On the parity of coefficients of eta powers

https://arxiv.org/pdf/2411.17638

(Joint with Lukas Mauth and Steven Charlton)

Anna Medvedovsky

University of Arizona

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1. q-Series supported on an arithmetic progression

We say that a q-series

$$f = \sum a_n q^n \in \mathbb{Z}[\![q]\!]$$

is **supported on arithmetic progression** b **modulo** m
if $a_n = 0$ unless $n \equiv b \mod m$.

Equivalently, $f \in q^b \mathbb{Z}[\![q^m]\!]$, at least if b is minimal, or if b is the **order at infinity** of f.

Motivating example: Eta powers

With the Dedekind eta function $\eta := q^{\frac{1}{24}} \prod_k (1 - q^k)$, set

$$\eta^{\underline{r}} := \eta^r(q^{m_r}),$$

where $m_r := \frac{24}{\gcd(24,r)}$ makes $\eta^{\underline{r}}$ a power series (integral exponents).

Then $\eta^{\underline{r}}$ is supported on $b_r := \frac{r}{\gcd(r,24)}$ modulo m_r.

2. Eta powers in characteristic zero

r	b_r	m_r	$\underline{\eta^r}$
1	1	24	$q - q^{25} - q^{49} + q^{121} + q^{169} - q^{289} - q^{361} + O(q^{529})$
2	1	12	$q - 2q^{13} - q^{25} + 2q^{37} + q^{49} + 2q^{61} - 2q^{73} + O(q^{97})$
3	1	8	$q - 3q^{9} + 5q^{25} - 7q^{49} + 9q^{81} - 11q^{121} + O(q^{169})$
4	1	6	$q - 4q^{7} + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} - 10q^{37} + O(q^{43})$
5	5	24	$q^{5} - 5q^{29} + 5q^{53} + 10q^{77} - 15q^{101} - 6q^{125} + O(q^{149})$
6	1	4	$q - 6q^{5} + 9q^{9} + 10q^{13} - 30q^{17} + 11q^{25} + O(q^{29})$
7	7	24	$q^{7} - 7q^{31} + 14q^{55} + 7q^{79} - 49q^{103} + 21q^{127} + O(q^{151})$
8	1	3	$q - 8q^{4} + 20q^{7} - 70q^{13} + 64q^{16} + 56q^{19} + O(q^{25})$
9	3	8	$q^{3} - 9q^{11} + 27q^{19} - 12q^{27} - 90q^{35} + 135q^{43} + O(q^{51})$
10	5	12	$q^{5} - 10q^{17} + 35q^{29} - 30q^{41} - 105q^{53} + O(q^{65})$

3. Mod-p order of infinity of $U_\ell f$ and its minimality

Recall: for ℓ prime the "pulling in" operator U_ℓ on q-series, with

$$U_\ell\left(\sum a_n q^n\right) = \sum a_{\ell n} q^n.$$

If $f = \sum a_n q^n$ is supported on b modulo m, then

$$U_\ell f = a_N q^{N/\ell} + (\text{higher-order terms}),$$

where $N := N(f, m, \ell)$ is the least index that is both b mod m and divisible by ℓ. Then N/ℓ is the **formal order at infinity of $U_\ell f$**.

If $a_N \not\equiv 0$ modulo p for a prime p, then

$$\boxed{\text{the mod-}p\text{ order at infinity of } U_\ell f \text{ is minimal.}}$$

4. Twisted density, and our first result

Definition (Ono? + ε)

Let f be a q-series supported on the arithmetic progression b modulo m. The **twisted density** $D_p(f, m)$ *of f is the natural density of the set of primes*

$$S_p(f, m) := \{\ell \text{ prime} : a_{N(f,m,\ell)} \not\equiv 0 \mod p\}.$$

In other words, $D_p(f, m)$ is the density of the set of primes ℓ for which the order at infinity of $U_\ell f$ is minimal (that is, formal).

