

# INTEGRAL ASYMPTOTICS AND THE WKB APPROXIMATION

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ABSTRACT. In this paper, we investigate integral asymptotics – the asymptotic behavior as  $x \rightarrow \infty$  of some function  $I(x)$  with an integral representation. Using this theory, we study solutions to Airy's equation. Then we discuss the WKB Approximation method, which approximates the solution to a specific type of second order differential equation.

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## 1. INTRODUCTION

This paper is an exposition on asymptotic methods. In general, asymptotic methods aim to describe the behavior of solutions or functions in a limiting sense as some parameter changes in value. In this paper, we look at specific cases of integrals and differential equations. Solutions of such equations commonly lack closed-forms, and so asymptotic methods are often useful for determining their leading-order behavior.

We will begin by developing two common methods to study integral asymptotics. The first, Laplace's Method, was presented by Pierre-Simon Laplace in 1774 and handles integrals of real-valued functions. The second extends this method to handle to complex integrals and was used first by Riemann in 1863; it is known as the Method of Steepest Descent.

We then look at a specific differential equation containing a small parameter. The WKB Method will be used to find approximate solutions. The credit for derivation of the WKB Method is usually given to Wentzel, Kramers, and Brillouin, who developed it in 1926 [4]. Although it is sometimes referred to by other names when credit is given to those who formulated the method previously, we refer to it as the WKB Method in this paper. Its primary usefulness is seen in physics to derive approximate solutions to Schrödinger's Equation. We formulate an approximate solution valid on  $\mathbb{R}$  using results derived in the section on Method of Steepest Descent. For a more physical approach to the WKB Method, see Griffiths [3].

## 2. INTEGRAL ASYMPTOTICS

The general study of asymptotics looks at the behavior of a function  $I(x)$  as  $x \rightarrow \infty$  or  $x \rightarrow 0$ . Integral asymptotics focuses on functions which are represented in integral form, for example the Gamma function:

$$I(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

Later, we will look at the behavior of the Gamma function as  $x \rightarrow \infty$ . In general, the integrals studied can be complex- or real-valued. We look at two methods of finding the asymptotics of such integrals, one which applies to real-valued functions and another which extends to complex-valued functions.

### 2.1. LAPLACE'S METHOD.

Suppose the function  $I(x)$  has the form

$$I(x) = \int_a^b f(t) e^{xg(t)} dt$$

where all numbers are real-valued. As  $x \rightarrow \infty$ , one would expect the largest contribution of the integral to come from wherever  $g(t)$  is largest. If  $g(t)$  has a maximum at  $c \in [a, b]$  and  $f(c) \neq 0$ , then we can start by approximating

$$I(x) \approx \int_{c-\epsilon}^{c+\epsilon} f(t) e^{xg(t)} dt$$

if  $c \neq a, b$  or

$$I(x) \approx \int_c^{c+\epsilon} f(t) e^{xg(t)} dt$$

if  $c = a$  (with a similar form if  $c = b$ ). As  $x \rightarrow \infty$ , the rest of the original integral is exponentially small comparatively. Then, because our interval is restricted to near  $t = c$ , we can replace  $f(t)$  and  $g(t)$  with their Taylor expansions. The only step remaining is to remove the dependence on  $\epsilon$  in the final expansion.

#### 2.1.1. A Simple Example.

For a first example, let us consider the integral

$$I(x) = \int_0^1 \sin(t) e^{-xt} dt$$

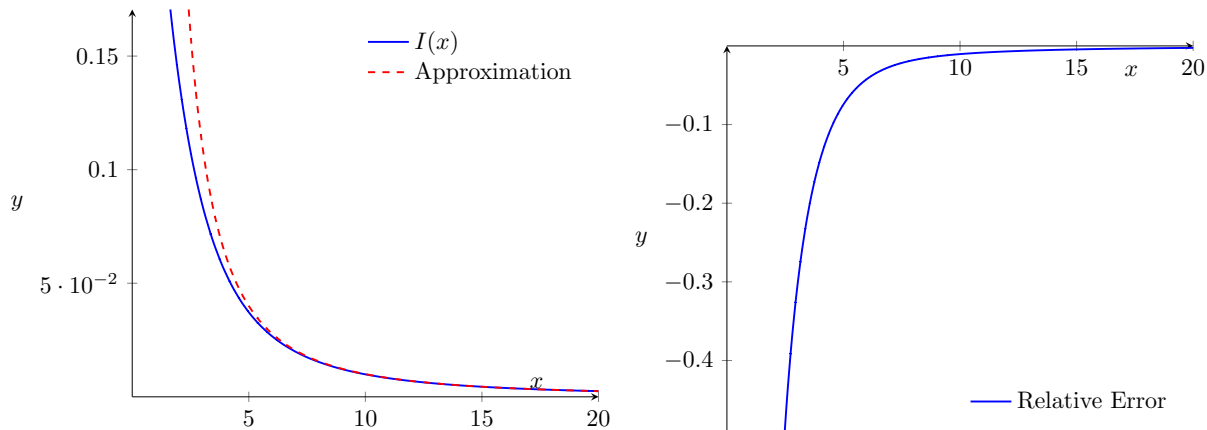


FIGURE 1. A plot of  $I(x) = \int_0^1 \sin(t)e^{-xt} dt$  and its leading order approximation,  $x^{-2}$ . As  $x \rightarrow \infty$ , the relative error,  $1 - x^{-2}/I(x)$ , goes to 0.

(Section 6.2, Exercise 4 in [2]). We see that  $g(t) = -t$  attains a maximum at  $t = 0$  over the range of integration, so as  $x \rightarrow \infty$ , the leading order behavior will be dominated by the integral near  $t = 0$ . Further, in this region  $\sin(t) \approx t$ . Thus, we can approximate

$$I(x) \approx \int_0^\epsilon t e^{-xt} dt.$$

We can now integrate by parts:

$$\begin{aligned} I(x) &\approx \left. \frac{t e^{-xt}}{-x} \right|_0^\epsilon + \frac{1}{x} \int_0^\epsilon e^{-xt} dt \\ &= \frac{\epsilon e^{-x\epsilon}}{-x} - \frac{e^{-x\epsilon}}{x^2} + \frac{1}{x^2}. \end{aligned}$$

For small  $\epsilon$ , we see that  $I(x) \approx x^{-2}$  to leading order as  $x \rightarrow \infty$ . In our next example, we will give a detailed analysis of the error incurred in each approximation. This example is meant as an illustration, although one could control the error. See Figure 1 for a comparison to the exact value. To obtain more terms for an asymptotic expansion, we would leave more terms in the  $\sin(t)$  expansion before evaluating any integrals.

Because the leading order is dominated by the region near  $t = 0$ , we could have just as easily approximated

$$I(x) \approx \int_0^\infty t e^{-xt} dt,$$

which would have provided the same answer because the integral on  $t \in [\epsilon, \infty)$  would be exponentially small compared the integral on  $t \in [0, \epsilon)$ .

### 2.1.2. An Extended Example: The Gamma Function.

The Gamma function is an extension of the factorial such that  $\Gamma(n+1) = n!$  and is defined as

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt.$$

Although this can be extended to the complex plane by letting  $x$  be complex, we consider only  $x > 0$  real and look for an approximation as  $x \rightarrow \infty$ .

As written, the Gamma Function does not fit our general Laplace integral form. Using  $t^x = \exp(x \log t)$ , we can rewrite our integral as

$$\Gamma(x+1) = \int_0^\infty \exp(-t + x \log t) dt.$$

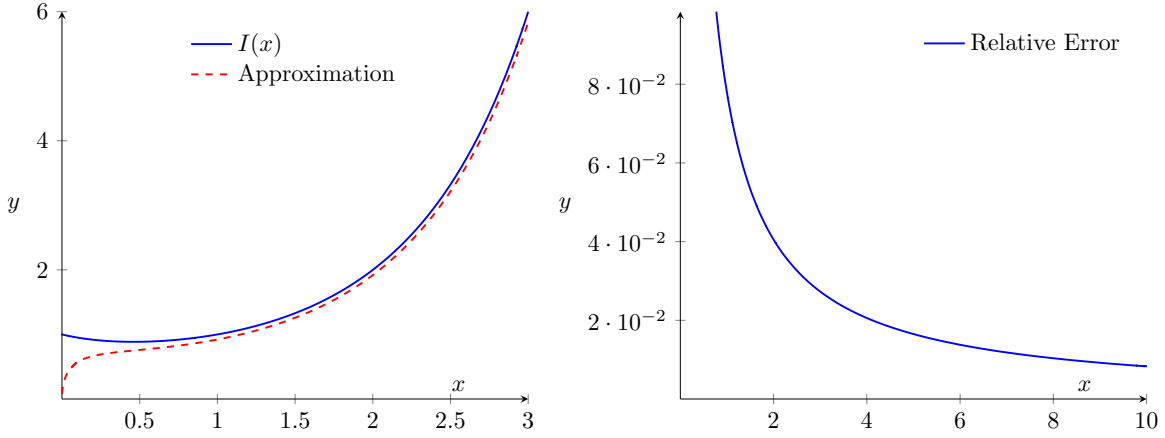


FIGURE 2.  $\Gamma(z)$  versus Stirling's approximation  $G(x)$ . As  $x \rightarrow \infty$ ,  $1 - G(x)/\Gamma(x) \rightarrow 0$ .

One might already guess, due to the integrand not being in the form  $f(t) \exp(xg(t))$ , that we might have a problem. Let us define

$$h(t) = -t + x \log t$$

and investigate where our integrand is maximal. We see from

$$h'(t) = -1 + \frac{x}{t} = 0$$

that the critical point depends on  $x$ ; specifically it is at  $t = x$ . To avoid such complications, we make the change of variables  $t = xu$  into the original integral. This gives us

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty \exp(-xu) (xu)^x x \, du \\ &= x^{x+1} \int_0^\infty \exp(-xu + x \log u) \, du \\ &= x^{x+1} e^{-x} \int_0^\infty \exp(x - xu + x \log u) \, du. \end{aligned}$$

We then have an integrand of the form  $f(t) \exp(xg(t))$  with  $f(t) = 1$  and  $g(u) = 1 - u + \log u$ . Taylor expanding around  $u = 1$ , the critical point of  $g(u)$ , gives  $g(u) = -(u-1)^2/2 + \mathcal{O}((u-1)^3)$  (used in step (2) below). Then applying Laplace's Method,

$$\begin{aligned} (1) \quad \Gamma(x+1) &\approx x^{x+1} e^{-x} \int_{1-\epsilon}^{1+\epsilon} \exp(xg(u)) \, du \\ (2) \quad &\approx x^{x+1} e^{-x} \int_{1-\epsilon}^{1+\epsilon} \exp(-x(u-1)^2/2) \, du \\ (3) \quad &\approx x^{x+1} e^{-x} \int_{-\infty}^{\infty} \exp(-x(u-1)^2/2) \, du \\ &= x^{x+1} e^{-x} \sqrt{\frac{2}{x}} \int_{-\infty}^{\infty} \exp(-s^2) \, ds \\ &= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \\ &= G(x). \end{aligned}$$

This result is known as Stirling's Approximation (see Figure 2). We made three approximations: (1) restricting the range of integration, (2) truncating the Taylor series of  $g(t)$ , and (3) expanding the range of

integration. In this extended example, we will now look at the error introduced in each of these approximations.

