Survey of recent results in asymptotic analysis of Riemann--Hilbert problems in Random Matrices

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Random Matrices

Consider the probability measure on $N \times N$ Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp\left\{-N \operatorname{Tr}\left[\quad V(M) \qquad \right]\right\} dM$$

$$dM = \prod_{j<k} dM_{jk}^{R} \, dM_{jk}^{I} \prod_{j=1}^{N} dM_{jj}$$

$$\hat{Z}_N = \int \exp\left\{-N \operatorname{Tr}\left[\qquad V(M) \quad \right]\right\} dM$$

interest is in the probabilistic description of the eigenvalues, as $N \to \infty$

Typically interested in eigenvalues, whose induced probability density is:

$$\frac{1}{Z_N} \exp\left(-N \sum_{j=1}^{N} V(\lambda_j) \right) \prod_{1 \le j < k \le N} \left(\lambda_k - \lambda_j \right)^2 ,$$

Partition Function: $$Z_N = \int \exp\left(-N \sum_{j=1}^{N} V(\lambda_j) \right) \prod_{1 \le j < k \le N} \left(\lambda_k - \lambda_j \right)^2 d^N \lambda$$

Closely related to polynomials $\left\{ p_j(x) \right\}_{j \ge 0}$ orthogonal wrt $e^{-NV(x)} dx$

Mean density of particles

Consider the random variable $\frac{1}{N}\#\{\text{evals} \ < \ x\}$

The mean density is defined via

$$\rho_1(x) = \frac{d}{dx}\left\langle \frac{1}{N}\#\{\text{evals} \ < x\}\right\rangle$$

The connection to orthogonal polynomials:

$$\rho_1(x) = K_N(x,x)$$

$$K_N(x,y) = e^{-\frac{N(V(x)+V(y))}{2}}\sum_{\ell=0}^{N-1} p_\ell(x)p_\ell(y)$$

All statistical properties can be expressed in terms of the orthogonal polynomials!

E.G.

$$\text{Prob}\{\text{no } \lambda_j\text{'s in } (a,b)\} = \det(1-\mathbf{K_N})_{L^2(a,b)}$$

$$(\mathbf{K_N} f)(x) = \int_a^b K_N(x,y) f(y) dy$$

we are interested in the behavior for $N \to \infty$

Basic asymptotic result: Under rather weak assumptions on V, the following limit exists.

(Johansson, 1998)

$$\lim_{N\to\infty} \rho_1^{(N)}(x) = \psi(x),$$ where $\psi \geq 0$ solves a well-known variational problem.

$$\sup_{\substack{0\leq d\mu, \\ \int d\mu = 1}} \left[-\int V d\mu + \iint \log|x-y| \, d\mu(x) d\mu(y) \right]$$

V real analytic with suitable growth $\Longrightarrow$ ψ is supported on finitely many intevals, and is analytic on the interior of each one.

(Deift, Kriecherbauer, McL '98)

Orthogonal polynomials on $\mathbf{R}$: characterization by Riemann-Hilbert problem

Discovered by Fokas, Its and Kitaev, 90s

$$Y(z)=\begin{pmatrix} \frac{1}{\kappa_{j,j}}p_j(z) & \frac{1}{2\pi i}\int_{\mathbf{R}}\frac{\frac{1}{\kappa_{j,j}}p_j(s)w(s)}{s-z}ds \\ -2\pi i\kappa_{j-1,j-1}p_{j-1}(z) & \int_{\mathbf{R}}\frac{-\kappa_{j-1,j-1}p_{j-1}(s)w(s)}{s-z}ds \end{pmatrix},\quad z \text{ off } \mathbf{R}.$$

(1) Y analytic in $\mathbf{C}\setminus\mathbf{R}$

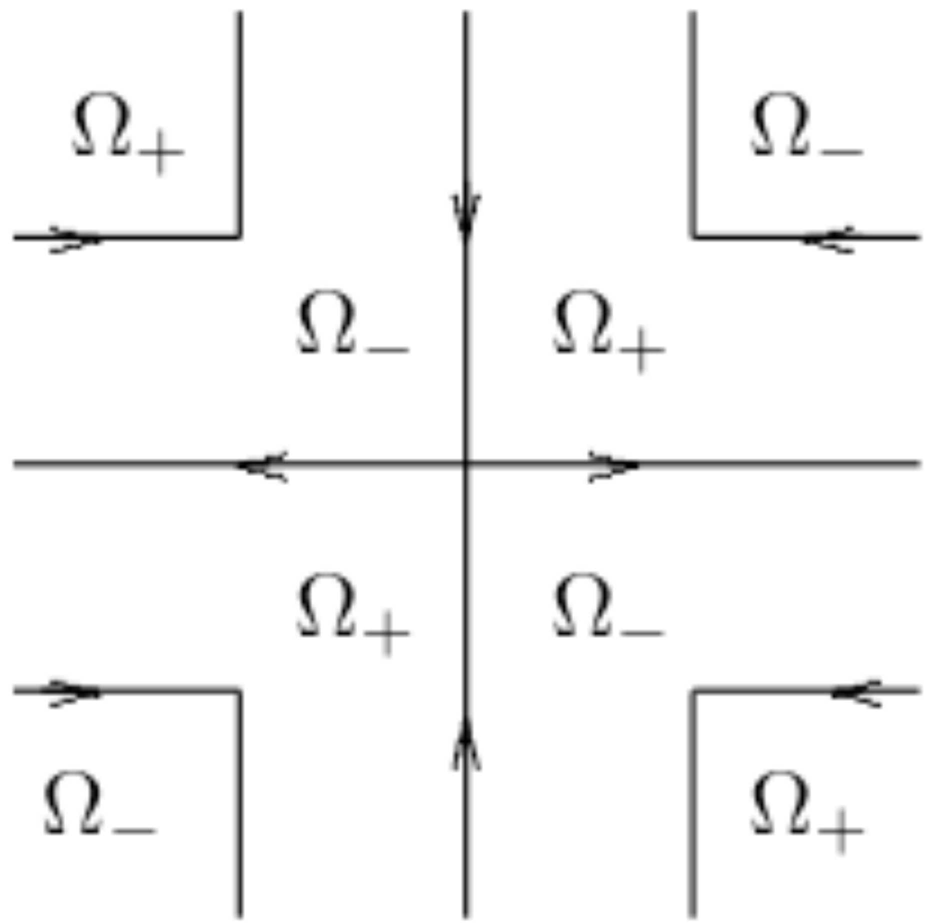
(2) $Y=\left(I+O\left(z^{-1}\right)\right)\begin{pmatrix} z^{j} & 0 \\ 0 & z^{-j}\end{pmatrix}$ as $z\to\infty$

(3) $Y_+(x)=Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1\end{pmatrix},\quad x\in\mathbf{R}$

Riemann Hilbert problems and Integral equations

Σ: Oriented contour, a finite union of smooth curves

$\overline{\mathbb{C}} \setminus \Sigma$: finite number of components.