Theorem A (CMM)

If f is a classical modular form, then the twisted density $D_p(f, m)$ always exists, and is a rational number.

Method: we show that $S_p(f, m)$ is a **frobenian** set: that is, there is a finite Galois extension $E/\mathbb{Q}$ so that whether a prime $\ell \in S_p(f, m)$ depends only on the conjugacy class of Frob_ℓ in $\mathrm{Gal}(E/\mathbb{Q})$.

5. Specializing to eta powers modulo 2

We now specialize to the case $p = 2$ and $f = \eta^{\underline{r}}$. We use the arithmetic progression modulus $m = m_r$.

Write $\boxed{D(r) := D_2(\eta^{\underline{r}}, m_r)}$. This is our main object of study.

Specializing Theorem A tells us that $D(r)$ is a *dyadic* fraction: the denominator is a power of 2.

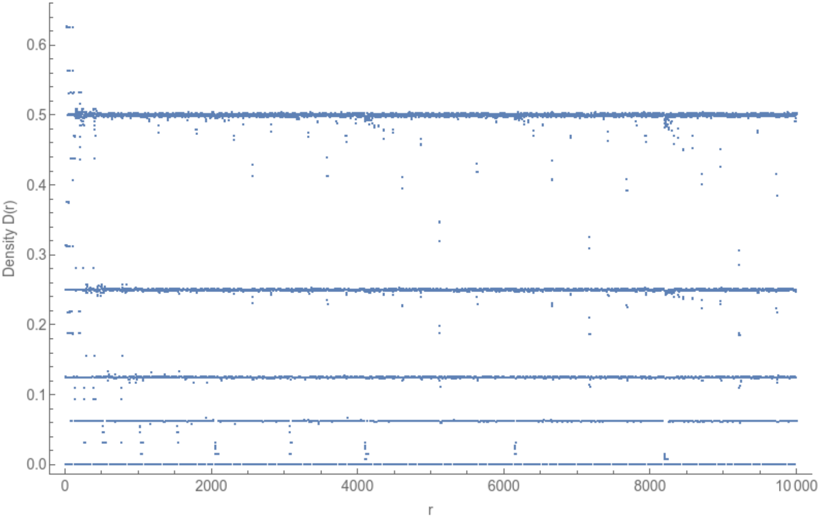
6. $D(r)$ computationally (with 10^6 coefficients of η)

r	$D(r)$	$\mathbb{Z}[\frac{1}{2}]$?	r	$D(r)$	$\mathbb{Z}[\frac{1}{2}]$?	r	$D(r)$	$\mathbb{Z}[\frac{1}{2}]$?
1	0.00009	0	15	0.25026	1/4	29	0.31215	5/16
2	0.00004	0	16	0.00001	0	30	0.25038	1/4
3	0.00003	0	17	0.31251	5/16	31	0.31228	5/16
4	0.00001	0	18	0.25044	1/4	32	0.00000	0
5	0.12483	1/8	19	0.31305	5/16	33	0.24982	1/4
6	0.00001	0	20	0.12511	1/8	34	0.18765	3/16
7	0.12507	1/8	21	0.62564	5/8	35	0.21882	7/32
8	0.00000	0	22	0.31239	5/16	36	0.25019	1/4
9	0.25022	1/4	23	0.62518	5/8	37	0.50012	1/2
10	0.12502	1/8	24	0.00000	0	38	0.18798	3/16
11	0.31205	5/16	25	0.31245	5/16	39	0.62503	5/8
12	0.00000	0	26	0.12511	1/8	40	0.12525	1/8
13	0.12513	1/8	27	0.37488	3/8	41	0.56321	9/16
14	0.12511	1/8	28	0.12520	1/8	42	0.37561	3/8

