d\rho \\ &= -2\mu_1(0\pm, s, t) + \frac{4}{3}\kappa \\ &= 0 \end{aligned}$$

implies $\mu_1(0\pm, s, t) = 2\kappa(s, t)/3$.

To see how the interface moves at leading order, we continue.

$$\mathcal{O}(\epsilon^2) : \begin{cases} r_{0t} U_{0\rho} = M_{2\rho\rho} + \kappa M_{1\rho} + (\Delta r)_1 M_{1\rho} + \Delta s M_{0s} + M_{0ss} |\nabla s|^2, \\ M_2 = -U_{2\rho\rho} - \kappa U_{1\rho} - (\Delta r)_1 U_{0\rho} - \Delta s U_{0s} - U_{0ss} |\nabla s|^2 + 6U_0^2 U_2 + 6U_0 U_1^2 - 2U_2, \\ U_2(\pm\infty, s, t) = u_2(0\pm, s, t), \\ M_2(\pm\infty, s, t) = \mu_2(0\pm, s, t). \end{cases}$$

Because $M_0 = 0$, $M_1(\rho, s, t) = \mu_1(0\pm, s, t)$, the first equation is

$$r_{0t}U_{0\rho} = M_{2\rho\rho}.$$

We note that r_t is constant with respect to ρ , so we can integrate with respect to ρ . To see this,

$$r = \epsilon\rho \quad \Rightarrow \quad r_\rho = \epsilon \quad \Rightarrow \quad r_{\rho t} = 0.$$

Integrating once with respect to ρ , we get

$$(1) \quad r_{0t}U_0 = M_{2\rho} + C(s, t).$$

Our matching conditions give us $M_2(\rho, s, t) \sim \mu_{1r}(0\pm, s, t)\rho$ as $\rho \rightarrow \pm\infty$. Then $M_{2\rho}(\rho, s, t) \sim \mu_{1r}(0\pm, s, t)$. Taking the limit $\rho \rightarrow \infty$ and the limit $\rho \rightarrow -\infty$ of (1), we get

$$r_{0t} = \mu_{1r}(0+, s, t) + C(s, t), \quad -r_{0t} = \mu_{1r}(0-, s, t) + C(s, t).$$

Thus, the movement of the interface to leading order is

$$r_{0t} = \frac{1}{2}(\mu_{1r}(0+, s, t) - \mu_{1r}(0-, s, t)).$$

4. MODIFIED CAHN-HILLIARD

As a third example, we look at the equation

$$u_t = \Delta(-\Delta u + \epsilon^{-2}g'(u)) + \epsilon^{-3}(f - u),$$

where f is a constant. This is the Cahn-Hilliard equation with an added source term. Note that $u(x, t; \epsilon) = f$ is a steady-state solution:

$$\frac{d}{dt}f = \Delta(-\Delta f + \epsilon^{-2}g'(f)) + 0 = 0.$$

With the added restriction, $\frac{d}{dt} \int_{\Omega} (u(x, t) - f) dx = 0$, we see that f represents the average value of u . We use domain \mathbb{R} and, as in the Cahn-Hilliard case, we rewrite our equation in coupled form

$$\begin{aligned} \epsilon^3 u_t &= \mu_{xx} + f - u, \\ \mu &= -\epsilon^3 u_{xx} + \epsilon g'(u). \end{aligned}$$

For boundary conditions, we take $u(\pm\infty, t; \epsilon) = f$ and $\mu_x(\pm\infty, t; \epsilon) = 0$. The second condition again restricts flux coming into the system.