INTERVAL RESTRICTION. The error introduced in (1) is

$$E_1(x, \epsilon) = x^{x+1} e^{-x} \int_{1+\epsilon}^{\infty} \exp(xg(u)) du + x^{x+1} e^{-x} \int_0^{1-\epsilon} \exp(xg(u)) du,$$

which we want to show is small compared to our final approximation. Let  $E_1^+(x, \epsilon)$  denote the first term and  $E_1^-(x, \epsilon)$  the second. We begin by looking at  $E_1^+(x, \epsilon)$ . Because  $g(u)$  is concave on  $(0, \infty)$ , as seen in Figure 3, we see that  $g(u)$  is less than its tangent line at  $u = 1 + \epsilon$ ,  $y(u) = a + b(u - 1 - \epsilon)$ , on our interval of interest,  $(1 + \epsilon, \infty)$ . Here  $a = \log(1 + \epsilon) - \epsilon$  and  $b = -\epsilon/(1 + \epsilon)$ . Thus,

$$\begin{aligned} E_1^+(x, \epsilon) &\leq x^{x+1} e^{-x} \int_{1+\epsilon}^{\infty} \exp(xy(u)) du \\ &= x^{x+1} \exp\left(x(-1 + a - b(1 + \epsilon))\right) \int_{1+\epsilon}^{\infty} \exp(xbu) du \\ &= -x^{x+1} \frac{\exp(-x + xa)}{xb}. \end{aligned}$$

Note that  $a = \log(1 + \epsilon) - \epsilon < 0$ . In particular, by Taylor series expansion

$$a = \log(1 + \epsilon) - \epsilon = -\epsilon^2/2 + \epsilon^3/3 - \epsilon^4/4 + \dots \leq -\epsilon^2/2 + \epsilon^3/3 = -\epsilon^2/6.$$

Then we have

$$E_1^+(x, \epsilon) \leq -\frac{x^x}{b} \exp(-x) \exp(-x\epsilon^2/6).$$

Thus  $E_1^+(x, \epsilon)$  is exponentially small compared to  $G(x) = \sqrt{2\pi x} (x/e)^x$ , i.e.

$$\frac{E_1^+(x, \epsilon)}{G(x)} \leq \frac{1 + \epsilon}{\epsilon\sqrt{2\pi x}} \exp(-x\epsilon^2/6).$$

We now show that  $E_1^-(x, \epsilon)$  is similarly small. We use the tangent line at  $u = 1 - \epsilon$ ,  $y(u) = a + b(u - 1 + \epsilon)$ , which again dominates  $g(u)$ . Here  $a = \log(1 - \epsilon) + \epsilon$  and  $b = \epsilon/(1 - \epsilon)$ . Then

$$\begin{aligned} E_1^-(x, \epsilon) &\leq x^{x+1} e^{-x} \int_0^{1-\epsilon} \exp(xy(u)) du \\ &= x^{x+1} \exp\left(x(-1 + a - b(1 - \epsilon))\right) \int_0^{1-\epsilon} \exp(xbu) du \\ &\leq x^{x+1} \exp\left(x(-1 + a - b(1 - \epsilon))\right) \int_{-\infty}^{1-\epsilon} \exp(xbu) du \\ &= x^{x+1} \frac{\exp(-x + xa)}{xb} \\ &< \frac{x^x}{b} \exp(-x) \exp(-x\epsilon^2/2). \end{aligned}$$

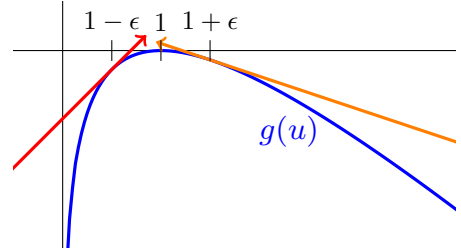


FIGURE 3.  $g(u)$  and tangent lines at  $1 + \epsilon$  and  $1 - \epsilon$ . Notice that  $g(u)$  is concave on  $(0, \infty)$  and so is less than its tangent line taken at any point from  $(0, \infty)$ .

Here we used that  $a = \log(1 - \epsilon) + \epsilon < -\epsilon^2/2 < 0$  which we get by Taylor expanding,

$$a = \log(1 - \epsilon) + \epsilon = -\epsilon^2/2 - \epsilon^3/3 - \epsilon^4/4 - \dots < -\epsilon^2/2.$$

Then

$$\frac{E_1^-(x, \epsilon)}{G(x)} < \frac{1 - \epsilon}{\epsilon\sqrt{2\pi x}} \exp(-x\epsilon^2/2).$$

TAYLOR SERIES TRUNCATION. The error introduced in (2) is

$$E_2(x, \epsilon) = x^{x+1} e^{-x} \int_{1-\epsilon}^{1+\epsilon} \left[ \exp(xg(u)) - \exp(-x(u-1)^2/2) \right] du.$$

Note that  $g(u) > -(u-1)^2/2$  for  $u \in [1-\epsilon, 1+\epsilon]$ , evident due to its Taylor series being an alternating series, and so this expression is positive. The difficulty in calculating this quantity is that we cannot explicitly calculate the integral of the first term. To remedy this, we look for an exact transformation  $f(u) = z$  such that  $g(u) = 1 - u + \log(u) = -z^2/2$ . We can then express

$$\int_{1-\epsilon}^{1+\epsilon} \exp(xg(u)) du = \int_{f(1-\epsilon)}^{f(1+\epsilon)} \exp(-xz^2/2) \xi'(z) dz$$

where  $\xi(z) = f^{-1}(z) = u$ . We would like to show that  $\xi(z)$  exists and is analytic near  $f(1) = 0$ . If this is the case, then we can use Taylor's Remainder Theorem to get

$$(4) \quad \xi'(z) = \xi'(0) + \xi''(0)z + \xi'''(c(z))z^2/2$$

where  $f(1-\epsilon) < c < f(1+\epsilon)$ . We can then bound  $E_2(x, \epsilon)$ .

We start by defining  $f(u) = \sqrt{-2g(u)}$ , taking branch cuts for the square root and logarithm along the negative real axis. Although  $g(1) = 0$ , we shall prove that  $f(u)$  is analytic near  $u = 1$  and does not have a branch point.

First,  $g(u)$  has Taylor series

$$g(u) = - \sum_{k=2}^{\infty} \frac{(-1)^k (u-1)^k}{k} = -\frac{1}{2}(u-1)^2 + \frac{1}{3}(u-1)^3 - \frac{1}{4}(u-1)^4 + \dots$$

Then

$$-2g(u) = (u-1)^2 \left[ 1 - \frac{2}{3}(u-1) + \frac{1}{2}(u-1)^2 + \dots \right] = (u-1)^2 \nu(u).$$

Letting  $u-1 = r \exp(i\theta)$  with  $\theta \in (-\pi, \pi]$ , we see  $\sqrt{(u-1)^2} = \sqrt{r^2 \exp(2i\theta)} = r \exp(i\theta)$  is analytic. Also, we see  $\nu(u) = 1 + \sum_{k=1}^{\infty} 2(-1)^k (u-1)^k / (k+2)$  is analytic in  $u$  and  $\nu(u) \neq 0$  or  $\infty$  near  $u = 1$ , so there is no branch point at  $u = 1$ . Because we do not need to worry about the branch cuts of the square root or logarithm along the negative real axis, we see  $f(u)$  is analytic near  $u = 1$  and can be written as

$$(5) \quad f(u) = (u-1) \nu^{1/2}(u).$$

We now show that  $\xi(z) = f^{-1}(z)$  exists and is analytic near  $z = 0$ . Because  $f(u)$  has a simple zero at  $u = 1$  and  $f'(1) \neq 0$ , there exists a contour  $C$  in the  $u$ -plane around  $u = 1$  inside which  $f(u) \neq 0$  except at  $u = 1$  and  $f'(u) \neq 0$ . Because  $f(u)$  is analytic near  $u = 1$ , the Argument Principle gives us

$$\frac{1}{2\pi i} \oint_C \frac{f'(u)}{f(u)} du = 1.$$

Let  $s$  be a point in the  $z$ -plane near  $z = 0$ . Letting  $g_s(u) = f(u) - s$ , there exists a contour  $C'$  within  $C$  such that

$$(6) \quad I(s) = \frac{1}{2\pi i} \oint_{C'} \frac{g'_s(u)}{g_s(u)} du = \frac{1}{2\pi i} \oint_C \frac{f'(u)}{f(u) - s} du = 1$$

is constant for all  $s$  inside  $C'$  due to Hurwitz's Theorem. Then, because  $u$  is analytic in the  $u$ -plane,  $uf'(u)/(f(u) - s)$  has the same poles as  $f'(u)/(f(u) - s)$ , of which there is only one at  $u = \xi(s)$ . The residue at this location given in (6) now gets multiplied by  $u$  evaluated at  $\xi(s)$ . Thus, by the Residue Theorem we have

$$\frac{1}{2\pi i} \oint_{C'} \frac{uf'(u)}{f(u) - s} du = \xi(s).$$

Thus  $\xi(z)$  exists and is analytic near  $z = 0$  and the following formula is revealing:

$$\frac{1}{2\pi i} \oint_C \frac{\xi(w)}{w-s} dw = \frac{1}{2\pi i} \oint_{C'} \frac{\xi(f(u))f'(u)}{f(u)-s} du = \frac{1}{2\pi i} \oint_{C'} \frac{uf'(u)}{f(u)-s} du = \xi(s).$$

Because we now know that  $\xi(z)$  is analytic near  $z = 0$ , we know it has a Taylor series around  $z = 0$ . We solve for the Taylor coefficients  $\xi^{(n)}(z)$  in (4). First, we have  $f(1) = 0$ ,  $f'(1) = 1$ ,  $f''(1) = 2$ . Next, through differentiation and the chain rule of  $f(\xi(z)) = z$ , we find  $\xi(0) = 1$ ,  $\xi'(0) = 1$ ,  $\xi''(0) = -2$ , so

$$\xi'(z) = 1 - 2z + \xi'''(c(z))z^2/2.$$

Using this expansion for  $\xi'(z)$  and letting  $M = \sup_{c \in (f(1-\epsilon), f(1+\epsilon))} \xi'''(c)$ ,

$$\begin{aligned} E_2(x, \epsilon) &= x^{x+1}e^{-x} \left( \int_{f(1-\epsilon)}^{f(1+\epsilon)} \exp(-xz^2/2) (1 - 2z + \xi'''(c(z))z^2/2) dz - \int_{-\epsilon}^{\epsilon} \exp(-xz^2/2) dz \right) \\ &= x^{x+1}e^{-x} \left( \int_{f(1-\epsilon)}^{-\epsilon} \exp(-xz^2/2) dz + \int_{\epsilon}^{f(1+\epsilon)} \exp(-xz^2/2) dz \right. \\ &\quad \left. + \int_{f(1-\epsilon)}^{f(1+\epsilon)} \exp(-xz^2/2) (-2z + \xi'''(c(z))z^2/2) dz \right) \\ &\leq x^{x+1}e^{-x} \left| \int_{-\infty}^{-\epsilon} \exp(-xz^2/2) dz + \int_{\epsilon}^{\infty} \exp(-xz^2/2) dz \right| + x^{x+1}e^{-x} \left| \int_{-\infty}^{\infty} \exp(-xz^2/2) (-2z + Mz^2/2) dz \right| \\ &= E_3(x, \epsilon) + x^{x+1}e^{-x} M \sqrt{\frac{\pi}{2x^3}} \end{aligned}$$

where  $E_3(x, \epsilon)$  is the same as the error introduced in (3) and will be bounded next.

Thus,

$$\frac{E_2(x, \epsilon)}{G(x)} \leq \frac{|\xi'''(c)|}{2x} + \frac{E_3(x, \epsilon)}{G(x)}.$$

$E_2(x, \epsilon)$  is the dominating error but is still small compared to our final approximation  $G(x)$ .

INTERVAL EXPANSION. Finally, we would like to show that the error introduced in (3),

$$E_3(x, \epsilon) = x^{x+1}e^{-x} \int_{1+\epsilon}^{\infty} \exp(-x(u-1)^2/2) du + x^{x+1}e^{-x} \int_{-\infty}^{1-\epsilon} \exp(-x(u-1)^2/2) du,$$

is small compared to our final approximation. We label the first term  $E_3^+(x, \epsilon)$  and the second  $E_3^-(x, \epsilon)$ .