7. More $D(r)$ computationally (with 10^6 coefficients of η)

r	?$D(r)$	r	?$D(r)$	r	?$D(r)$	r	?$D(r)$	r	?$D(r)$	r	?$D(r)$
1	0	23	5/8	45	1/2	67	7/32	89	1/2	111	9/16
2	0	24	0	46	5/16	68	3/16	90	1/4	112	1/8
3	0	25	5/16	47	17/32	69	1/2	91	1/2	113	7/16
4	0	26	1/8	48	0	70	7/32	92	1/8	114	1/4
5	1/8	27	3/8	49	5/16	71	17/32	93	1/2	115	17/32
6	0	28	1/8	50	5/16	72	1/4	94	1/4	116	1/8
7	1/8	29	5/16	51	3/8	73	7/16	95	1/2	117	1/2
8	0	30	1/4	52	1/8	74	3/16	96	0	118	1/4
9	1/4	31	5/16	53	1/2	75	1/2	97	7/32	119	1/2
10	1/8	32	0	54	3/8	76	1/16	98	3/16	120	1/4
11	5/16	33	1/4	55	1/2	77	1/2	99	3/16	121	15/32
12	0	34	3/16	56	1/8	78	1/4	100	3/16	122	1/8
13	1/8	35	7/32	57	5/8	79	1/2	101	1/4	123	1/2
14	1/8	36	1/4	58	5/16	80	1/8	102	1/8	124	1/8
15	1/4	37	1/2	59	9/16	81	1/2	103	13/32	125	1/2
16	0	38	3/16	60	1/4	82	1/4	104	1/8	126	1/4
17	5/16	39	5/8	61	5/16	83	1/2	105	5/8	127	7/16
18	1/4	40	1/8	62	5/16	84	1/8	106	5/16	128	0
19	5/16	41	9/16	63	5/8	85	1/2	107	9/16	129	1/8
20	1/8	42	3/8	64	0	86	1/4	108	1/8	130	3/32
21	5/8	43	9/16	65	7/32	87	1/2	109	17/32	131	7/64
22	5/16	44	3/16	66	1/4	88	1/16	110	5/16	132	1/8

8. Plot of $D(r)$ for $r < 10k$



9. Our results on $D(r)$: theory

Theorem B (CMM)

$D(r) = 0$ if and only if r is a divisor or multiple of 32 or 48.

Callback to data

Theorem C (CMM)

1. *$D(r) < 1$ for all r*
2. *$D(2r) < \frac{1}{2}$ for all r*
3. *$D(4r) \leq \frac{1}{4}$, with equality if and only if $r = 9, 15, 18, 30$.*

Callback to data

Theorem D (CMM; see arXiv for precise statement)

Closed form for $D(r)$ for $r = 3 \cdot 2^k(1 + 2^n)$ and $r = 2^k(1 + 2 \cdot 4^n)$ for $k = 0, 1, 2, 3$ and $n \geq 1$.

Sample result: $D\big(3(1 + 4^n)\big) = \frac{3}{2^{n+1}}$ *if $n \geq 2$.*

10. Our results on $D(r)$: practice. Green = proven

r	$?D(r)$	r	$?D(r)$	r	$?D(r)$	r	$?D(r)$	r	$?D(r)$	r	$?D(r)$
1	0	23	5/8	45	1/2	67	7/32	89	1/2	111	9/16
2	0	24	0	46	5/16	68	3/16	90	1/4	112	1/8
3	0	25	5/16	47	17/32	69	1/2	91	1/2	113	7/16
4	0	26	1/8	48	0	70	7/32	92	1/8	114	1/4
5	1/8	27	3/8	49	5/16	71	17/32	93	1/2	115	17/32
6	0	28	1/8	50	5/16	72	1/4	94	1/4	116	1/8
7	1/8	29	5/16	51	3/8	73	7/16	95	1/2	117	1/2
8	0	30	1/4	52	1/8	74	3/16	96	0	118	1/4
9	1/4	31	5/16	53	1/2	75	1/2	97	7/32	119	1/2
10	1/8	32	0	54	3/8	76	1/16	98	3/16	120	1/4
11	5/16	33	1/4	55	1/2	77	1/2	99	3/16	121	15/32
12	0	34	3/16	56	1/8	78	1/4	100	3/16	122	1/8
13	1/8	35	7/32	57	5/8	79	1/2	101	1/4	123	1/2
14	1/8	36	1/4	58	5/16	80	1/8	102	1/8	124	1/8
15	1/4	37	1/2	59	9/16	81	1/2	103	13/32	125	1/2
16	0	38	3/16	60	1/4	82	1/4	104	1/8	126	1/4
17	5/16	39	5/8	61	5/16	83	1/2	105	5/8	127	7/16
18	1/4	40	1/8	62	5/16	84	1/8	106	5/16	128	0
19	5/16	41	9/16	63	5/8	85	1/2	107	9/16	129	1/8
20	1/8	42	3/8	64	0	86	1/4	108	1/8	130	3/32
21	5/8	43	9/16	65	7/32	87	1/2	109	17/32	131	7/64
22	5/16	44	3/16	66	1/4	88	1/16	110	5/16	132	1/8