We again have conservation of mass:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u dx &= \int_{\mathbb{R}} u_t dx \\ &= \int_{\mathbb{R}} \epsilon^{-3} \mu_{xx} dx + \int_{\mathbb{R}} \epsilon^{-3} (f - u) dx \\ &= \epsilon^{-3} \mu_x \Big|_{-\infty}^{\infty} \\ &= 0. \end{aligned}$$

4.1. GRADIENT FLOW.

Define the energy functional

$$E_\epsilon(u) = \int_{\Omega} \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^2} g(u) + \frac{1}{2\epsilon^3} (f - u)v \right] dx$$

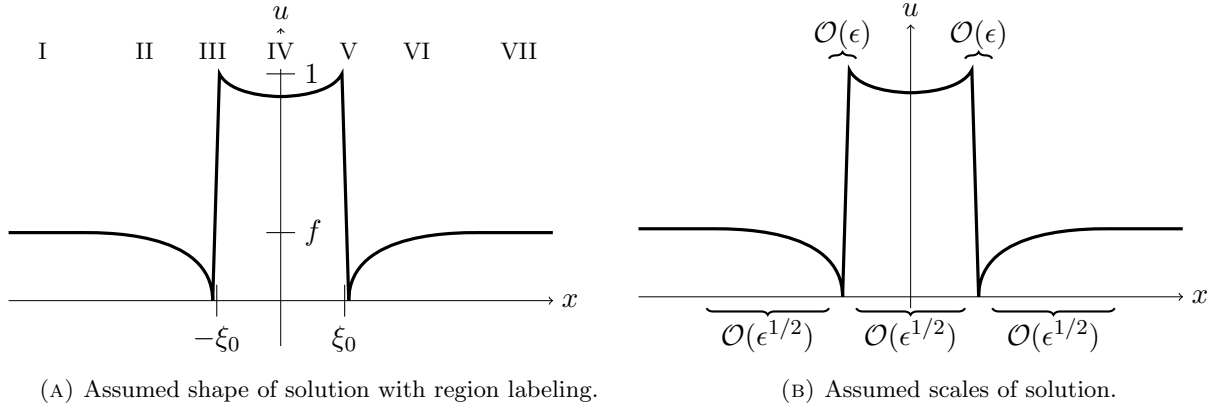


FIGURE 4. Assumptions on the solution $u(x, t; \epsilon)$. Note that $\xi_0 = \mathcal{O}(\epsilon^{1/2})$.

where $\Delta v = u - f$. Then, calculating the first variation, we show that $u_t = \Delta(\delta E_\epsilon / \delta u)$. First, we rewrite $f - u = -\Delta v$ and note that Δ^{-1} is a distributive and self-adjoint operator.

$$\begin{aligned}
\left\langle \frac{\partial E_\epsilon}{\partial u}, \varphi \right\rangle &= \lim_{h \rightarrow 0} \frac{E_\epsilon(u + h\varphi) - E_\epsilon(u)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \int_{\Omega} \left[\frac{1}{2} |\nabla(u + h\varphi)|^2 - \frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^2} g(u + h\varphi) - \frac{1}{\epsilon^2} g(u) \right. \\
&\quad \left. + \frac{1}{2\epsilon^3} (f - u - h\varphi) \Delta^{-1}(u + h\varphi - f) - \frac{1}{2\epsilon^3} (f - u) \Delta^{-1}(u - f) \right] dx \\
&= \int_{\Omega} \left[-\Delta u \varphi + \frac{1}{\epsilon^2} g_u(u) \varphi + \frac{1}{2\epsilon^3} (f \Delta^{-1} \varphi - u \Delta^{-1} \varphi - \varphi \Delta^{-1} u + \varphi \Delta^{-1} f) \right] dx \\
&= \int_{\Omega} \left[-\Delta u \varphi + \frac{1}{\epsilon^2} g_u(u) \varphi + \frac{1}{2\epsilon^3} ((f - u) \Delta^{-1} \varphi + \varphi \Delta^{-1} (f - u)) \right] dx \\
&= \int_{\Omega} \left[-\Delta u \varphi + \frac{1}{\epsilon^2} g_u(u) \varphi + \frac{1}{2\epsilon^3} (-\Delta v \Delta^{-1} \varphi - \varphi v) \right] dx \\
&= \int_{\Omega} \left[-\Delta u \varphi + \frac{1}{\epsilon^2} g_u(u) \varphi - \frac{1}{\epsilon^3} \varphi v \right] dx \\
&= \langle \Delta^{-1} u_t, \varphi \rangle.
\end{aligned}$$