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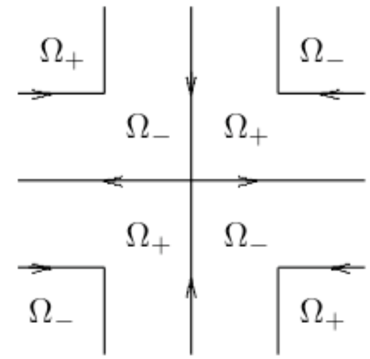

Cauchy operator:

$$Ch(z) = C_\Sigma h(z) \equiv \int_\Sigma \frac{h(s)}{s-z}\frac{ds}{2\pi i}, \qquad z \in \mathbb{C}\backslash\Sigma.$$

Cauchy "projection":

$$C^{\pm}h(z) \equiv \lim_{\substack{z' \to z \\ z' \in (\pm)-\text{side of } \Sigma}} (Ch)(z')$$

On what spaces are these operators defined???

$$L^p(\Sigma, |dz|), 1 \leq p < \infty$$

$$Lip^\nu(\Gamma) \qquad \|f\|_{Lip^\nu(\Gamma)} := \sup_{z\in\Gamma} \|f(z)\| + \sup_{z_1,z_2\in\Gamma} \frac{\|f(z_2) - f(z_1)\|}{\|z_2 - z_1\|^\nu}$$

$$\mathcal{W} = \{f \in C(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R})\} \qquad \|f\|_{\mathcal{W}} := \left\|\widehat{f}\right\|_{L^1}$$

For each of these spaces, C, C_+ and C_- are all bounded operators:

$$\|C_\pm h\| \leq c\|h\|$$

(1) M analytic in $\mathbf{C} \setminus \Sigma$

(2) $M = \mathbf{I} + \mathbf{O}\left(z^{-1}\right)$ as $z \to \infty$

(3) $M_+(z) = M_-(z)V(z)\ ,\ \ z \in \Sigma$

A related integral operator: $C_V(f) := C_-\left(f\left(V - I\right)\right)$

$$
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The fundamental integral equation:

$$(\mathbf{I} - C_V)\gamma = C_V(I)$$

If you can solve this integral equation, then you can solve the RHP:

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$$M(z) = I + C\left(\left(I + \gamma\right)\left(V - I\right)\right)$$

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If you can solve this integral equation, then you can solve the RHP:

$$M(z) = I + \tfrac{1}{2\pi i}\int_\Sigma \frac{\left(I + \gamma\right)\left(V - I\right)ds}{s - z}$$

A related integral operator: $C_V(f) := C_-\left(f\left(V - I\right)\right)$

The fundamental integral equation: $(\mathrm{I} - C_V)\gamma = C_V(I)$

$$\left\|C_V(f)\right\| \leq c\left\|f\left(V - I\right)\right\|$$

A related integral operator: $C_V(f) := C_-\big(f(V-I)\big)$

The fundamental integral equation: $(\mathrm{I} - C_V)\gamma = C_V(I)$

$$\|C_V(f)\| \le c\big\|f(V-I)\big\|$$

$Lip^{\nu}(\Gamma)$ and $\mathcal{W}$ are algebras: $\le c\|V-I\|\,\|f\|$

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$$\|C_V(f)\| \leq c\big\|f(V-I)\big\|$$

$Lip^{\nu}(\Gamma)$ and $\mathcal{W}$ are algebras: $\leq c\|V-I\|\ \|f\|$

For L^2: $\leq c\|V-I\|_{\infty}\ \|f\|_2$

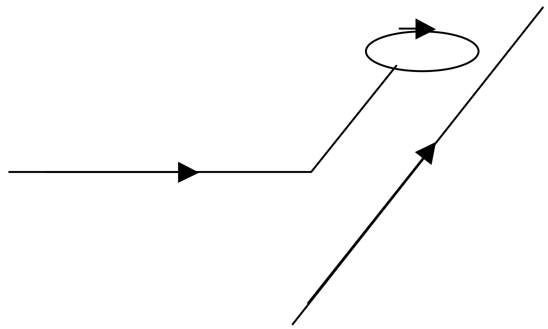
The point: $\|V-I\|$ Small means Neumann Series!

Asymptotic analysis: Guiding Principle

(1) M analytic in $\mathbf{C} \setminus \Sigma$

(2) $M = \mathbf{I} + \mathbf{O}\left(z^{-1}\right)$ as $z \to \infty$

(3) $M_{+}(z) = M_{-}(z)\left(\mathbf{I} + \mathbf{E}\right), \; z \in \Sigma$



Either $\|\mathbf{E}\|$, or $\|\mathbf{E}\|_{L^2}, \|\mathbf{E}\|_{L^\infty}$ asymptotically small $\left(\le \mathrm{cN}^{-1}\right)$

One can prove directly that a solution M exists, and $\left|M - \mathbf{I}\right| \le C\left\|\mathbf{E}\right\|$

Goal: find a sequence of explicit transformations starting from RHP for Y arriving at a version of the above RHP.

$$Y \to B \to D \to E \to M$$

Guiding principle II: oscillations

$$B = \mathbf{I} + \mathbf{O}\left(z^{-1}\right) \text{ as } z \to \infty$$

$$B_+(z) = B_-(z)\mathbf{V}_B(z), \quad z \in \Sigma$$

•On green portion of contour:

$$\mathbf{V}_B = \begin{pmatrix} e^{-iN\Omega_j} & 0 \\ 0 & e^{iN\Omega_j} \end{pmatrix} + \mathbf{O}\left(e^{-cN}\right)$$