We bound  $E_3^+(x, \epsilon)$ . First,

$$\begin{aligned} I(y) &= \int_y^{\infty} \exp(-s^2) ds \\ &= \int_0^{\infty} \exp(-(s+y)^2) ds \\ &= \exp(-y^2) \int_0^{\infty} \exp(-u^2 - 2uy) du \\ &\leq \exp(-y^2) \int_0^{\infty} \exp(-2uy) du \\ &= \frac{\exp(-y^2)}{2y}. \end{aligned}$$

Then, by a change of variables  $s = \sqrt{x/2}(u - 1)$ ,

$$E_3^+(x, \epsilon) = x^{x+1} e^{-x} \sqrt{\frac{x}{2}} I\left(\sqrt{\frac{x}{2}} \epsilon\right) \leq \frac{1}{2\epsilon} x^{x+1} \exp(-x) \exp(-x\epsilon^2/2),$$

which is exponentially small compared to our final approximation, i.e.

$$\frac{E_3^+(x, \epsilon)}{G(x)} \leq \sqrt{\frac{x}{8\pi}} \exp(-x\epsilon^2/2).$$

Similarly, by the change of variables  $s = \sqrt{x/2}(1 - u)$ ,

$$E_3^-(x, \epsilon) = x^{x+1} e^{-x} \sqrt{\frac{x}{2}} I\left(\sqrt{\frac{x}{2}} \epsilon\right) \leq \frac{1}{2\epsilon} x^{x+1} \exp(-x) \exp(-x\epsilon^2/2)$$

and we again see

$$\frac{E_3^-(x, \epsilon)}{G(x)} \leq \sqrt{\frac{x}{8\pi}} \exp(-x\epsilon^2/2).$$

## 2.2. METHOD OF STEEPEST DESCENT.

In general, the integral under consideration may be complex-valued. The Method of Steepest Descent, also known as the Saddle-Point Method, generalizes the idea of Laplace's Method to complex integrals of the form

$$\begin{aligned} I(x) &= \int_{\Gamma} f(z) \exp(xg(z)) dz \\ &= \int_{\Gamma} f(z) \exp\left(x(\phi(z) + i\psi(z))\right) dz \end{aligned}$$

where  $\Gamma$  is a contour in the complex-plane, and  $\phi(z)$ ,  $\psi(z)$ , and  $x$  are real-valued. Unfortunately, we can no longer claim that the main contribution comes from around wherever  $g(z)$  attains a real or absolute maximum along  $\Gamma$  – now, rapid oscillations in the imaginary component  $\psi(z)$  can lead to cancellation.

Instead, this method usually prescribes deforming  $\Gamma$  into a contour along which  $\psi(z)$  is constant. It can be shown these paths of constant phase are also the paths of steepest ascent or decent of  $\phi(z)$  [1]. From this family of contours, we choose one on which  $\phi(z)$  attains a maximum on the interior, which requires  $g'(z) = 0$  at some point on the contour. We then linearly approximate the contour at the saddle point and apply the ideas of Laplace's Method on the remaining integral.

### 2.2.1. Airy's Equation.

As an example, we will look at deriving the asymptotic behavior of solutions to the equation

$$y''(x) - xy(x) = 0,$$

also known as the Airy Equation. To find the solutions, we start by supposing they take the form

$$y(x) = \int_{\Gamma} f(z) \exp(xz) dz$$

where  $\Gamma$  is some contour in the complex plane.

Assuming we can differentiate under the integral and then integrating by parts, we obtain

$$\begin{aligned} y''(x) - xy &= \int_{\Gamma} z^2 f(z) \exp(xz) dz - \int_{\Gamma} x f(z) \exp(xz) dz \\ &= \int_{\Gamma} z^2 f(z) \exp(xz) dz - f(z) \exp(xz)|_{\Gamma} + \int_{\Gamma} f'(z) \exp(xz) dz. \end{aligned}$$

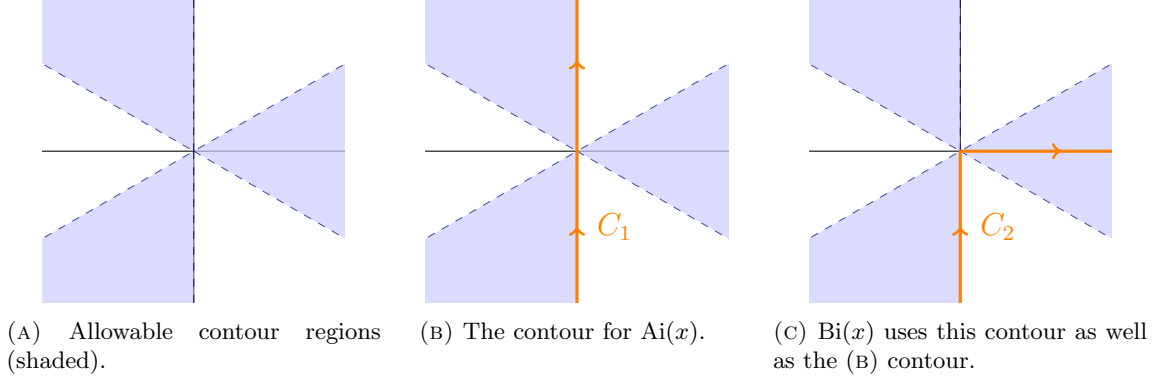


FIGURE 4. Contours of solutions of the Airy equation.

Let us require that  $f(z) \exp(xz) \Big|_{\Gamma} = 0$ . We are then left with  $\int_{\Gamma} (z^2 f(z) + f'(z)) e^{xz} dz = 0$ . This reduces to the condition  $z^2 f(z) + f'(z) = 0$ , or equivalently  $f(z) = c \exp(-z^3/3)$ .

Returning to the boundary conditions,  $f(z) \exp(xz) \Big|_{\Gamma} = c \exp(-z^3/3 + xz) \Big|_{\Gamma} = 0$ . If we pick  $\Gamma$  to extend to infinity in the complex plane, the direction is restricted by  $\text{Re}[z^3] > 0$  as  $\Gamma \rightarrow \infty$ . If  $z = r \exp(i\theta)$ , this gives us  $\theta \in (-\pi/6, \pi/6) \cup (\pi/2, 5\pi/6) \cup (7\pi/6, 3\pi/2)$ . See Figure 4A.

Hence, solutions to Airy's Equation may be represented in integral form as

$$y(x) = \int_{\Gamma} c \exp(-z^3/3 + xz) dz$$

with appropriately chosen  $\Gamma$ .

Two of such solutions,  $\text{Ai}(x)$  and  $\text{Bi}(x)$ , are commonly chosen as a basis for all solutions of the Airy equation. These are known as the Airy and Bairy functions.  $\text{Ai}(x)$  uses the contour  $\Gamma = C_1$  shown in Figure 4B and  $c = -i/(2\pi)$  and can be simplified to

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos(t^3/3 + xt) dt.$$

$\text{Bi}(x)$  adds solutions using  $\Gamma = C_1$ ,  $c = -1/(2\pi)$ , and  $\Gamma = C_2$ ,  $c = 1/\pi$ , and simplifies to

$$\text{Bi}(x) = \frac{1}{\pi} \int_0^{\infty} \left( \exp(-t^3/3 + xt) + \sin(t^3/3 + xt) \right) dt.$$

$\text{Bi}(x)$  is chosen so that its phase differs by that of  $\text{Ai}(x)$  by  $\pi/2$  for negative  $x$  and has the same amplitude of oscillation as  $x \rightarrow -\infty$ .

Because these solutions form a basis for the solutions of Airy's equation, any solution can be written in the form  $c_1 \text{Ai}(x) + c_2 \text{Bi}(x)$ . Note that  $\text{Ai}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , which can be shown using the Reimann-Lebesgue Lemma, and  $\text{Bi}(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , so any solution with  $c_2 \neq 0$  will behave like  $c_2 \text{Bi}(x)$  for large  $x \gg 0$ .

### 2.2.2. Airy Function Asymptotics.

AIRY FUNCTION,  $\lambda > 0$ . Starting with the function

$$\text{Ai}(\lambda) = \frac{-i}{2\pi} \int_{C_1} \exp(-s^3/3 + \lambda s) ds$$

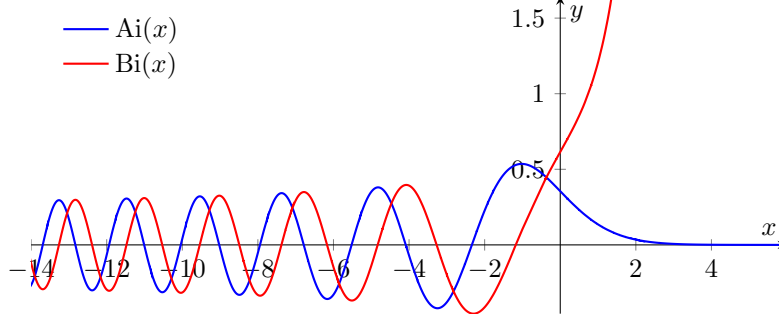


FIGURE 5.  $\text{Ai}(x)$  and  $\text{Bi}(x)$ .

where  $C_1$  is the imaginary axis as shown in Figure 4B, we look for the leading order behavior as  $\lambda \rightarrow +\infty$ . We first make the change of variables  $z = s\sqrt{\lambda}$ :

$$\text{Ai}(\lambda) = \frac{-i\sqrt{\lambda}}{2\pi} \int_{C_1} \exp\left(\lambda^{3/2}(-z^3/3 + z)\right) dz.$$

We look to deform  $C_1$  into a contour which passes through a saddle point of  $g(z) = -z^3/3 + z$  and on which  $g(z)$  has constant imaginary part. The saddle points of  $g(z)$  are at  $z = \pm 1$ . Letting  $z = x + iy$ , we have

$$g(z) = (-x^3/3 + xy^2 + x) + i(y^3/3 - x^2y + y).$$

In order for  $\text{Im}[g(z)]$  to be constant, we have  $y(y^2/3 - x^2 + 1) = C$ . Since  $\text{Im}[g(z)] = 0$  at the saddle points, we see  $C = 0$ . One possible curve which satisfies  $\text{Im}[g(z)] = 0$  and passes through a saddle point is  $y = 0$ , or  $z = x$ , shown as  $D_3$  in Figure 6A; however, this line cannot be deformed into  $C_1$ . The other two possible choices are the hyperbolas from  $y^2/3 - x^2 + 1 = 0$ , shown in Figure 6A as  $D_1$  and  $D_2$ . Using either of these hyperbolas gives  $\text{Re}[g(z)] = 8x^3/3 - 2x$ ; thus,  $D_1$  will attain a real maximum at  $z = -1$  and  $\text{Re}[g(z)] \rightarrow -\infty$  as  $x \rightarrow -\infty$  along the contour, which is what we want, whereas  $D_2$  will attain a real minimum at  $z = 1$ .

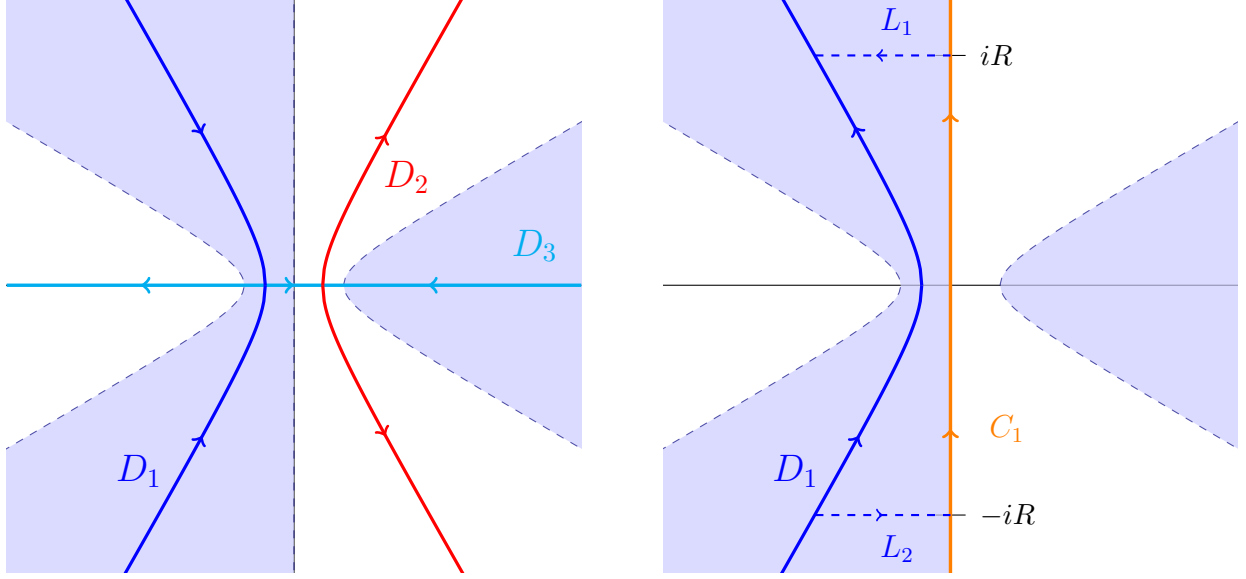
We deform  $C_1$  into  $D_1$  as shown in Figure 6B using  $L_1$  and  $L_2$  and Cauchy's Integral Theorem, since our integrand is analytic, by showing that  $\int_{L_1} \rightarrow 0$  and  $\int_{L_2} \rightarrow 0$  as  $R \rightarrow \infty$ .