For this example, we use

$$g(u) = \begin{cases} u^2 & \text{if } u < \frac{1}{2}, \\ (1 - u)^2 & \text{if } u > 1/2. \end{cases}$$

4.2. ASSUMPTIONS.

We assume that our problem started in some non-constant initial state and has evolved into some solution near equilibrium. On a long time scale, the solution will appear to be in a steady state. To see the motion, we will rescale time in order to view the motion occurring on a fast time scale.

We assume our solution looks similar to Figure 4a. In this, we assume:

- (1) f is small, i.e. $f \sim 0.1$,
- (2) our solution is symmetric about 0,
- (3) boundary conditions $\lim_{x \rightarrow \pm\infty} u(x, t) = f$,
- (4) the solution tends toward 0 and 1 on either side of $\pm\xi_0$ as depicted,

- (5) there are Cahn-Hilliard type interfaces, and
- (6) there are spatial scales of size $\mathcal{O}(1)$, $\mathcal{O}(\epsilon^{1/2})$, $\mathcal{O}(\epsilon)$.

Before, we had an interface Γ and tracked the motion through the variable r . Now, in one dimension, we center part of our solution around ξ_0 and track its motion to see the motion of the right “interface”. We label our three different spacial scales as follows: x , $\xi = x/\epsilon^{1/2}$, $z = (x - \epsilon^{1/2}\xi_0)/\epsilon$. Note that $\xi_0 = \mathcal{O}(\epsilon^{1/2})$, as can be seen in Figure 4, and that the z coordinate is centered around ξ_0 which will allow us to track the motion of ξ_0 . Specifically, we define ξ_0 to be the location where $u = 1/2$ in the right inner layer.

Because it has evolved for a long time, we also assume that u_τ is small, i.e. $u_\tau = \mathfrak{o}(1)$, in the outer and intermediate regions, expecting that only the inner region will be changing.

DOMINANT BALANCE.

In order to find a rescaling of time which will reveal the desired behavior, we rescale $\tau = \nu(\epsilon)t$.

$$\begin{aligned} \text{OUTER:} \quad \epsilon^3 \nu(\epsilon) u_\tau &= \mu_{xx} + f - u, \\ \mu &= -\epsilon^3 u_{xx} + \epsilon g'(u). \end{aligned}$$

Because are near steady-state in the outer region, we do not expect time to play a role in the leading order solution, hence $\nu(\epsilon) \leq \epsilon^{-3}$.

$$\begin{aligned} \text{INTERMEDIATE: } \xi = x/\epsilon^{1/2}. \quad \epsilon^4 \nu(\epsilon) u_\tau &= \mu_{\xi\xi} + \epsilon(f - u), \\ \mu &= -\epsilon^2 u_{\xi\xi} + \epsilon g'(u). \end{aligned}$$

$$\begin{aligned} \text{INNER: } z = (x - \epsilon^{1/2}\xi_0) / \epsilon. \quad U_t &= U_\tau \nu(\epsilon) - U_z \nu(\epsilon) \epsilon^{-1/2} \xi_0'(\tau). \\ \epsilon^5 \nu(\epsilon) U_\tau - \epsilon^{9/2} \nu(\epsilon) U_z \xi_0'(\tau) &= \mu_{zz} + \epsilon^2(f - u), \\ \mu &= -\epsilon u_{zz} + \epsilon g'(u). \end{aligned}$$

Here we expect to find the behavior of the moving interface at a low perturbative order, which suggests $\epsilon^{3/2} \leq \epsilon^{9/2} \nu(\epsilon) \leq \epsilon$, giving $\nu(\epsilon) = \epsilon^{-3}$ or $\nu(\epsilon) = \epsilon^{-7/2}$. This and outer solution together give $\nu(\epsilon) = \epsilon^{-3}$.