11. Method outline

Step 1: Reexpress $\eta^{\underline{r}}$ modulo $p = 2$ in terms of mod-2 modular forms of level 1 and level 9.

Step 2: Reinterpret the twisted density of a mod-p modular form f in terms of the density $\delta(f)$ as studied by Bellaïche. More precisely, we show $D_p(f, m)$ is a sum of $\varphi(m)$ terms of the form $\delta(T_u f)$.

Step 3: Apply Galois methods going back to Bellaïche (building on work of Serre and Nicolas) to study $\delta(f)$.

12. Step 1: Eta powers modulo 2, theory

Recall that $b_r = \frac{r}{\gcd(r,24)}$, $m_r = \frac{24}{\gcd(r,24)}$, and $\eta^{\underline{r}} = \eta^r(q^{m_r})$.

Lemma (CMM)

1. *If* $3 \mid r$ *then* $\eta^{\underline{r}} \equiv \Delta^{b_r}$ *modulo 2.*
2. *Otherwise* $\eta^{\underline{r}} \equiv C^{b_r}$ *modulo 2.*

Here $\Delta = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6) \in S_{12}(1)$ *and* $C = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + O(q^{19}) \in S_4\big(\Gamma_0(9)\big)$.

From now on, we work modulo 2, so that

$$\Delta := q + q^9 + q^{25} + q^{49} + q^{81} + q^{121} + q^{169} + q^{225} + O(q^{289}),$$

$$C := q + q^{25} + q^{49} + q^{121} + q^{169} + O(q^{289}),$$

both in $\mathbb{F}_2[\![q]\!]$.

13. Eta powers modulo 2, practice

r	b_r	m_r	η^{r}_{-}
1	1	24	$q + q^{25} + q^{49} + q^{121} + q^{169} + q^{289} + O(q^{361})$
2	1	12	$q + q^{25} + q^{49} + q^{121} + q^{169} + q^{289} + O(q^{361})$
3	1	8	$q + q^{9} + q^{25} + q^{49} + q^{81} + q^{121} + q^{169} + O(q^{225})$
4	1	6	$q + q^{25} + q^{49} + q^{121} + q^{169} + q^{289} + O(q^{361})$
5	5	24	$q^{5} + q^{29} + q^{53} + q^{101} + q^{149} + q^{173} + O(q^{197})$
6	1	4	$q + q^{9} + q^{25} + q^{49} + q^{81} + q^{121} + q^{169} + O(q^{225})$
7	7	24	$q^{7} + q^{31} + q^{79} + q^{103} + q^{127} + q^{151} + q^{175} + O(q^{199})$
8	1	3	$q + q^{25} + q^{49} + q^{121} + q^{169} + q^{289} + O(q^{361})$
9	3	8	$q^{3} + q^{11} + q^{19} + q^{43} + q^{59} + q^{67} + q^{75} + O(q^{83})$
10	5	12	$q^{5} + q^{29} + q^{53} + q^{101} + q^{149} + q^{173} + O(q^{197})$

14. Step 2: Twisted density vs. density

Definition (Bellaïche)

Let $f = \sum_n a_n q^n$ be a mod-p modular form. Then the **density** $\delta(f) = \delta_p(f)$ **of** f *is the density of*

$$S_f := \{\ell \text{ prime} : a_\ell \not\equiv 0 \mod p\}$$

as a set of primes.