Define

$$I_1(\lambda, R) = \frac{-i\sqrt{\lambda}}{2\pi} \int_{L_1} \exp\left(\lambda^{3/2}(-z^3/3 + z)\right) dz.$$

Along  $L_1$ ,  $z = -x + iR$ . Let  $\alpha_R = \sqrt{1 + R^2/3}$  and  $x = \alpha_R \tilde{x}$ .

$$\begin{aligned} I_1(\lambda, R) &= \frac{i\sqrt{\lambda}}{2\pi} \int_0^{\alpha_R} \exp\left(\frac{\lambda^{3/2}}{3}(x^3 - 3xR^2 + iR^3 - 3ix^2R - 3x + 3iR)\right) dx \\ &= \frac{i\sqrt{\lambda}}{2\pi} \int_0^1 \alpha_R \exp\left(\frac{\lambda^{3/2}}{3}[\alpha_R^3 \tilde{x}^3 - 3\alpha_R \tilde{x}R^2 - \alpha_R \tilde{x}]\right) \exp\left(\frac{i\lambda^{3/2}}{3}[R^3 - 3\alpha_R^2 \tilde{x}^2 R + R]\right) d\tilde{x}. \end{aligned}$$



(A) Contours for  $\text{Im}[-z^3/3 + z] = 0$  where arrows indicate direction of increasing real part.

(B) Steepest descent contour  $D_1$  for  $\text{Ai}(\lambda)$ ,  $\lambda > 0$ .  $C_1$  is deformed into  $D_1$  by  $L_1$  and  $L_2$ . Arrows indicate direction of traversal.

FIGURE 6. Contours for  $\text{Ai}(\lambda)$ ,  $\lambda > 0$ . Shading indicates where  $\text{Re}[-z^3/3 + z] < 0$ .

Then

$$\begin{aligned}
\left| I_1(\lambda, R) \right| &\leq \frac{\sqrt{\lambda}}{2\pi} \int_0^1 \alpha_R \exp\left(\frac{\lambda^{3/2}}{3} [\alpha_R^3 \tilde{x}^3 - 3\alpha_R \tilde{x} R^2 - \alpha_R \tilde{x}]\right) d\tilde{x} \\
&\leq \frac{\sqrt{\lambda}}{2\pi} \int_0^1 \alpha_R \exp\left(\frac{\tilde{x} \lambda^{3/2}}{3} [\alpha_R^3 - 3\alpha_R R^2 - \alpha_R]\right) d\tilde{x} \\
&= \frac{\sqrt{\lambda}}{2\pi} \frac{3\alpha_R}{\lambda^{3/2} (\alpha_R^3 - 3\alpha_R R^2 - \alpha_R)} \left( \exp\left(\frac{\lambda^{3/2}}{3} [\alpha_R^3 - 3\alpha_R R^2 - \alpha_R]\right) - 1 \right) \\
&\sim \frac{\sqrt{\lambda}}{2\pi} \frac{3}{\lambda^{3/2} (-8R^2/3 - 1)} \left( \exp\left(\frac{\lambda^{3/2}}{3\sqrt{3}} [-8R^3/3 - R]\right) - 1 \right) \quad \text{using } \alpha_R \sim R/\sqrt{3} \text{ as } R \rightarrow \infty \\
&\rightarrow 0 \text{ as } R \rightarrow \infty.
\end{aligned}$$

An almost identical argument holds for  $L_2$  where  $z = -x - iR$ . Thus, we have deformed the contour  $C_1$  into  $D_1$  along which  $\text{Im}[g(z)] = 0$ , so

$$\begin{aligned}
\text{Ai}(\lambda) &= \frac{-i\sqrt{\lambda}}{2\pi} \int_{D_1} \exp\left(\lambda^{3/2} (-z^3/3 + z)\right) dz \\
&= \frac{-i\sqrt{\lambda}}{2\pi} \int_{D_1} \exp\left(\lambda^{3/2} \text{Re}[-z^3/3 + z]\right) dz.
\end{aligned}$$

Along  $D_1$ , we have a maximum at  $z = -1$  and can apply the ideas behind Laplace's Method. First, everything away from  $z = -1$  on  $D_1$  is negligible. We approximate  $D_1$  near  $z = -1$  by the line  $z = -1 + i\tau$  where  $\tau$  is real-valued, shown as  $\tilde{D}_1$  in Figure 7. Observe that  $z = -1$  corresponds to  $\tau = 0$ , and so everything away from  $\tau = 0$  is negligible. Then

$$-z^3/3 + z = -2/3 - \tau^2 + i\tau^3/3,$$

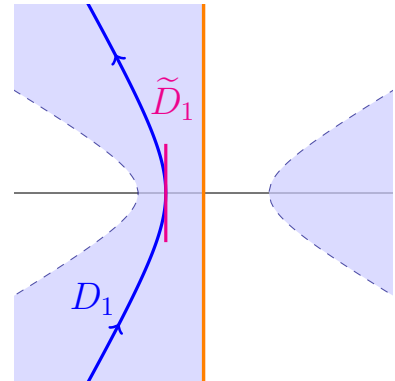


FIGURE 7. We approximate the steepest descent contour  $D_1$  near  $z = -1$  by  $\tilde{D}_1$ .

and so  $\operatorname{Re}[g(z)] = -2/3 - \tau^2$  on  $\tilde{D}_1$ . Applying Laplace's Method, for large  $\lambda > 0$  we have

$$\begin{aligned}
\operatorname{Ai}(\lambda) &= \frac{-i\sqrt{\lambda}}{2\pi} \int_{D_1} \exp\left(\lambda^{3/2} \operatorname{Re}[-z^3/3 + z]\right) dz \\
&\approx \frac{-i\sqrt{\lambda}}{2\pi} \int_{D_1 \text{ near } z=-1} \exp\left(\lambda^{3/2} \operatorname{Re}[-z^3/3 + z]\right) dz \\
&\approx \frac{\sqrt{\lambda}}{2\pi} \exp\left(-\frac{2}{3}\lambda^{3/2}\right) \int_{-\epsilon}^{\epsilon} \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\
&\approx \frac{\sqrt{\lambda}}{2\pi} \exp\left(-\frac{2}{3}\lambda^{3/2}\right) \int_{-\infty}^{\infty} \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\
&= \frac{1}{2\sqrt{\pi}\lambda^{1/4}} \exp\left(-\frac{2}{3}\lambda^{3/2}\right).
\end{aligned}$$

This is the leading order approximation for  $\operatorname{Ai}(\lambda)$  as  $\lambda \rightarrow \infty$ .

AIRY FUNCTION,  $\lambda < 0$ . In this case, we let  $z = s\sqrt{-\lambda}$ . Then

$$\operatorname{Ai}(\lambda) = \frac{-i\sqrt{-\lambda}}{2\pi} \int_{C_1} \exp\left((- \lambda)^{3/2} (-z^3/3 - z)\right) dz.$$

We again look to deform  $C_1$ . The saddle points of  $g(z) = -z^3/3 - z$  are at  $z = \pm i$ . Letting  $z = x + iy$ ,

$$g(z) = (-x^3/3 + xy^2 - x) + i(y^3/3 - x^2y - y).$$

Requiring  $\operatorname{Im}[g(z)]$  to be constant gives us  $y^3/3 - x^2y - y = C$ . Because  $\operatorname{Im}[g(z)] = \mp 2/3$  at the saddle points, we require  $y^3/3 - x^2y - y = \mp 2/3$ , which leads to the solutions shown in Figure 8A and 8B. In each figure the solutions share a constant imaginary part and, noting the arrows which indicate the direction of increasing  $\operatorname{Re}[g(z)]$ , we build a steepest descent curve by taking the solid paths. We then deform the contour  $C_1$  into  $E_1 + E_2$ , as shown in Figure 9, where  $\operatorname{Re}[g(z)]$  attains a maximum at  $z = -i$  along  $E_1$  and at  $z = i$  along  $E_2$ .

To do this, one can show, similar to before, that the integral along  $L_1$  and  $L_2$  goes to zero as  $R \rightarrow \infty$ .

We again apply the ideas of Laplace's Method. Along  $E_2$ , we have a maximum at  $z = i$  and  $\operatorname{Im}[g(z)] = -2/3$ . Again, away from  $z = i$  the contribution from the integral along  $E_2$  is negligible, and near  $z = i$  we approximate  $E_2$  by the line  $z = -\tau + i(\tau + 1)$  with  $\tau$  real. Then on  $\tilde{E}_2$ ,

$$-z^3/3 - z = -2\tau^2 - 2\tau^3/3 + i(-2/3 - 2\tau^3/3).$$

Thus,  $\operatorname{Re}[g(z)] \approx -2\tau^2$  near  $z = i$  or  $\tau = 0$ . Following the same procedure as before,

$$\begin{aligned}
I_2(\lambda) &= \frac{-i\sqrt{-\lambda}}{2\pi} \int_{E_2} \exp\left((- \lambda)^{3/2} (-z^3/3 - z)\right) dz \\
&= \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(i(-\lambda)^{3/2} \operatorname{Im}[-z^3/3 - z]\right) \int_{E_2} \exp\left((- \lambda)^{3/2} \operatorname{Re}[-z^3/3 - z]\right) dz \\
&\approx \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{E_2 \text{ near } z=i} \exp\left((- \lambda)^{3/2} \operatorname{Re}[-z^3/3 - z]\right) dz \\
&\approx \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{-\epsilon}^{\epsilon} \exp\left(-2(-\lambda)^{3/2}\tau^2\right) (-1 + i)d\tau \\
&\approx \frac{(1+i)\sqrt{-\lambda}}{2\pi} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{-\infty}^{\infty} \exp\left(-2(-\lambda)^{3/2}\tau^2\right) d\tau \\
&= \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right).
\end{aligned}$$