Rescaling by $\tau = \epsilon^{-3}t$ gives us:

$$\begin{aligned} \text{OUTER:} \quad u_\tau &= \mu_{xx} + f - u, \\ \mu &= -\epsilon^3 u_{xx} + \epsilon g'(u). \end{aligned}$$

$$\begin{aligned} \text{INTERMEDIATE: } \xi = x/\epsilon^{1/2}. \quad \epsilon u_\tau &= \mu_{\xi\xi} + \epsilon(f - u), \\ \mu &= -\epsilon^2 u_{\xi\xi} + \epsilon g'(u). \end{aligned}$$

$$\begin{aligned} \text{INNER: } z = (x - \epsilon^{1/2}\xi_0) / \epsilon. \quad U_t &= U_\tau \epsilon^{-3} - U_z \epsilon^{-3} \epsilon^{-1/2} \xi_0'(\tau). \\ \epsilon^2 U_\tau - \epsilon^{3/2} U_z \xi_0'(\tau) &= \mu_{zz} + \epsilon^2(f - u), \\ \mu &= -\epsilon u_{zz} + \epsilon g'(u). \end{aligned}$$

4.3. MATCHING CONDITIONS.

Because we have different powers of ϵ in our equation, we expand

$$u(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} u_n(x, t), \quad \mu(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} \mu_n(x, t)$$

for the outer solution;

$$\phi(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} \phi_n(\xi, t), \quad \Lambda(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} \Lambda_n(\xi, t)$$

for the intermediate solution; and

$$U(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} U_n(x, t), \quad M(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{n/2} M_n(x, t)$$

for the inner solution. Note that there are two distinct inner regions: region III with solution U^L, M^L and region V with solution U^R, M^R . Because our solution is symmetric, we focus on U^R, M^R centered around ξ_0 and denote this solution by U, M . We must match each region, from I to VII, to its neighboring region(s).

We begin by matching I and II and matching VI and VII. For x near 0,

$$\begin{aligned} u_i(x, t) &= u_i(0\pm, t) + xu_{ix}(0\pm, t) + \frac{x^2}{2}u_{ixx}(0\pm, t) + \mathcal{O}(x^3) \\ &= u_i(0\pm, t) + \epsilon^{1/2}\xi u_{ix}(0\pm, t) + \frac{\epsilon\xi^2}{2}u_{ixx}(0\pm, t) + \mathcal{O}(\epsilon^{3/2}). \end{aligned}$$

Equating powers of ϵ as $\epsilon \rightarrow 0$ in $\sum \epsilon^{n/2} \phi_n(\xi, t) = \sum \epsilon^{n/2} u_n(x, t)$, our first two conditions are as $\xi \rightarrow \pm\infty$,

$$\begin{aligned} \phi_0(\pm\infty, t) &= u_0(0\pm, t), \\ \phi_1(\rho, t) &\sim u_1(0\pm, t) + \xi u_{0x}(0\pm, t). \end{aligned}$$

To match regions V and VI, for ξ near ξ_0 ,

$$\begin{aligned} \phi_i(x, t) &= \phi_i(\xi_0+, t) + (\xi - \xi_0+)\phi_{i\xi}(\xi_0+, t) + \frac{(\xi - \xi_0+)^2}{2}\phi_{i\xi\xi}(\xi_0+, t) + \mathcal{O}((\xi - \xi_0+)^3) \\ &= \phi_i(\xi_0+, t) + \epsilon^{1/2}z\phi_{i\xi}(\xi_0+, t) + \frac{\epsilon z^2}{2}\phi_{i\xi\xi}(\xi_0+, t) + \mathcal{O}(\epsilon^{3/2}). \end{aligned}$$

Equating powers of ϵ as $\epsilon \rightarrow 0$ in $\sum \epsilon^{n/2} U_n(\xi, t) = \sum \epsilon^{n/2} \phi_n(x, t)$, our first two conditions are as $z \rightarrow +\infty$,

$$\begin{aligned} U_0(+\infty, t) &= \phi_0(\xi_0+, t), \\ U_1(z, t) &\sim \phi_1(\xi_0+, t) + z\phi_{0\xi}(\xi_0+, t). \end{aligned}$$

Similarly, matching IV and V, we get as $z \rightarrow -\infty$,

$$\begin{aligned} U_0(-\infty, t) &= \phi_0(\xi_0-, t), \\ U_1(z, t) &\sim \phi_1(\xi_0-, t) + z\phi_{0\xi}(\xi_0-, t). \end{aligned}$$

There are similar matching conditions for region III.