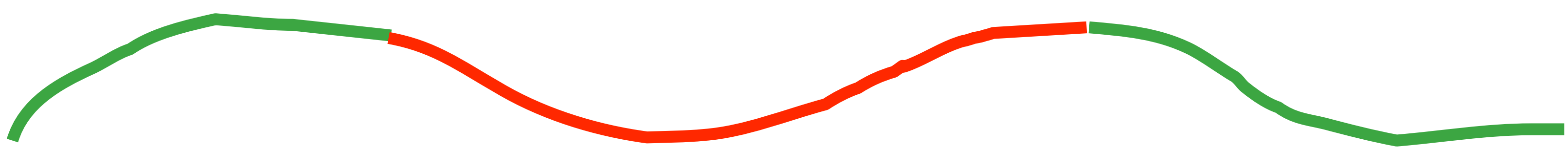
•On red portion:

$$\mathbf{V}_B = \begin{pmatrix} e^{-iN\theta(z)} & 1 \\ 0 & e^{iN\theta(z)} \end{pmatrix}$$

θ is purely real on red portion of Σ

θ has analytic extension to nbd of red portion

θ is increasing along Σ



From this vantage point, one can easily arrive at the 'Guiding Principle" RHP.

Arriving at this oscillatory RHP is the challenge.

Asymptotic analysis for OPs on $\mathbf{R}$

$w(x) = e^{-NV(x)}$: V real analytic, sufficient growth at infinity

1. Oscillations: use equilibrium measure $\psi(x)dx$ which solves the variational problem

$$\sup_{\substack{0 \le d\mu, \\ \int d\mu = 1}} \left[-\int V d\mu + \iint \log|x-y| d\mu(x) d\mu(y) \right]$$

Assumptions on V imply that $\psi(x)dx$ Is supported on finitely many intervals (red), and

$\psi(x)$ is analytic on the interior of each interval of its support.

$$A(z) = \begin{pmatrix} e^{-\frac{N\ell}{2}} & 0 \\ 0 & e^{\frac{N\ell}{2}} \end{pmatrix} Y(z) \begin{pmatrix} e^{-N\left(\int \log(z-s)\psi(s)ds - \frac{\ell}{2}\right)} & 0 \\ 0 & e^{N\left(\int \log(z-s)\psi(s)ds - \frac{\ell}{2}\right)} \end{pmatrix}$$

Off support of $\psi(x)$ $\mathbf{V}_A = \begin{pmatrix} e^{-N\pi i\Omega_\nu} & e^{N\left(\int \log|z-s|\psi(s)ds - V - \ell\right)} \\ 0 & e^{N\pi i\Omega_\nu} \end{pmatrix}$

Exponentially small

Constants

On support of $\psi(x)$ $\mathbf{V}_A = \begin{pmatrix} e^{-N\pi i\int_z^T \psi(s)ds} & 1 \\ 0 & e^{N\pi i\int_z^T \psi(s)ds} \end{pmatrix}$

Exponentially small

Off support of $\psi(x)$ $\mathbf{V}_A = \begin{pmatrix} e^{-N\pi i \Omega_\nu} & e^{N\left(\int \log|z-s|\psi(s)ds - V - \ell\right)} \\ 0 & e^{N\pi i \Omega_\nu} \end{pmatrix}$

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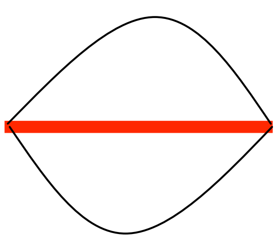
On support of $\psi(x)$ $\mathbf{V}_A = \begin{pmatrix} e^{-N\pi i \int_z^T \psi(s)ds} & 1 \\ 0 & e^{N\pi i \int_z^T \psi(s)ds} \end{pmatrix}$

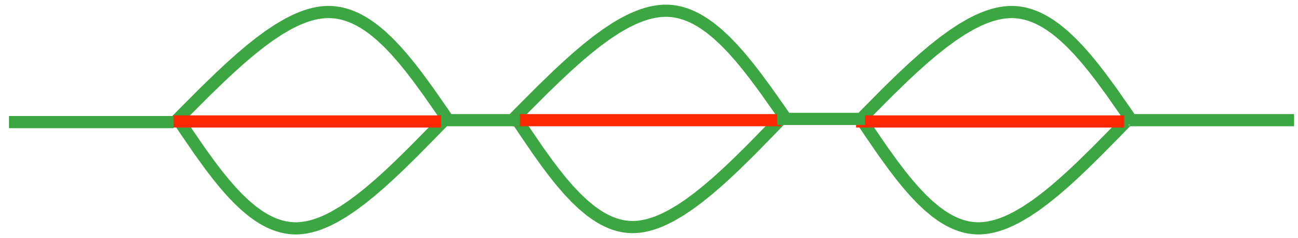
Step 2:

$$= \begin{pmatrix} 1 & 0 \\ e^{N\pi i \int_z^T \psi(s)ds} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-N\pi i \int_z^T \psi(s)ds} & 1 \end{pmatrix}$$

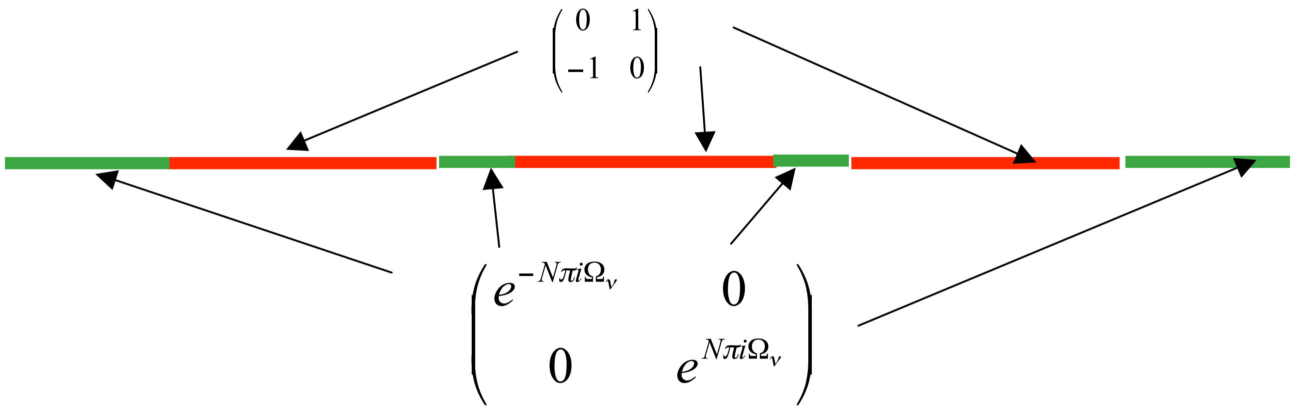


$$\left(A(z) \begin{pmatrix} 1 & 0 \\ -e^{-N\pi i \int_z^T \psi(s)ds} & 1 \end{pmatrix} \right)_+ = \left(A(z) \begin{pmatrix} 1 & 0 \\ e^{N\pi i \int_z^T \psi(s)ds} & 1 \end{pmatrix} \right)_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