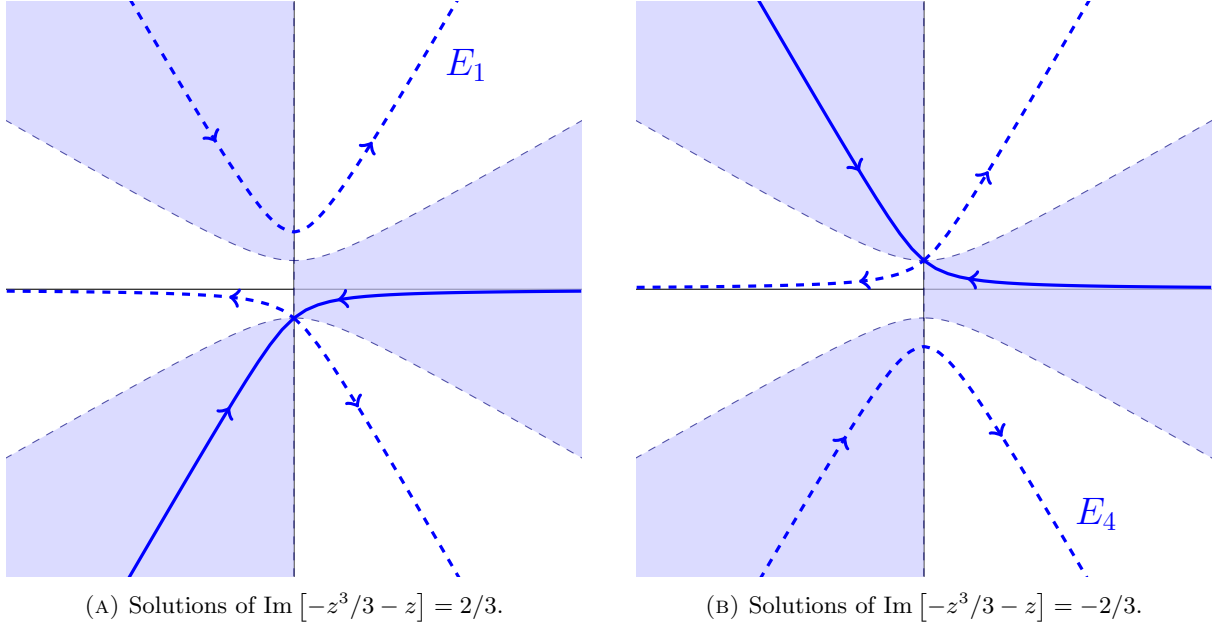


FIGURE 8. Solutions of  $\text{Im} [-z^3/3 - z] = \pm 2/3$  with arrows indicating direction of increasing real part. Shading indicates where  $\text{Re} [-z^3/3 - z] < 0$ . To build a path of steepest descent for  $\text{Ai}(\lambda)$ ,  $\lambda < 0$ , we take the solid paths. See Figure 9. Dashing indicates paths not taken.

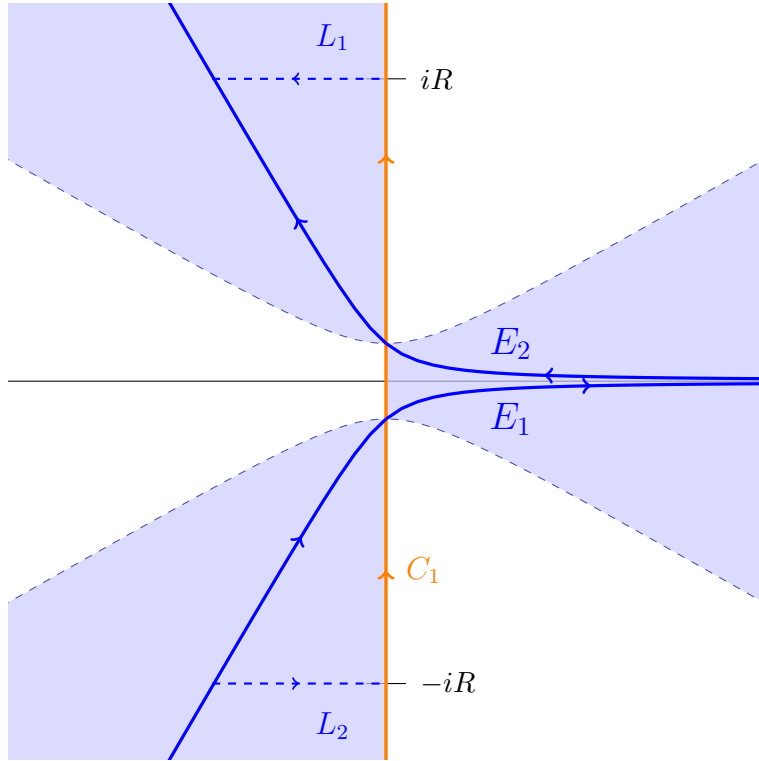


FIGURE 9. Steepest descent contours  $E_1$  and  $E_2$  for  $\text{Ai}(\lambda)$ ,  $\lambda < 0$ . Arrows indicate direction of traversal. Shading indicates where  $\text{Re} [-z^3/3 - z] < 0$ .  $C_1$  is deformed into  $E_2 + E_3$  by  $L_1$  and  $L_2$ .

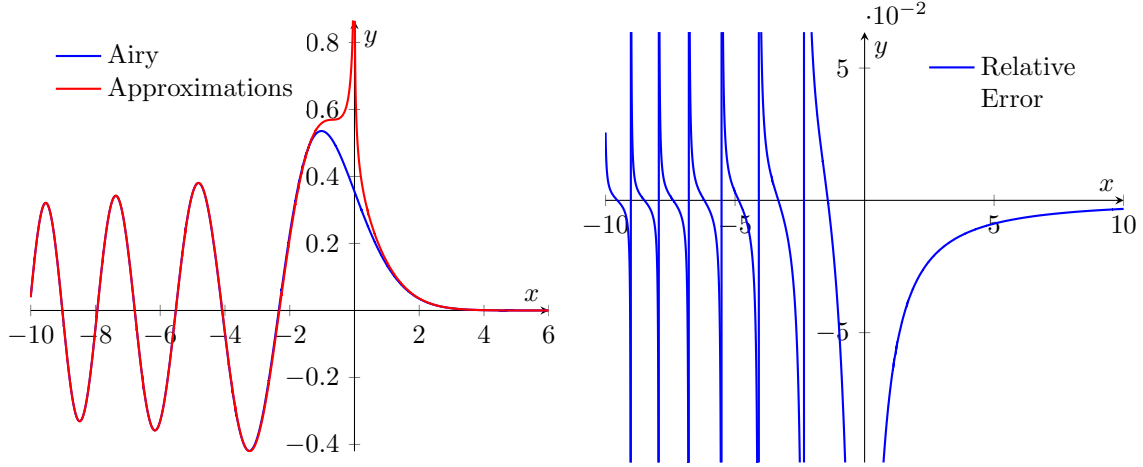


FIGURE 10. A plot of  $\text{Ai}(x)$  and its leading order approximations for  $x < 0$  and  $x > 0$ . The relative error is also plotted. The zeros of the approximation for  $x < 0$  do not exactly coincide with the zeros of  $\text{Ai}(x)$ , hence the relative error blows up at the zeros of  $\text{Ai}(x)$ .

Along  $E_1$ , we have a maximum at  $z = -i$  and have  $\text{Im}[g(z)] = 2/3$ . Near  $z = -i$ , we approximate  $E_1$  by  $z = \tau + i(\tau - 1)$  with  $\tau$  real. Then along  $\tilde{E}_1$ ,

$$-z^3/3 - z = -2\tau^2 + 2\tau^3/3 + i(2/3 - 2\tau^3/3).$$

Thus, we approximate  $\text{Re}[g(z)] \approx -2\tau^2$ .

$$\begin{aligned} I_1(\lambda) &= \frac{-i\sqrt{-\lambda}}{2\pi} \int_{E_1} \exp\left((- \lambda)^{3/2} (-z^3/3 - z)\right) dz \\ &= \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(i(-\lambda)^{3/2} \text{Im}[-z^3/3 - z]\right) \int_{E_1} \exp\left((- \lambda)^{3/2} \text{Re}[-z^3/3 - z]\right) dz \\ &\approx \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{E_1 \text{ near } z=-i} \exp\left((- \lambda)^{3/2} \text{Re}[-z^3/3 - z]\right) dz \\ &\approx \frac{-i\sqrt{-\lambda}}{2\pi} \exp\left(i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{-\epsilon}^{\epsilon} \exp\left(-2(-\lambda)^{3/2}\tau^2\right) (1+i)d\tau \\ &\approx \frac{(1-i)\sqrt{-\lambda}}{2\pi} \exp\left(i\frac{2}{3}(-\lambda)^{3/2}\right) \int_{-\infty}^{\infty} \exp\left(-2(-\lambda)^{3/2}\tau^2\right) d\tau \\ &= \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right). \end{aligned}$$

Thus, as  $\lambda \rightarrow -\infty$ ,

$$\begin{aligned} \text{Ai}(\lambda) &= I_1(\lambda) + I_2(\lambda) \\ &\approx \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right) + \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right). \end{aligned}$$

#### A NOTE ON NOTATION

Although it is frequently written as such, it is technically incorrect to write

$$\text{Ai}(\lambda) \sim \frac{1}{\sqrt{\pi}(-\lambda)^{1/4}} \cos\left(-\frac{2}{3}(-\lambda)^{3/2} + \frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}(-\lambda)^{1/4}} \sin\left(\frac{2}{3}(-\lambda)^{3/2} + \frac{\pi}{4}\right)$$

as  $\lambda \rightarrow -\infty$  because this would imply

$$\text{Ai}(\lambda) = \frac{1}{\sqrt{\pi}(-\lambda)^{1/4}} \sin\left(\frac{2}{3}(-\lambda)^{3/2} + \frac{\pi}{4}\right) (1 + o(1)).$$

This is false, because the zeros of  $\text{Ai}(\lambda)$  do not exactly coincide with those of  $\sin\left(\frac{2}{3}(-\lambda)^{3/2} + \frac{\pi}{4}\right)$ . Instead,

$$\text{Ai}(\lambda) \sim \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right) + \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right)$$

because this statement allows flexibility in the location of zeros:

$$\text{Ai}(\lambda) = \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right) (1 + o(1)) + \frac{1}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right) (1 + o(1)).$$

Similarly, it is often falsely stated that as  $\lambda \rightarrow -\infty$ ,

$$\text{Bi}(\lambda) \sim \frac{1}{\sqrt{\pi}(-\lambda)^{1/4}} \sin\left(-\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}(-\lambda)^{1/4}} \cos\left(\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right).$$

BAIRY FUNCTION,  $\lambda > 0$ . We start with

$$\text{Bi}(\lambda) = \frac{-1}{2\pi} \int_{C_1} \exp(-s^3/3 + \lambda s) ds + \frac{1}{\pi} \int_{C_2} \exp(-s^3/3 + \lambda s) ds.$$

Since we've already analyzed  $C_1$ , from which we get

$$\frac{-1}{2\pi} \int_{C_1} \exp(-s^3/3 + \lambda s) ds \sim \frac{-i}{2\sqrt{\pi}\lambda^{1/4}} \exp\left(-\frac{2}{3}\lambda^{3/2}\right),$$

we look at deforming  $C_2$ .