4.4. OUTER LAYER.

After rescaling by $\tau = \epsilon^{-3}t$, our outer equation becomes

$$\begin{aligned} u_\tau &= \mu_{xx} + f - u, \\ \mu &= -\epsilon^3 u_{xx} + \epsilon g'(u). \end{aligned}$$

If $u < 1/2$, as in regions I and VII, then $g'(u) = 2u$.

$$\mathcal{O}(1) : \begin{cases} u_{0\tau} = \mu_{0xx} + f - u_0, \\ \mu_0 = 0, \end{cases} \quad \mathcal{O}(\epsilon^{1/2}) : \begin{cases} u_{1\tau} = \mu_{1xx} - u_1, \\ \mu_1 = 0, \end{cases} \quad \mathcal{O}(\epsilon) : \begin{cases} u_{2\tau} = \mu_{2xx} - u_2, \\ \mu_2 = 2u_0. \end{cases}$$

Through these and the boundary conditions, we get $u_0 = f, \mu_0 = 0, u_1 = 0, \mu_1 = 0, u_2 = 0, \mu_2 = 2f$.

4.5. INTERMEDIATE LAYER.

After our rescaling $\tau = \epsilon^{-3}t$ and $\xi = x/\epsilon^{1/2}$ we have

$$\begin{aligned}\epsilon\phi_\tau &= \Lambda_{\xi\xi} + \epsilon(f - \phi), \\ \Lambda &= -\epsilon^2\phi_{\xi\xi} + \epsilon g'(\phi).\end{aligned}$$

If $\phi < 1/2$ as in regions II and VI, then $g'(\phi) = 2\phi$.

$$\mathcal{O}(1) : \begin{cases} 0 = \Lambda_{0\xi\xi}, \\ \Lambda_0 = 0, \end{cases} \quad \mathcal{O}(\epsilon^{1/2}) : \begin{cases} 0 = \Lambda_{1\xi\xi}, \\ \Lambda_1 = 0, \end{cases} \quad \mathcal{O}(\epsilon) : \begin{cases} \phi_{0\tau} = \Lambda_{2\xi\xi} + f - \phi_0, \\ \Lambda_2 = 2\phi_0. \end{cases}$$

So $\Lambda_0 = 0$ and $\Lambda_1 = 0$. Also, taking $\phi_{0\tau}$ to be small due to our near-steady state solution, $\phi_0 - 2\phi_{0\xi\xi} = f$. So

$$\phi_0(\xi, \tau) = f + c_1(\tau) \exp(\xi/\sqrt{2}) + c_2(\tau) \exp(-\xi/\sqrt{2}).$$

In region VI, we have the matching conditions $\phi \rightarrow f$ as $\xi \rightarrow \infty$ and $\phi \rightarrow 0$ as $\xi \rightarrow \xi_0$, which gives us $c_1(\tau) = 0$, $c_2(\tau) = -f \exp(\xi_0/\sqrt{2})$. Thus

$$\phi_0(\xi, t) = f \left(1 - \exp(-(\xi - \xi_0)/\sqrt{2}) \right), \quad \Lambda_2 = 2f \left(1 - \exp(-(\xi - \xi_0)/\sqrt{2}) \right).$$