Again, S_f is a frobenian set: membership is determined by the conjugacy class of Frobenius in a Galois extension E_f of $\mathbb{Q}$.

Proposition (CMM)

Le f be a mod-p modular form supported on b mod m, and suppose the Hecke operators preserve the mod-m grading. For c prime to m, let u_c be minimial with $u_c \equiv bc^{-1}$ modulo m. Then

$$D(f, m) = \sum_{c<m \text{ and } \gcd(\mathrm{c,m})=1} \delta(T_{u_c} f).$$

15. Step 2: Twisted density vs. density, continued

Corollary (CMM)

Therefore we have the following expressions for $D(r)$.

r	m_r	η^r_-	$D(r)$
$3s$, s odd	8	Δ^s	$\delta(f)+\delta(T_3 f)+\delta(T_5 f)+\delta(T_7 f)$
$6s$, s odd	4	Δ^s	$\delta(f)+\delta(T_3 f)$
$12s$, s odd	2	Δ^s	$\delta(f)$
$24s$	1	Δ^s	$\delta(f)$
s with $2,3 \nmid s$	24	C^s	$\delta(f)+\delta(T_5 f)+\delta(T_7 f)+\delta(T_{11} f)+$ $+\delta(T_{13} f)+\delta(T_{17} f)+\delta(T_{19} f)+\delta(T_{23} f)$
$2s$ with $2,3 \nmid s$	12	C^s	$\delta(f)+\delta(T_5 f)+\delta(T_7 f)+\delta(T_{11} f)$
$4s$ with $2,3 \nmid s$	6	C^s	$\delta(f)+\delta(T_5 f)$
$8s$ with $3 \nmid s$	3	C^s	$\delta(f)+\delta(U_2 f)$

16. Step 3: Galois methods going back to Bellaïche

Theorem (Swinnerton-Dyer, Serre, Nicolas, Bellaïche)

SD *$M_1 = \mathbb{F}_2[\Delta]$ is the space of mod-2 modular forms of level 1.*

S-N *$K_1 := \langle \Delta^n : n \text{ odd} \rangle \subset M(1)$ is a Hecke-invariant subspace.*

S-N *For n odd, Δ^n is supported on n modulo 8.*

S-N *$A_1 := \mathit{Hecke}(K_1) \simeq \mathbb{F}_2[\![T_3, T_5]\!]$. Then A_1 is dual to K_1, and any $f \in K_1$ is a linear form $L_f : A_1 \to \mathbb{F}_2$ sending T_ℓ to $a_\ell(f)$.*

B *$G_1 :=$ Galois group of maximal pro-2 extension of $\mathbb{Q}$ unramified outside 2. Then there is a unique universal representation $r_1 : G_1 \to \mathrm{SL}_2(A_1)$ with $\mathrm{tr}\, r_1(\mathrm{Frob}_\ell) = T_\ell$.*

B *Given $f \in K_1$, there exists an finite algebra quotient A_f of A_1 and a finite group quotient G_f of G_1 with $G_f \xrightarrow{\mathrm{tr}\, r_1} A_f \xrightarrow{L_f} \mathbb{F}_2$ sending Frob_ℓ to $a_\ell(f)$. Therefore $\ell \mapsto a_\ell(f)$ is frobenian.*

Expectation (Bellaïche): If Δ^n not abelian/dihedral, then $\delta(\Delta^n) = \frac{1}{8}$.
Similar analysis in level 9: expect $\delta(C^n) = \frac{1}{16}$ for most n (if $2, 3 \nmid n$).

17. Thank you!

Thank you!