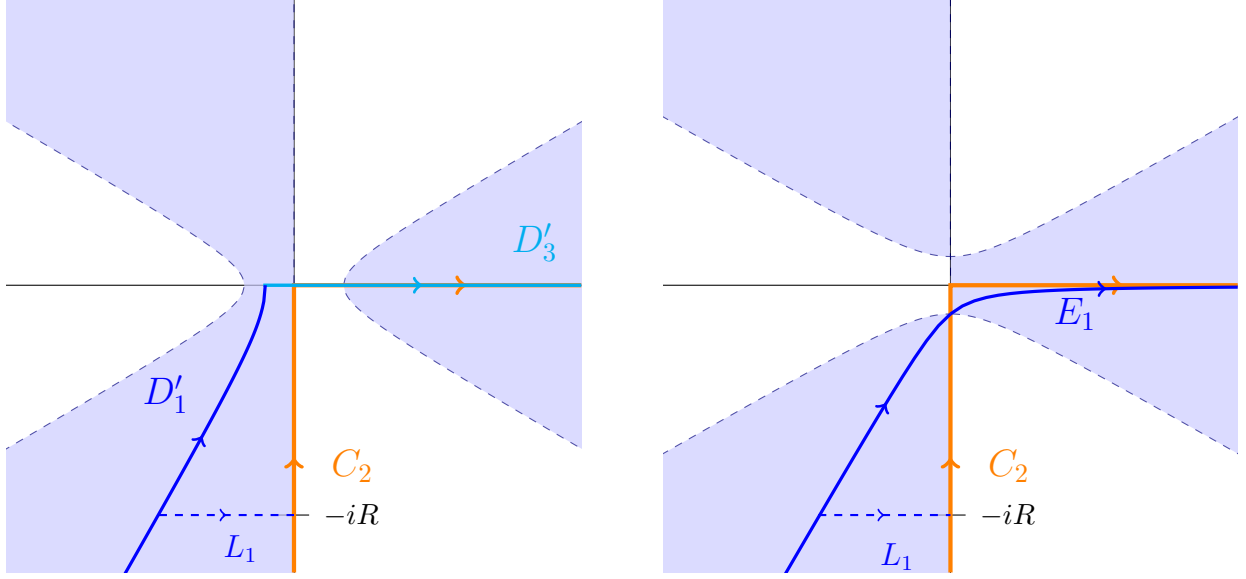
After a change of variables  $z = s\sqrt{\lambda}$  for  $\lambda > 0$ , we recall the saddle points of  $g(z) = -z^3/3 + z$  are at  $z = \pm 1$ . This time, we see that we can use  $D_3$  to help build a contour along which  $\text{Im}[g(z)]$  is constant, as shown in Figure 11A. After noting that the integral along  $L_1$  goes to 0 as  $R \rightarrow \infty$ , we have

$$\frac{1}{\pi} \int_{C_2} \exp(-s^3/3 + \lambda s) ds = \frac{\sqrt{\lambda}}{\pi} \int_{D'_1} \exp\left(\lambda^{3/2}(-z^3/3 + z)\right) dz + \frac{\sqrt{\lambda}}{\pi} \int_{D'_3} \exp\left(\lambda^{3/2}(-z^3/3 + z)\right) dz.$$

Similar to the Airy Function,  $\lambda > 0$  case, we approximate  $D'_1$  by the line  $z = -1 + i\tau$  where  $\tau$  ranges from  $-\epsilon$  to 0 — this is the bottom half of  $\tilde{D}_1$  from Figure 7. Repeating the calculation process from before,

$$\begin{aligned} \text{Bi}(\lambda) &= \frac{\sqrt{\lambda}}{\pi} \int_{D'_1} \exp\left(\lambda^{3/2} \text{Re}[-z^3/3 + z]\right) dz \\ &\approx \frac{i\sqrt{\lambda}}{\pi} \exp\left(-\frac{2}{3}\lambda^{3/2}\right) \int_{-\epsilon}^0 \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\ &\approx \frac{\sqrt{i\lambda}}{\pi} \exp\left(-\frac{2}{3}\lambda^{3/2}\right) \int_{-\infty}^0 \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\ &= \frac{i}{2\sqrt{\pi}\lambda^{1/4}} \exp\left(-\frac{2}{3}\lambda^{3/2}\right). \end{aligned}$$

Along  $D'_3$ , we have saddle points at  $z = 1$  and  $z = -1$ . Note  $g(-1) = -2/3$  is a local minimum and  $g(1) = 2/3$  is a global maximum of  $\text{Re}[g(z)]$  along  $D'_3$ , so the contribution from  $z = 1$  is dominant. Around  $z = 1$ , we make the exact change of variables  $z = 1 + \tau$  with  $\tau$  real. Then  $\text{Re}[g(z)] = 2/3 - \tau^2 - \tau^3/3 \approx 2/3 - \tau^2$



(A) Steepest descent contour built from  $D'_1$  and  $D'_3$  for the  $C_2$  part of  $\text{Bi}(\lambda)$ ,  $\lambda > 0$ . Shading indicates where  $\text{Re}[-z^3/3 + z] < 0$ .  
 (B) Steepest descent contour  $E_1$  for the  $C_2$  part of  $\text{Bi}(\lambda)$ ,  $\lambda < 0$ . Shading indicates where  $\text{Re}[-z^3/3 + z] < 0$ .

FIGURE 11. Steepest descent contours for  $C_2$  part of  $\text{Bi}(\lambda)$ ,  $\lambda < 0$ , where arrows indicate direction of traversal.

is maximal around  $\tau = 0$  and recall  $\text{Im}[g(z)] = 0$ . Then

$$\begin{aligned}
 \frac{\sqrt{\lambda}}{\pi} \int_{D'_3} \exp\left(\lambda^{3/2}(-z^3/3 + z)\right) dz &= \frac{\sqrt{\lambda}}{\pi} \exp\left(i\lambda^{3/2}\text{Im}[-z^3/3 + z]\right) \int_{D'_3} \exp\left(\lambda^{3/2}\text{Re}[-z^3/3 + z]\right) dz \\
 &\approx \frac{\sqrt{\lambda}}{\pi} \int_{D'_3 \text{ near } z=1} \exp\left(\lambda^{3/2}\text{Re}[-z^3/3 + z]\right) dz \\
 &\approx \frac{\sqrt{\lambda}}{\pi} \exp\left(\frac{2}{3}\lambda^{3/2}\right) \int_{-\epsilon}^{\epsilon} \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\
 &\approx \frac{\sqrt{\lambda}}{\pi} \exp\left(\frac{2}{3}\lambda^{3/2}\right) \int_{-\infty}^{\infty} \exp\left(-\lambda^{3/2}\tau^2\right) d\tau \\
 &= \frac{1}{\sqrt{\pi}\lambda^{1/4}} \exp\left(\frac{2}{3}\lambda^{3/2}\right).
 \end{aligned}$$

Adding these three approximations, we see as  $\lambda \rightarrow +\infty$ ,

$$\text{Bi}(\lambda) \sim \frac{1}{\sqrt{\pi}\lambda^{1/4}} \exp\left(\frac{2}{3}\lambda^{3/2}\right).$$

BAIRY FUNCTION,  $\lambda < 0$ . We have again already analyzed  $C_1$  in the case of  $\lambda < 0$  and get

$$\frac{-1}{2\pi} \int_{C_1} \exp(-s^3/3 + \lambda s) ds \sim \frac{-i}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right) - \frac{i}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right).$$

Looking at Figure 11B, we see that we can deform  $C_2$  into  $E_1$ . Using our previous analysis,

$$\frac{1}{\pi} \int_{C_2} \exp(-s^3/3 + \lambda s) ds \sim \frac{i}{\sqrt{\pi}\lambda^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\pi/4\right).$$

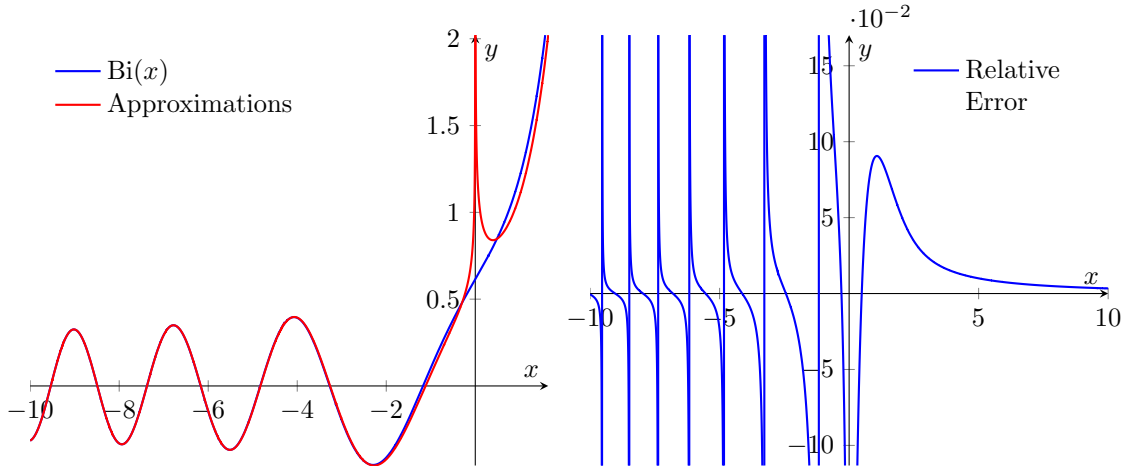


FIGURE 12. A plot of  $\text{Bi}(x)$  and its leading order approximations for  $x < 0$  and  $x > 0$ . The relative error is also plotted. Again, the zeros of the approximation for  $x < 0$  do not exactly coincide with the zeros of  $\text{Bi}(x)$ , hence the relative error blows up at the zeros of  $\text{Bi}(x)$ .

Thus,

$$\begin{aligned} \text{Bi}(\lambda) &= \frac{-1}{2\pi} \int_{C_1} \exp(-s^3/3 + \lambda s) ds + \frac{1}{\pi} \int_{C_2} \exp(-s^3/3 + \lambda s) ds \\ &\sim \frac{-i}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(-i\frac{2}{3}(-\lambda)^{3/2} + i\frac{\pi}{4}\right) + \frac{i}{2\sqrt{\pi}(-\lambda)^{1/4}} \exp\left(i\frac{2}{3}(-\lambda)^{3/2} - i\frac{\pi}{4}\right). \end{aligned}$$

### 3. THE WKB APPROXIMATION

The WKB Method finds approximate solutions to the second-order differential equation

$$\epsilon^2 y''(x) - V(x)y(x) = 0$$

where  $\epsilon$  is a small parameter. For a general potential,  $V(x)$ , this equation does not have an exact solution. For example,  $V(x) = x^2 + x^3$ . Even when the equation does have an exact solution, it may not be manageable, such as how using  $V(x) = x^2$  gives a solution in terms of the parabolic cylinder function. Hence, we look for an approximate solution in the limit of small  $\epsilon$ .

EXTENSION TO A MORE GENERAL EQUATION.

Note that any second order equation of the form

$$z''(x) + p(x)z'(x) + q(x)z(x) = 0$$

with  $p(x) \neq 0$  can be converted into WKB form

$$\epsilon^2 y''(x) - V(x)y(x) = 0$$

through the change of variables  $z = y \exp\left(-\frac{1}{2} \int p(x) dx\right)$  and  $V(x) = \epsilon^2 \left[p'(x)/2 + p^2(x)/4 - q(x)\right]$ .

This method is most often used to localize approximate solutions to the time-independent, one-dimensional Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x)\psi = E\psi.$$

Without exactly solving the equation, one can find the allowable energy levels for a given potential. Griffiths [3] has a thorough presentation of this method in the context of the Schrödinger equation. In this paper, we return to our more general equation.

### 3.1. FORMULATION.

For intuition's sake, let us start by looking at the case where  $V(x) = V_0$  is a constant. Then our equation becomes  $\epsilon^2 y''(x) - V_0 y(x) = 0$ . For solutions, we have three separate cases:

$$y(x) = \begin{cases} a_1 \exp\left(-x\sqrt{V_0}/\epsilon\right) + a_2 \exp\left(x\sqrt{V_0}/\epsilon\right) & \text{if } V_0 > 0, \\ c_1 \exp\left(-ix\sqrt{V_0}/\epsilon\right) + c_2 \exp\left(ix\sqrt{V_0}/\epsilon\right) & \text{if } V_0 < 0, \\ c_1 x + c_2 & \text{if } V_0 = 0. \end{cases}$$

This suggests that for a general  $V(x)$  we use the ansatz  $y(x) = A(x) \exp(\eta(x)/\epsilon)$  when  $V(x) > 0$  and  $y(x) = A(x) \exp(i\eta(x)/\epsilon)$  when  $V(x) < 0$ . Here  $A(x)$  is the amplitude function and  $\eta(x)$  is the phase function.