In region II we have $\phi \rightarrow f$ as $\xi \rightarrow -\infty$ and $\phi \rightarrow 0$ as $\xi \rightarrow -\xi_0$, so

$$\phi_0(\xi, t) = f \left(1 - \exp((\xi + \xi_0)/\sqrt{2}) \right), \quad \Lambda_2 = 2f \left(1 - \exp((\xi + \xi_0)/\sqrt{2}) \right).$$

If $\phi > 1/2$ as in region IV, then $g'(\phi) = 2\phi - 2$.

$$\mathcal{O}(1) : \begin{cases} 0 = \Lambda_{0\xi\xi}, \\ \Lambda_0 = 0, \end{cases} \quad \mathcal{O}(\epsilon^{1/2}) : \begin{cases} 0 = \Lambda_{1\xi\xi}, \\ \Lambda_1 = 0, \end{cases} \quad \mathcal{O}(\epsilon) : \begin{cases} \phi_{0\tau} = \Lambda_{2\xi\xi} + f - \phi_0, \\ \Lambda_2 = 2\phi_0 - 2. \end{cases}$$

So $\Lambda_0 = 0$, $\Lambda_1 = 0$, and $\phi_0 - 2\phi_{0\xi\xi} = f$. Here we have the matching conditions $\phi \rightarrow 1$ as $\xi \rightarrow \xi_0$ and $\phi \rightarrow 1$ as $\xi \rightarrow -\xi_0$. This gives us

$$c_1(t) = c_2(t) = \frac{1-f}{\exp(\xi_0/\sqrt{2}) + \exp(-\xi_0/\sqrt{2})}, \quad \phi_0(\xi, t) = f + \frac{(1-f) \cosh(\xi/\sqrt{2})}{\cosh(\xi_0/\sqrt{2})}.$$

4.6. INNER LAYER.

In region V, we expect a Cahn-Hilliard type solution at leading order. At the perturbative order, we expect to catch the motion of $\xi_0(t)$.

Making the change of variables $\tau = \epsilon^{-3}t$ and $z = (x - \epsilon^{1/2}\xi_0)/\epsilon$ and using

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial \tau} \frac{d\tau}{dt} + \frac{\partial U}{\partial z} \frac{dz}{d\tau} \frac{d\tau}{dt} = \epsilon^{-3}U_\tau - \epsilon^{-7/2}U_z \xi_0'(\tau),$$

we have

$$\begin{aligned}\epsilon^2 U_\tau - \epsilon^{3/2} U_z \xi_0'(\tau) &= M_{zz} + \epsilon^2 (f - U), \\ M &= -\epsilon U_{zz} + \epsilon g'(U).\end{aligned}$$

If $U < 1/2$ as when $z > 0$, then $g'(U) = 2U$.

$$\mathcal{O}(1) : \begin{cases} 0 = M_{0zz}, \\ M_0 = 0. \end{cases} \quad \mathcal{O}(\epsilon^{1/2}) : \begin{cases} 0 = M_{1\xi\xi}, \\ M_1 = 0, \end{cases} \quad \mathcal{O}(\epsilon) : \begin{cases} 0 = M_{2zz}, \\ M_2 = -U_{0zz} + 2U_0. \end{cases}$$

So $M_0 = 0$, $M_1 = 0$, $M_2 = a(\tau)z + b(\tau)$. Then

$$U_0(z, \tau) = \frac{az}{2} + \frac{b}{2} + c_1(\tau) \exp(z\sqrt{2}) + c_2(\tau) \exp(-z\sqrt{2}).$$

Using the matching condition $U_0 \rightarrow 0$ as $z \rightarrow \infty$, we see $a = 0$, $c_1 = 0$, $b = 0$, and so $M_2 = 0$ and $U_0(z, \tau) = c_2 \exp(-z\sqrt{2})$ when $z > 0$.