B_+ $\qquad$ B_-

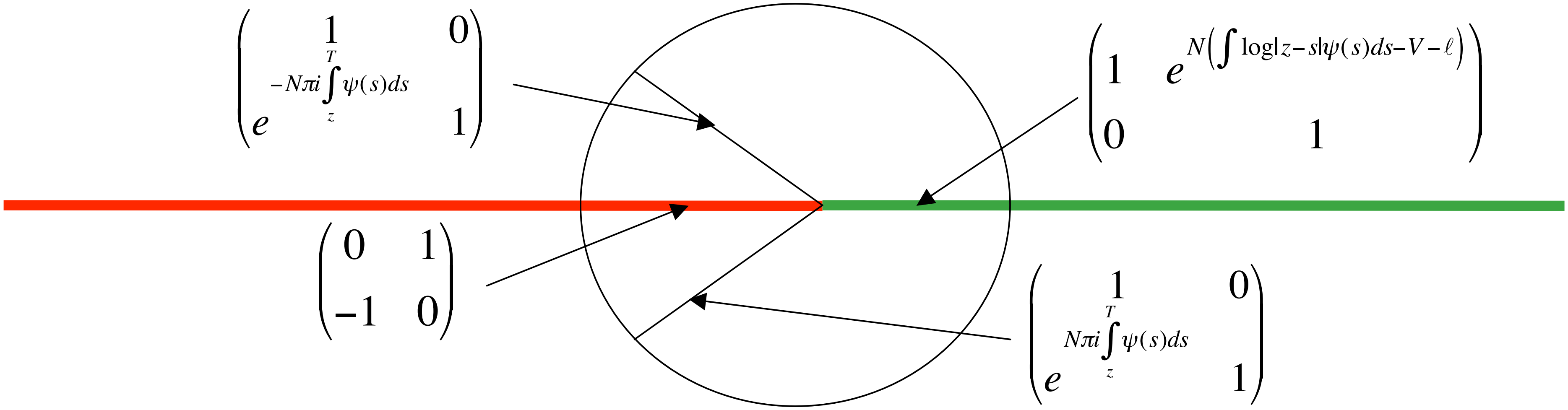


Step 3: ignore all exponentially small terms, solve approximate RHP that remains.



$$\text{Soln}: \quad B_{\text{outer}}$$

Step 4: for full problem: no exponential decay rate near transition points: build local approximation



(1) B_a analytic in open components

(2) $B_{a,+}(z) = B_{a,-}(z) V_D(z)$, $z \in \Sigma \cap \text{disk}$

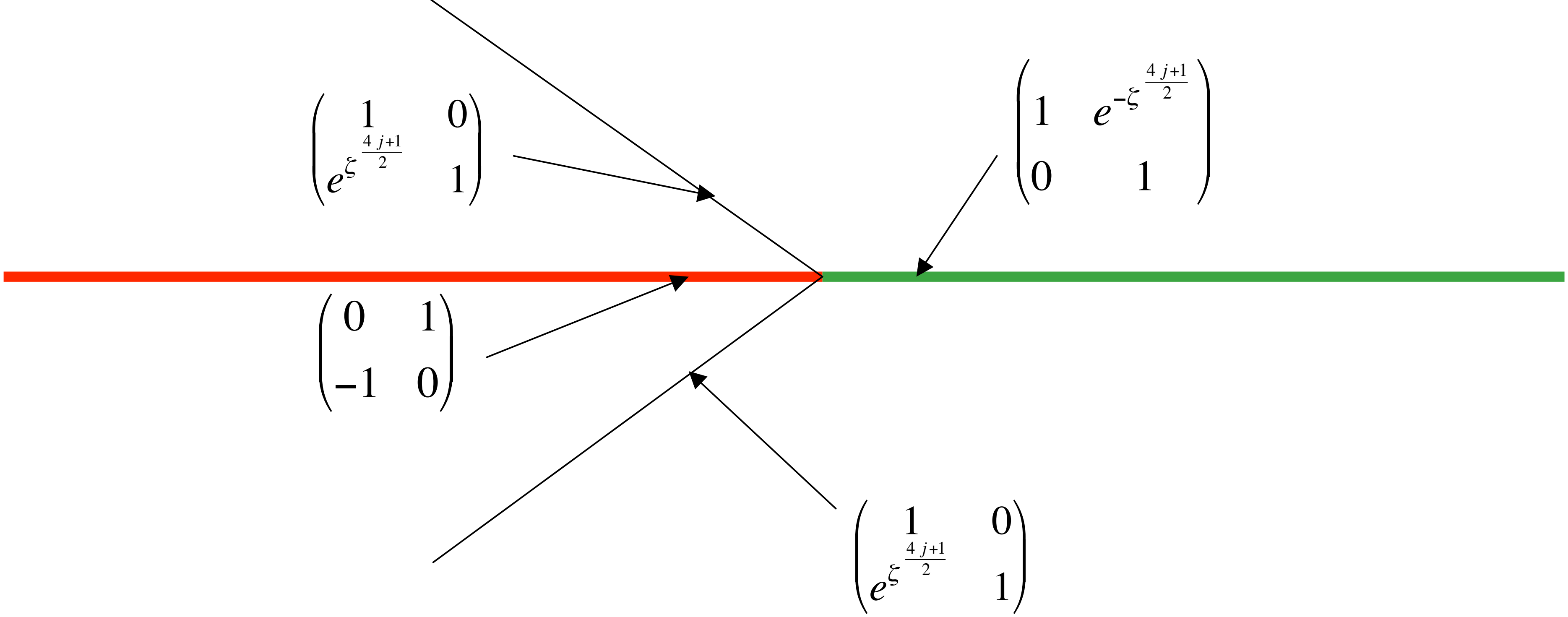
(3) $B_a \left(B_{\text{outer}}\right)^{-1} = I + O\left(N^{-1}\right)$ on the circle