We already know that we will not be able to find an exact solution for general  $V(x)$ , so we expand  $A(x)$  in a power series in powers of  $\epsilon$ :

$$A(x) = a_0(x) + \epsilon a_1(x) + \epsilon^2 a_2(x) + \dots$$

For shorthand, we will abbreviate  $A' = A'(x)$ . When  $V(x) > 0$ , we compute

$$\begin{aligned} y'(x) &= [A' + A\eta'/\epsilon] \exp(\eta/\epsilon), \\ y''(x) &= [A'' + 2A'\eta'/\epsilon + A\eta''/\epsilon + A(\eta')^2/\epsilon^2] \exp(\eta/\epsilon). \end{aligned}$$

Plugging in the expansion for  $A(x)$  and grouping by orders of  $\epsilon$ , our equation becomes:

$$\left[ (\eta')^2 a_0 + \epsilon(2\eta' a_0' + \eta'' a_0 + (\eta')^2 a_1) + \epsilon^2(a_0'' + 2\eta' a_1' + \eta'' a_1 + (\eta')^2 a_2) + \mathcal{O}(\epsilon^3) \right] - V(x) \left[ a_0 + \epsilon a_1 + \epsilon^2 a_2 + \mathcal{O}(\epsilon^3) \right] = 0.$$

Because our equation holds in the limit as  $\epsilon \rightarrow 0$ , it is valid to equate orders of  $\epsilon$ , from which we get:

$$\begin{aligned} \mathcal{O}(1): \quad & (\eta')^2 a_0 = V(x) a_0 \\ & \implies \eta = \pm \int \sqrt{V(x)} dx. \\ \mathcal{O}(\epsilon): \quad & 2\eta' a_0' + \eta'' a_0 + (\eta')^2 a_1 = V(x) a_1 \\ & \implies 2\eta' a_0' + \eta'' a_0 = 0 \\ & \implies a_0 = c/\sqrt{\eta'}. \end{aligned}$$

The first of these conditions,  $\eta = \pm \int \sqrt{V(x)} dx$ , is known as the Eikonal equation. The second condition,  $2\eta' a_0' + \eta'' a_0 = 0$ , is known as the transport equation.

We could continue this process to solve for each  $a_i(x)$ , but stopping here gives us a leading-order approximation of  $y(x)$ .

Hence, when  $V(x) > 0$  our approximate solution is

$$y_+(x) = \alpha_1 (V(x))^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_0^x \sqrt{V(x)} dx\right) + \alpha_2 (V(x))^{-1/4} \exp\left(\frac{1}{\epsilon} \int_0^x \sqrt{V(x)} dx\right).$$

Doing an almost identical analysis, we find that when  $V(x) < 0$  our approximate solution is

$$y_-(x) = \beta_1 (-V(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_0^x \sqrt{-V(x)} dx\right) + \beta_2 (-V(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_0^x \sqrt{-V(x)} dx\right).$$

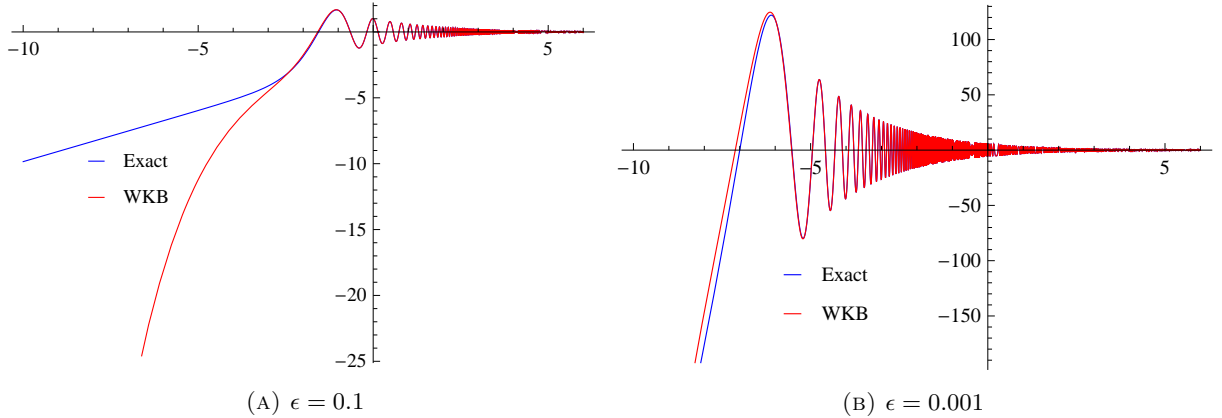


FIGURE 13. Example plot of exact solution, which is in terms of Bessel functions of the first and second kind, along with the WKB approximation where  $V(x) = -e^{2x}$ . As  $V(x) \rightarrow 0$ , or  $x \rightarrow -\infty$ , the WKB approximation worsens. As  $\epsilon \rightarrow 0$ , the approximation improves. Note that the exact solution also changes with  $\epsilon$ . Plotted via Mathematica.

For a purely positive or negative  $V(x)$ , we can use these as our approximate solutions.

To get an idea of the error incurred in leaving out the rest of the series in our approximation, we could set  $a_1(x) = a_0(x)w(x)$ . Solving for  $w(x)$ , we find that  $w(x)$  gets large as  $V(x) \rightarrow 0$ :

$$w(x) = k + \frac{V'(x)}{8(V(x))^{3/2}} + \frac{1}{32} \int \frac{(V'(x))^2}{(V(x))^{5/2}} dx.$$

The details of this derivation can be found in Holmes [4]. For this paper, it suffices to understand that the error of our approximation grows as  $V(x)$  approaches 0. This is to be expected because our approximate solutions blow up as  $V(x) \rightarrow 0$ .

### 3.1.1. A Simple Example.

For an example, we look at  $V(x) = -e^{2x}$ , which is purely negative. Figures 13a and 13b show the exact solutions along with the approximate solutions for two different values of  $\epsilon$ . We can see that as  $\epsilon \rightarrow 0$ , our approximate solution approaches the exact solution, although the solution differs for each  $\epsilon$ . Also, as  $x \rightarrow -\infty$  for any value of  $\epsilon$ ,  $V(x) \rightarrow 0$  and our approximate solution diverges from the exact.

## 3.2. TURNING POINTS AND THE CONNECTION FORMULA.

Suppose we have a true solution,  $y(x)$ , to the equation  $\epsilon^2 y''(x) - V(x)y(x) = 0$  for  $x \in \mathbb{R}$ . Our goal is to build an approximation to  $y(x)$  valid for all  $x \in \mathbb{R}$ . Because we will have different approximations in different regions for a general  $V(x)$ , we denote the approximation of  $y(x)$  when  $V(x) > 0$  as  $y_+(x)$  and when  $V(x) < 0$  as  $y_-(x)$ . When  $V(x)$  is near 0, we will approximate  $y(x)$  by  $y_0(x)$ . Then, combining these approximations, we will have a global approximation to  $y(x)$ . To approximate  $y(x)$  when  $V(x)$  is near 0, we will approximate  $V(x)$  by a straight line.

### 3.2.1. Connection Formula.

Assume that  $V(x)$  has a simple zero and positive slope at  $x = 0$ . Then near zero we find an approximate  $y_0(x)$  using  $V(x) = ax$  where  $a = V'(0) > 0$ . Our equation becomes

$$(7) \quad \epsilon^2 y_0''(x) - axy_0(x) = 0.$$

Let us consider a rescaling of this equation with  $\tilde{x} = \alpha x$ . Then we have

$$\epsilon^2 \alpha^2 \frac{d^2 y_0}{d\tilde{x}^2} - \frac{a}{\alpha} \tilde{x} y_0 = 0.$$

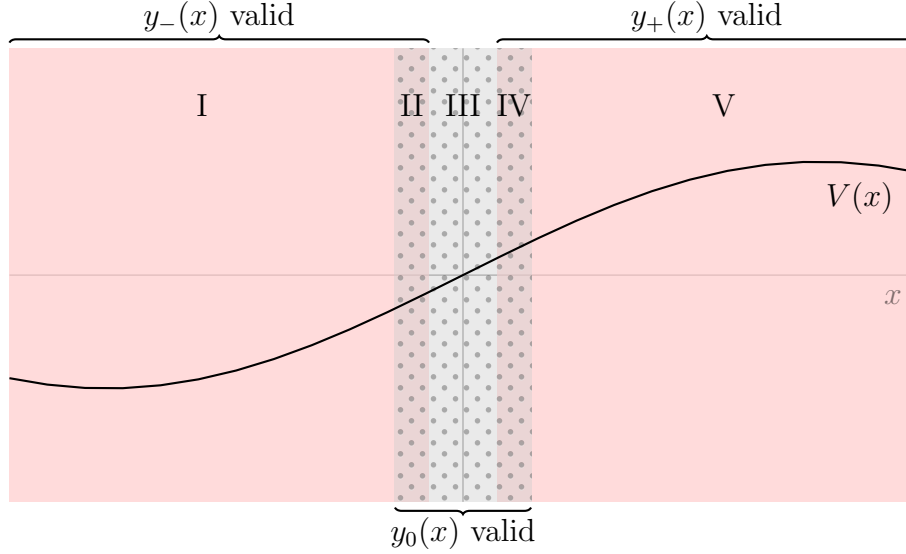


FIGURE 14. Connection regions.

Letting  $\alpha = a^{1/3}/\epsilon^{2/3}$ , we arrive at Airy's Equation

$$\frac{d^2 y_0}{d\tilde{x}^2} - \tilde{x} y_0 = 0$$

with solution  $y_0(\tilde{x}) = c_1 \text{Ai}(\tilde{x}) + c_2 \text{Bi}(\tilde{x})$ . Hence, (7) has solutions

$$y_0(x) = c_1 \text{Ai}\left(\frac{a^{1/3}}{\epsilon^{2/3}} x\right) + c_2 \text{Bi}\left(\frac{a^{1/3}}{\epsilon^{2/3}} x\right).$$

There exists an overlap in regions of validity for approximations  $y_0(x)$  and  $y_+(x)$ , and similarly an overlap for  $y_0(x)$  and  $y_-(x)$ . Hence, in order to have a consistent global approximation, we “match” the solutions by making sure they agree asymptotically in these regions. We will refer to the regions as shown in Figure 14. Note the regions of overlap are regions II and IV. In the limit of small  $\epsilon$ , it becomes valid to use the asymptotics of Ai and Bi for our solution  $y_0(x)$  for any fixed  $x$ . We start by assuming we know  $\alpha_1$  and  $\alpha_2$  and work to determine  $c_1, c_2, \beta_1, \beta_2$ .

In region IV,  $x > 0$  and so

$$\text{Ai}(\alpha x) \sim \frac{\epsilon^{1/6} \exp\left(-\frac{2}{3} a^{1/2} \epsilon^{-1} x^{3/2}\right)}{2\sqrt{\pi} a^{1/12} x^{1/4}}, \quad \text{Bi}(\tilde{x}) \sim \frac{\epsilon^{1/6} \exp\left(\frac{2}{3} a^{1/2} \epsilon^{-1} x^{3/2}\right)}{\sqrt{\pi} a^{1/12} \tilde{x}^{1/4}}.$$

Substituting  $V(x) = ax$  into  $y_+(x)$ , we have

$$y_+(x) = \alpha_1 (ax)^{-1/4} \exp\left(-\frac{2}{3} a^{1/2} \epsilon^{-1} x^{3/2}\right) + \alpha_2 (ax)^{-1/4} \exp\left(\frac{2}{3} a^{1/2} \epsilon^{-1} x^{3/2}\right).$$

Assuming  $y_0(x)$  and  $y_+(x)$  are both valid approximations of  $y(x)$  in this region, it must be true that  $\alpha_2 a^{-1/4} = c_2 \pi^{-1/2} \epsilon^{1/6} a^{-1/12}$ . For all physically-relevant problems  $\alpha_2 = 0$ , which we assume for the rest of this paper. It is then visible that  $\alpha_1 a^{-1/4} = c_1 2^{-1} \pi^{-1/2} \epsilon^{1/6} a^{-1/12}$ .

#### A NOTE ON COEFFICIENTS

Suppose  $\alpha_2 \neq 0$ . Our argument for determining  $c_1, c_2$  from  $\alpha_1, \alpha_2$  relies on the approximations having the same asymptotic behavior in region IV. Unfortunately,  $\alpha_1 \text{Ai}(x) + \alpha_2 \text{Bi}(x) \sim \alpha_2 \text{Bi}(x)$  asymptotically and so for any choice of  $c_1$ , our approximations will be asymptotically equivalent. Thus, we will be unable to

determine the relation of  $c_1$  to  $\alpha_1$  and  $\alpha_2$ . After finding the relation of  $c_1, c_2$  to  $\beta_1, \beta_2$ , we will be left with three coefficients:  $\alpha_1, \alpha_2, c_1$ .