If $U > 1/2$ as when $z < 0$, then $g'(U) = 2U - 2$.

$$\mathcal{O}(1) : \begin{cases} 0 = M_{0zz}, \\ M_0 = 0. \end{cases} \quad \mathcal{O}(\epsilon^{1/2}) : \begin{cases} 0 = M_{1\xi\xi}, \\ M_1 = 0, \end{cases} \quad \mathcal{O}(\epsilon) : \begin{cases} 0 = M_{2zz}, \\ M_2 = -U_{0zz} + 2U_0 - 2. \end{cases}$$

So $M_0 = 0$, $M_1 = 0$, $M_2 = a(\tau)z + b(\tau)$. Then

$$U_0(z, \tau) = \frac{az}{2} + \frac{b}{2} + 1 + c_1(\tau) \exp(z\sqrt{2}) + c_2(\tau) \exp(-z\sqrt{2}).$$

Using the matching condition $U_0 \rightarrow 1$ as $z \rightarrow -\infty$, we see that $a = 0$, $c_2 = 0$, $b = 0$, and so when $z < 0$ we have $U_0(z, \tau) = 1 + c_1 \exp(z\sqrt{2})$.

To patch these into a continuous solution in region V, we see $1 + c_1 = c_2$. From this and our requirement that $U_0(0, \tau) = 1/2$,

$$U_0(z, \tau) = \begin{cases} 1 - \frac{1}{2} \exp(z\sqrt{2}), & z < 0, \\ \frac{1}{2} \exp(-z\sqrt{2}), & z \geq 0. \end{cases}$$

We compute one further order to detect the movement of ξ_0 .

$$\mathcal{O}(\epsilon^{3/2}) : \begin{cases} -U_{0z}\xi'_0(\tau) = M_{3zz}, \\ M_3 = -U_{1zz} + 2U_1 \end{cases} \quad \text{when } z > 0, \quad \text{and} \quad \mathcal{O}(\epsilon^{3/2}) : \begin{cases} -U_{0z}\xi'_0(\tau) = M_{3zz}, \\ M_3 = -U_{1zz} + 2U_1 \end{cases} \quad \text{when } z < 0.$$

We integrate

$$\int_{-\infty}^{\infty} -U_{0z}\xi'_0(\tau) dz = \int_{-\infty}^{\infty} (-U_{1zz} + 2U_1)_{zz} dz.$$

Note that as $z \rightarrow \pm\infty$, $U_1(z, t) \sim \phi_1(\xi_0 \pm, t) + \phi_{0\xi}(\xi_0 \pm, t)z$, so $U_{1z}(z, t) \sim \phi_{0\xi}(\xi_0 \pm, t)$; then $U_{1zz}(z, t) \rightarrow 0$ and $U_{1zzz}(z, t) \rightarrow 0$ as $z \rightarrow \pm\infty$. Then $-\xi'_0(\tau) = 2(\phi_{0\xi}(\xi_0+, t) - \phi_{0\xi}(\xi_0-, t)) = -2$, so $\xi'_0(\tau) = 2$.

5. EXTENSIONS

This method of analysis can be used on many other problems. For the above examples, one can compute further orders to find a more precise description of the solution and interface behavior. For example, one can check that the motion by curvature derived for the Allen-Cahn equation is actually true to second order.

REFERENCES

- [1] Fife, P. (1988). *Dynamics of Internal Layers and Diffusive Interfaces*. Cbms-Nsf Regional Conference Series in Applied Mathematics Series. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104).
URL <http://books.google.com/books?id=D9oHfyYfjVMC>
- [2] Gavish, N., Hayrapetyan, G., Promislow, K., & Yang, L. (2011). Curvature driven flow of bi-layer interfaces. *Physica D: Nonlinear Phenomena*, 240(7), 675 – 693.
URL <http://www.sciencedirect.com/science/article/pii/S0167278910003362>
- [3] Pego, R., & Pego, R. (1989). Front migration in the nonlinear cahn-hilliard equation. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 422(1863), 261–278.
- [4] Rubinstein, J., Sternberg, P., & Keller, J. (1989). Fast reaction, slow diffusion, and curve shortening. *SIAM Journal on Applied Mathematics*, 49(1), 116–133.
URL <http://epubs.siam.org/doi/abs/10.1137/0149007>