Transform to canonical model RHP

$$z \mapsto \zeta(z) \text{ so that } N\pi i \int_z^T \psi(s)ds = \zeta^{\frac{4j+1}{2}}$$

Transform to canonical model RHP

$$z \mapsto \zeta(z) \;\text{ so that }\; \mathrm{N}\pi i \int_z^T \psi(s)\,ds = \zeta^{\frac{4j+1}{2}}$$



(1) P analytic off the rays

(2) $P_+(\zeta) = P_-(\zeta) V_P(\zeta)\,,\quad \varsigma \in \Sigma$

(3) $P\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} = I + O\left(\zeta^{-1}\right)$ for $\zeta \to \infty$

$$B_a(z) = \mathrm{E}(z) P\big(\zeta(z)\big)$$

One has constructed a global approximation to D, exterior to the ovals, and interior as well.

Step 5: study RHP for
$$M = \begin{cases} BB_a^{-1} & \text{inside the ovals} \\ BB_{\text{outer}}^{-1} & \text{outside the ovals} \end{cases}$$

And use the guiding principle.

$$M = I + \mathbf{O}\left(n^{-\kappa}\right)$$ Uniformly, and differentiable away from the final contours

$$Y(z) = \begin{pmatrix} e^{\frac{N\ell}{2}} & 0 \\ 0 & e^{-\frac{N\ell}{2}} \end{pmatrix} \left(M(z) B_{outer}(z) \begin{pmatrix} 1 & 0 \\ e^{-N\pi i \int_z^T \psi(s)ds} & 1 \end{pmatrix} \right) \begin{pmatrix} e^{N\left(\int \log(z-s)\psi(s)ds - \frac{\ell}{2}\right)} & 0 \\ 0 & e^{-N\left(\int \log(z-s)\psi(s)ds - \frac{\ell}{2}\right)} \end{pmatrix}$$

Recent results and challenges: characterization results

characterization results: Janossy densities

$$\mathcal{J}_{k,I}(x_1,\ldots x_k)\mu(dx_1)\cdots\mu(dx_k) = \Pr\{\text{there are exactly } k \text{ particles in } I, \text{ one in each of the } k \text{ infinitesimal intervals } (x_i, x_i+dx_i)\}.$$

K_n: Integral operator with kernel $e^{-\frac{(V(x)+V(y))}{2}}\sum_{\ell=0}^{n-1} p_\ell(x)p_\ell(y)$ restricted to the interval I.

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THEN: $L_I = K_I(Id - K_I)^{-1}$ is an integral operator with kernel

$$e^{-\frac{(V(x)+V(y))}{2}}\sum_{\ell=0}^{n-1} \tilde{p}_\ell(x)\tilde{p}_\ell(y)$$

where $\int_{\mathbf{R}\setminus I} \tilde{p}_j(x)\tilde{p}_k(x)e^{-V(x)}dx = \delta_{jk}$

[Borodin-Soshnikov math-ph/0212063]

Asymptotic analysis: need new parametrix at endpoints of the interval I.

characterization results: Multiple orthogonal polynomials

Multiple orthogonal polynomial with index $\boldsymbol{n} = (n_1, \ldots, n_p) \in \mathbb{N}_p$

Orthogonal w.r.t. $\mathbf{w}(z) = \left(w_1(z), \ldots, w_p(z)\right)$

$$\int_{\Gamma_1} Q_{\mathbf{n}}(z) z^{\nu} w_1(z)\,dz = 0, \quad \nu = 0, \ldots, n_1 - 1, \qquad \int_{\Gamma_2} Q_{\mathbf{n}}(z) z^{\nu} w_2(z)\,dz = 0, \quad \nu = 0, \ldots, n_2 - 1, \quad \cdots$$

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Orthogonal w.r.t. $\mathbf{w}(z) = \left(w_1(z), \ldots, w_p(z)\right)$

$$\int_{\Gamma_\alpha} Q_{\mathbf{n}}(z) z^{\nu} w_\alpha(z)\,dz = 0, \quad \nu = 0, \ldots, n_\alpha - 1, \quad \alpha = 1, \ldots, p$$

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Riemann-Hilbert problem [Geronimo, Kuijlaars, and van Assche 2001]

A *is analytic on* $\mathbb{C} \setminus \cup_{\alpha=1}^{p} \Gamma_\alpha$

$$A(z) = \left(I + O\left(\frac{1}{z}\right)\right) \operatorname{diag}\left(z^{|\mathbf{n}|}, z^{-n_1}, z^{-n_2}, \ldots, z^{-n_p}\right),$$

$$A_+(x) = A_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) & \cdots & \cdots & w_{p-1}(x) & w_p(x) \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & & & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

characterization results: Coupled Random Matrices

$$\frac{1}{Z_N}\exp(-Tr(V(M_1)+W(M_2)-2\tau M_1 M_2))dM_1 dM_2$$

Assume W is a polynomial of degree d. Then take Γ=**R** and

$$w_j(z)=\int_{\mathbf{R}} y^{j-1}e^{-V(z)-W(y)+2\tau zy}dy,\ \ j=1,\ \ldots\ ,\mathrm{d}-1,\ \ \text{and } p=d.$$

Eigenvalue statistics can be described in terms of Multiple Orthogonal Polynomials.

Various slightly different Riemann--Hilbert characterizations exist:
Kuijlaars-McL, Bertola-Eynard-Harnad, and Kapaev.

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Assume W is a polynomial of degree d. Then take $\Gamma = \mathbf{R}$ and

$$w_j(z) = \int_{\mathbf{R}} y^{j-1} e^{-V(z)-W(y)+2\tau z y} dy, \quad j = 1, \ldots, d-1, \quad \text{and } p = d.$$

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Various slightly different Riemann--Hilbert characterizations exist:
Kuijlaars-McL, Bertola-Eynard-Harnad, and Kapaev.