Similarly, if we were to start with knowing  $\beta_1, \beta_2$ , we could find their relation to  $c_1, c_2$ ; we could then determine  $\alpha_2$  but then fail to find  $\alpha_1$ 's relation to anything.

In region II,  $x < 0$  and so

$$\text{Ai}(\alpha x) \sim \frac{\epsilon^{1/6}}{2\sqrt{\pi}a^{1/12}(-x)^{1/4}} \left[ \exp\left(-i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} + i\frac{\pi}{4}\right) + \exp\left(i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} - i\frac{\pi}{4}\right) \right],$$

$$\text{Bi}(\alpha x) \sim \frac{\epsilon^{1/6}}{2i\sqrt{\pi}a^{1/12}(-x)^{1/4}} \left[ \exp\left(-i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} + i\frac{\pi}{4}\right) - \exp\left(i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} - i\frac{\pi}{4}\right) \right].$$

Using  $\alpha_2 = 0$  and our relation of  $c_1$  to  $\alpha_1$ ,  $y_0(x)$  has the approximation

$$y_0(x) \sim \alpha_1(-ax)^{-1/4} \left[ \exp\left(-i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} + i\frac{\pi}{4}\right) + \exp\left(i\frac{2}{3}a^{1/2}(-x)^{3/2}\epsilon^{-1} - i\frac{\pi}{4}\right) \right].$$

For  $x$  near 0,  $y_-(x)$  has the approximation

$$y_-(x) \sim \beta_1(-ax)^{-1/4} \exp\left(-i\frac{2}{3}a^{1/2}\epsilon^{-1}(-x)^{3/2}\right) + \beta_2(-ax)^{-1/4} \exp\left(\frac{2}{3}a^{1/2}\epsilon^{-1}(-x)^{3/2}\right).$$

We can see that for both approximations to be valid we need  $\beta_1 = \alpha_1 \exp(i\pi/4)$  and  $\beta_2 = \alpha_1 \exp(-i\pi/4)$ .

Thus we have the full approximation

$$y(x) \sim \begin{cases} \alpha_1 (V(x))^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_0^x \sqrt{V(x)} dx\right) & \text{if } x \gg 0, \\ 2\sqrt{\pi}(\epsilon a)^{-1/6} \alpha_1 \text{Ai}\left(\epsilon^{-2/3} a^{1/3} (x - A)\right) & \text{if } x \approx 0, \\ \alpha_1 (-V(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_0^x \sqrt{-V(x)} dx + i\frac{\pi}{4}\right) \\ \quad + \alpha_1 (-V(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_0^x \sqrt{-V(x)} dx - i\frac{\pi}{4}\right) & \text{if } x \ll 0. \end{cases}$$

### 3.2.2. Turning Point Specifications.

Suppose  $V(x)$  crosses 0 at  $A$  with non-zero slope  $a$ . We arrive at the similar form:

$$y(x) \sim \begin{cases} \alpha_1 (V(x))^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_A^x \sqrt{V(x)} dx\right) & \text{if } x \gg 0, \\ 2\sqrt{\pi}(\epsilon a)^{-1/6} \alpha_1 \text{Ai}\left(\epsilon^{-2/3} a^{1/3} x\right) & \text{if } x \approx 0, \\ \alpha_1 (-V(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_x^A \sqrt{-V(x)} dx + i\frac{\pi}{4}\right) \\ \quad + \alpha_1 (-V(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_x^A \sqrt{-V(x)} dx - i\frac{\pi}{4}\right) & \text{if } x \ll 0. \end{cases}$$

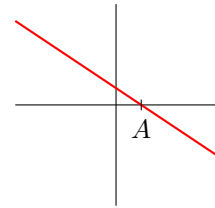


FIGURE 15.  
General turning point,  
 $V(x) = a(x - A)$ .

### 3.3. APPLICATION: THE EIGENVALUE PROBLEM.

Consider the corresponding eigenvalue problem associated with each  $V(x)$ :

$$(8) \quad \epsilon^2 y''(x) - V(x)y(x) = -\lambda y(x)$$

finding  $\lambda$  such that an eigenfunction  $y(x)$  exists which is bounded as  $x \rightarrow \pm\infty$ . This is equivalent to having  $\tilde{V}(x) = V(x) - \lambda$  in our original problem.

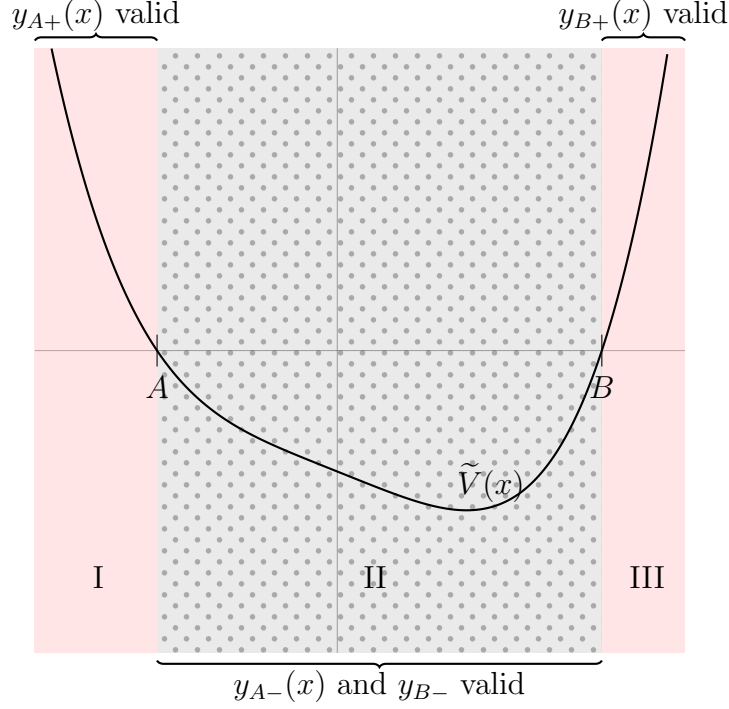


FIGURE 16. Plot of example  $\tilde{V}(x) = V(x) - \lambda$ .

If  $\tilde{V}(x) > 0$  does not cross the  $x$ -axis, then there is no eigenfunction corresponding to  $\lambda$  because our boundedness requirement would give  $\alpha_1 = \alpha_2 = 0$  on our  $y_+(x)$  approximation.

Now suppose we have  $\tilde{V}(x)$  which crosses the axis at multiple points. For simplification, assume  $\tilde{V}(x) = 0$  at only  $x = A, B$  and  $\tilde{V}(x) < 0$  on  $(A, B)$ , as in Figure 16. Suppose for our given  $\lambda$ , there exists a true eigenfunction  $y(x)$  to (8). Further, suppose around  $A$  we have approximations to  $y(x)$  of  $y_{A+}(x)$  in region I and  $y_{A-}(x)$  in region II, and around  $B$  we have approximations  $y_{B+}(x)$  in region III and  $y_{B-}$  in region II. We denote the coefficients for  $y_{A+}(x)$  by  $\alpha_1$  and  $\alpha_2$  and the coefficients for  $y_{B+}(x)$  by  $\gamma_1$  and  $\gamma_2$ . For boundedness, we see  $\alpha_2 = \gamma_2 = 0$ . In order to have a valid global approximation, we need  $y_{A-}(x)$  and  $y_{B-}(x)$  to agree in region II.

$$\begin{cases} y_{A+}(x) = \alpha_1 (\tilde{V}(x))^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_x^A \sqrt{\tilde{V}(x)} dx\right), & x < A, \\ y_{A-}(x) = \alpha_1 (-\tilde{V}(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_x^A \sqrt{-\tilde{V}(x)} dx + i\frac{\pi}{4}\right) + \alpha_2 (-\tilde{V}(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_x^A \sqrt{-\tilde{V}(x)} dx - i\frac{\pi}{4}\right), & A < x < B, \\ y_{B-}(x) = \gamma_1 (-\tilde{V}(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_x^B \sqrt{-\tilde{V}(x)} dx + i\frac{\pi}{4}\right) + \gamma_2 (-\tilde{V}(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_x^B \sqrt{-\tilde{V}(x)} dx - i\frac{\pi}{4}\right), & A < x < B, \\ y_{B+}(x) = \gamma_1 (\tilde{V}(x))^{-1/4} \exp\left(-\frac{1}{\epsilon} \int_B^x \sqrt{\tilde{V}(x)} dx\right), & B < x. \end{cases}$$

We now find conditions for  $y_{A-}(x) = y_{B-}(x)$  for  $x \in (A, B)$ . We begin by rewriting

$$\begin{aligned} y_{A-}(x) &= \alpha_1 (-\tilde{V}(x))^{-1/4} \exp\left(-\frac{i}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx + \frac{i}{\epsilon} \int_x^B \sqrt{-\tilde{V}(x)} dx + i\frac{\pi}{4}\right) \\ &\quad + \alpha_2 (-\tilde{V}(x))^{-1/4} \exp\left(\frac{i}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx - \frac{i}{\epsilon} \int_x^B \sqrt{-\tilde{V}(x)} dx - i\frac{\pi}{4}\right). \end{aligned}$$

Then in order for  $y_{A-}(x)$  and  $y_{B-}(x)$  to be functionally the same, we need

$$\alpha_1 \exp\left(-\frac{i}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx + i\frac{\pi}{4}\right) = \gamma_1 \exp\left(-i\frac{\pi}{4}\right)$$

and

$$\alpha_1 \exp\left(\frac{i}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx - i\frac{\pi}{4}\right) = \gamma_1 \exp\left(i\frac{\pi}{4}\right).$$

This gives us

$$\exp\left(2\frac{i}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx\right) = \exp(i\pi),$$

so  $\int_A^B \sqrt{-\tilde{V}(x)}/\epsilon dx = \pi n + \pi/2$ ,  $n \in \mathbb{Z}$ . For  $n$  odd we see  $\alpha_1 = \gamma_1$ , and for  $n$  even  $\alpha_1 = -\gamma_1$ . Thus we need

$$\gamma_1 = (-1)^{n+1}\alpha_1 \quad \text{and} \quad \frac{1}{\epsilon} \int_A^B \sqrt{-\tilde{V}(x)} dx = \pi n - \pi/2.$$

This last condition restricts our choice of  $\lambda$  for which such a global approximation, our eigenfunction, will exist. Recall that  $V(x) - \lambda < 0$  on  $(A, B)$  and hence  $\int_A^B \sqrt{-\tilde{V}(x)} dx > 0$ , so our possible  $n$  values are restricted to  $n = 1, 2, 3, \dots$

### 3.3.1. Particle in a Potential Well.

Solving this eigenvalue problem can be used to find allowable energy levels of a particle in a potential well along with their corresponding wavefunctions:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

As stated by Griffiths (pg. 333), “the result is extraordinarily powerful, for it enables us to calculate (approximate) allowed energies without ever solving the Schrödinger equation, by simply evaluating one integral.”

## 4. EXTENSIONS

We could further analyze the spectral properties of the WKB eigenvalue problem, such as finding the maximum or minimum eigenvalue, the spacing between eigenvalues for large  $x$ , or the different types of spectra. Though our eigenvalue restrictions are only accurate to order  $\epsilon$ , it is possible to find higher order correction terms. Also, our eigenvalue analysis can be easily extended to other forms of  $V(x)$ , such as those with more turning points like  $x^4 - x^2$  or  $\sin(x)$  using the same analysis as above at each turning point.

## REFERENCES

- [1] Bender, C., & Orszag, S. (1978). *Advanced Mathematical Methods for Scientists and Engineers*. International series in pure and applied mathematics. McGraw-Hill.
- [2] Carrier, G., Krook, M., & Pearson, C. (1983). *Functions of a Complex Variable: Theory and Technique*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.
- [3] Griffiths, D. (2005). *Introduction to Quantum Mechanics*. Pearson Prentice Hall.
- [4] Holmes, M. (1995). *Introduction to Perturbation Methods*. Texts in Applied Mathematics. Springer-Verlag.