But rigorous asymptotic analysis of any of the RHPs is still lacking. At issue is the existence of a suitable generalized equilibrium measure.

characterization results: Random Matrices with Source

$$\frac{1}{Z_n} e^{-\mathrm{Tr}(V(M)-AM)} dM$$

A: matrix with eigenvalues $a_1, \ldots, a_p$ (multiplicities $n_1, \ldots, n_p$). Then take **Γ=R** and

$$w_j(z) = e^{-(V(z)-a_i z)}, \quad j = 1, \ \ldots, \ p$$

Eigenvalue statistics can be described in terms of Multiple Orthogonal Polynomials.

Model due to Brezin-Hikami
Determinantal point process: Zinn-Justin
RHP characterization: Bleher-Kuijlaars

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Determinantal point process: Zinn-Justin
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Rigorous asymptotic analysis of the associated RHP: hang on…

In general: at issue is the existence of a suitable generalized equilibrium measure.

characterization results: Skew OPs for β=1,4

$$\langle f,g\rangle_1 = \iint_{\mathbb{R}^2} f(x)g(y)\epsilon(x-y)e^{-NV(x)-NV(y)}dxdy.$$

$$\epsilon(x) = \begin{cases} -1 & : x<0 \\ 1 & : x>0 \end{cases};$$

$$\begin{aligned} \langle p_{2k}(x), p_{2j}(y)\rangle_\beta &= 0 \\ \langle p_{2k}(x), p_{2j+1}(y)\rangle_\beta &= h_j\delta_{kj} \\ \langle p_{2k+1}(x), p_{2j}(y)\rangle_\beta &= -h_j\delta_{kj} \\ \langle p_{2k+1}(x), p_{2j+1}(y)\rangle_\beta &= 0, \end{aligned}$$

V. Pierce has described a RHP for these polynomials for *V* a polynomial of degree *d:*

$$w_j(x) = \int_{\mathbb{R}} y^j\epsilon(x-y)e^{-V(x)-V(y)}dy$$

$$\mathcal{W}(x) = e^{-2V(x)}.$$

characterization results: Skew OPs for β=1,4

$$w_j(x) = \int_{\mathbb{R}} y^j \epsilon(x-y) e^{-V(x)-V(y)} dy$$

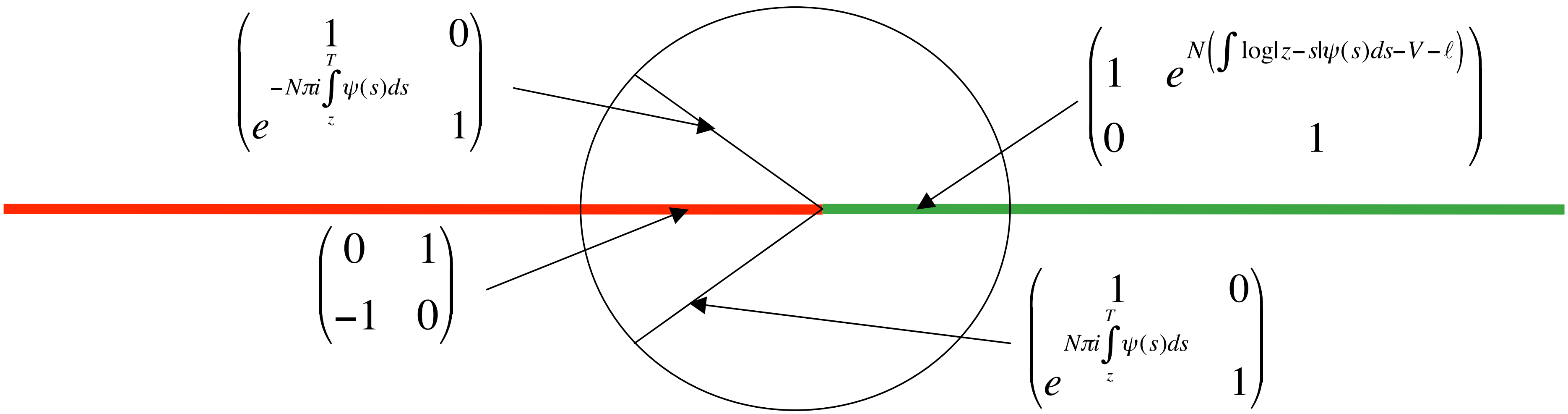
$$\mathcal{W}(x) = e^{-2V(x)}.$$

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^{2k} & & & & \\ & z^{-2k+d-1} & & & \\ & & z^{-1} & & \\ & & & \ddots & \\ & & & & z^{-1} \end{pmatrix}$$

$$Y_+ = Y_- \begin{pmatrix} 1 & \mathcal{W}(x) & w_0(x) & \dots & w_{d-2}(x) \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & & \dots & & 1 \end{pmatrix} \quad \text{for real } x.$$

Asymptotic analysis? Actually, the challenge is for varying weights.

Local analysis and parametrices: double scaling limits



(1) B_a analytic in open components

(2) $B_{a,+}(z) = B_{a,-}(z)V_D(z)$, $z \in \Sigma \cap \text{disk}$

(3) $B_a\left(B_{\text{outer}}\right)^{-1} = I + O\left(N^{-1}\right)$ on the circle

Transform to canonical model RHP

$$z \mapsto \zeta(z) \ \text{ so that } \ N\pi i\int_z^T \psi(s)ds = \zeta^{\frac{4j+1}{2}}$$

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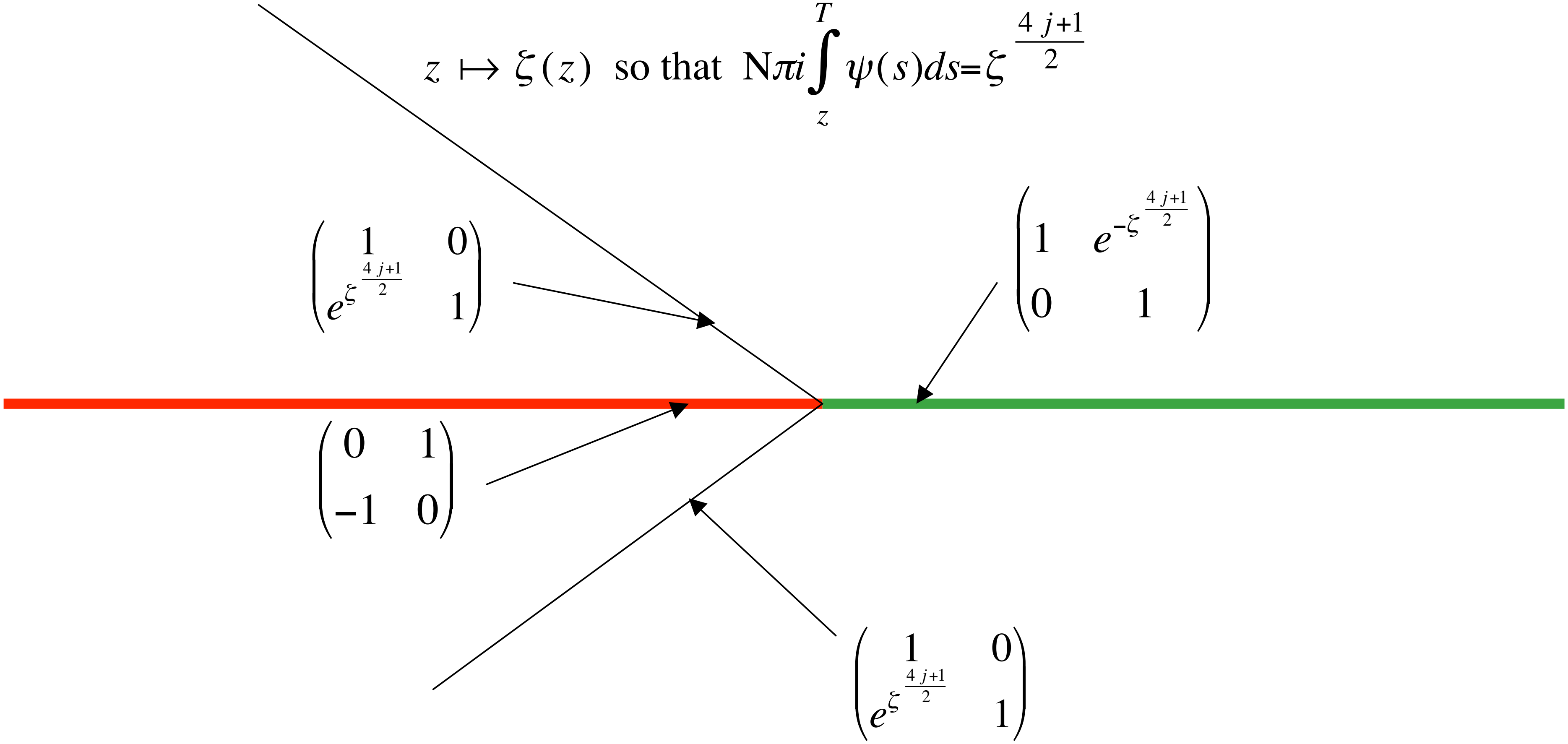


(1) P analytic off the rays

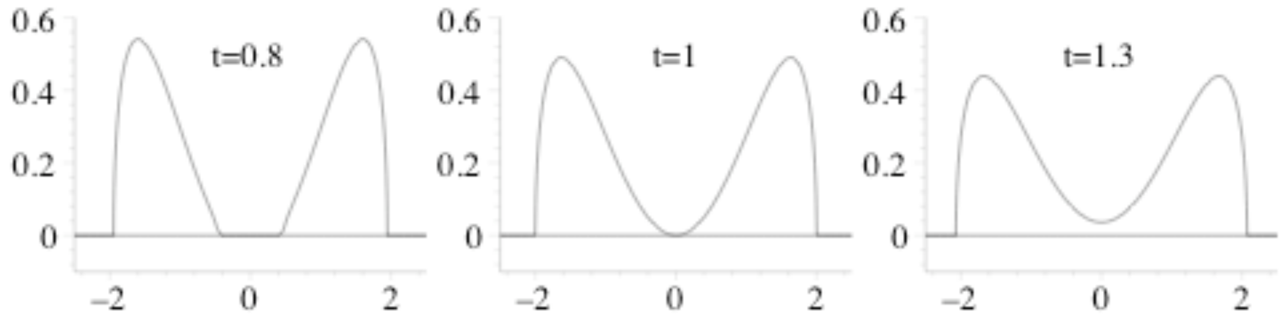
(2) $$P_+(\zeta) = P_-(\zeta)V_P(\zeta), \quad \varsigma \in \Sigma$$

(3) $$P\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} = I + O\left(\zeta^{-1}\right) \text{ for } \zeta \to \infty$$

$$B_a(z) = \mathbf{E}(z)P\left(\zeta(z)\right)$$

Local analysis and parametrices: double scaling limits

$$Z_n^{-1}\exp\left(-n\ \mathrm{Tr}\ t^{-1}V(M)\right)dM$$



Equilibrium measures for $t^{-1}(x^4/4 - x^2)$

$$K_{n,t}(x,y) = e^{-\frac{n}{2t}\left(V(x)+V(y)\right)}\sum_{k=0}^{n-1} p_{k,t}(x)p_{k,t}(y)$$

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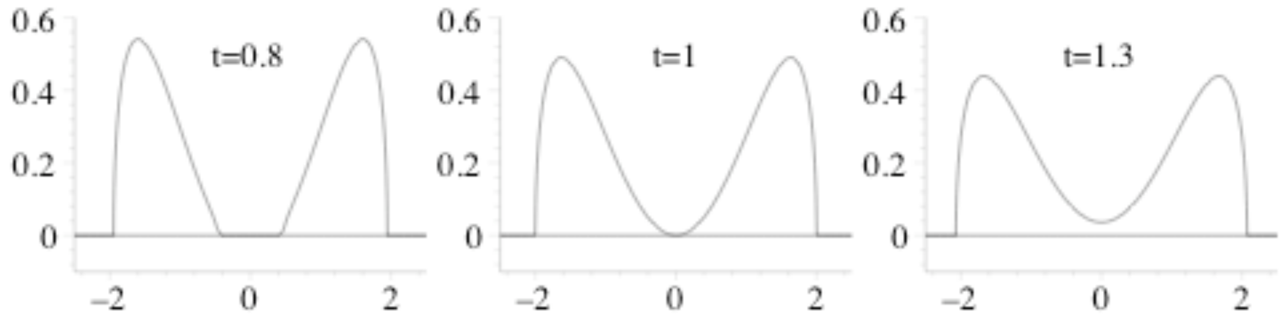

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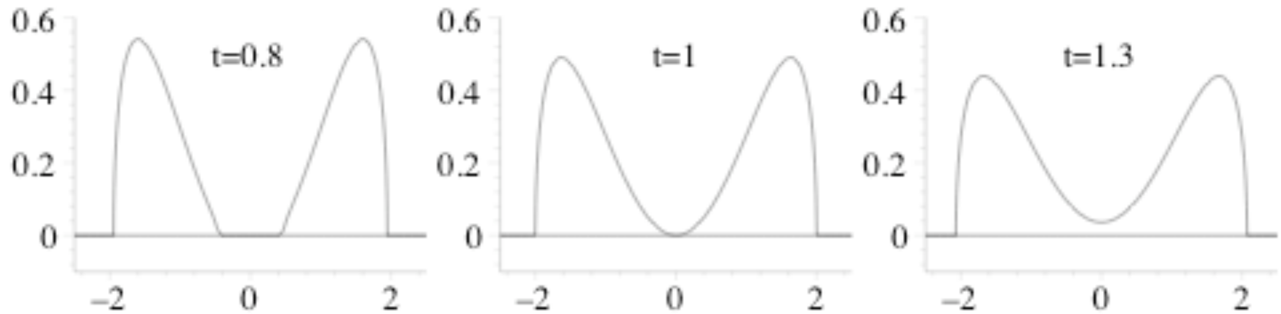

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Choose $t=t_n$ so that $\lim_{n\to\infty} n^{2/3}(t_n - 1) = L$. Then:

$$\lim_{n\to\infty} K_{n,t}\left(x^* + \frac{u}{(cn)^{1/3}}, y^* + \frac{v}{(cn)^{1/3}}\right) = K^{crit}(u,v,s)$$

$K^{crit}(u,v,s)$: kernel built using the RHP associated to Hastings-McLeod soln to PII.

Bleher and Its, CPAM 2003: $t^{-1}(x^4/4 - x^2)$, $x^*=0$.

Claeys and Kuijlaars math-ph/0501074: $\psi_V(x^*) = \psi_V'(x^*) = 0, \qquad \psi_V''(x^*) > 0.$

Local analysis and parametrices: singularities

$$\frac{1}{Z_n}|\det M|^{2\alpha} e^{-n \operatorname{tr} V(M)} dM \quad \alpha > -1/2$$

Assume

$$\begin{cases} V : \mathbb{R} \to \mathbb{R} \text{ is real analytic,} \\ \lim_{|x|\to\infty} \frac{V(x)}{\log(x^2+1)} = +\infty, \\ \psi(0) > 0, \end{cases}$$

Then

$$\frac{1}{n\psi(0)} K_n\left(\frac{u}{n\psi(0)}, \frac{v}{n\psi(0)}\right) = \pi\sqrt{u}\sqrt{v}\frac{J_{\alpha+\frac{1}{2}}(\pi u)J_{\alpha-\frac{1}{2}}(\pi v) - J_{\alpha-\frac{1}{2}}(\pi u)J_{\alpha+\frac{1}{2}}(\pi v)}{2(u-v)} + O\left(\frac{u^\alpha v^\alpha}{n}\right),$$

G. Akemann, Damgaard, Magnea, and Nishigak, Nucl. Phys. B ‘97i

Kuijlaars and Vanlessen, CMP2003, using Vanlessen’s construction of a parametrix: math-ph/02120014

Local analysis and parametrices: singularities

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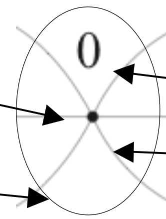
$$\begin{pmatrix} 0 & |x|^{2\alpha} \\ -|x|^{-2\alpha} & 0 \end{pmatrix}$$

$$P(z)\left(P^{(\infty)}\right)^{-1}(z) = I + O\left(\frac{1}{n}\right)$$



$$\begin{pmatrix} 1 & 0 \\ \omega(z)^{-1}e^{-2n\phi(z)} & 1 \end{pmatrix}$$

$\phi'(0) \neq 0$

$$P(z) = O\begin{pmatrix} 1 & |z|^{2\alpha} \\ 1 & |z|^{2\alpha} \end{pmatrix} \qquad \text{as } z \to 0 \ \ (\text{for } \alpha < 0)$$

for $\alpha > 0$,

$$P(z) = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from outside the lens,} \\ O\begin{pmatrix} |z|^{-2\alpha} & 1 \\ |z|^{-2\alpha} & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ from inside the lens.} \end{cases}$$

Equilibrium measures, existence, global analysis

Equilibrium measures, existence, global analysis

$$\lim_{N\to\infty} \rho_1^{(N)}(x) = \psi(x),$$ where $\psi \geq 0$ solves a well-known variational problem.

$$\sup_{\substack{0\leq d\mu, \\ \int d\mu=1}} \left[-\int V d\mu + \iint \log|x-y| d\mu(x) d\mu(y) \right]$$

Regularity results

V real analytic with suitable growth $\Longrightarrow$ ψ is supported on finitely many intevals, and is analytic on the interior of each one.

(Deift, Kriecherbauer, McL '98)

$\exists$ V $\in$ C^∞ so that ψ is supported on infinitely many disjoint intervals [Kuijlaars, McL]

Equilibrium measures, existence, global analysis

Grava and Tian: using techniques from singular limits of integrable systems:

$$V(\xi) = V_*(\xi) + t\, p(\xi)$$ With $V \in C^\infty$, p a monic polynomial

P odd: 1 interval in support for all $|t|$ sufficiently large.

P even and $t << -1$: *1* interval in support.

P even and $t >> 1$: *2* intervals in support.

[Comm. Math. Sci. 4(2006), 551-573]

Equilibrium measures, existence, global analysis

RMT with source: $\frac{1}{Z_n} e^{-\mathrm{Tr}(V(M)-AM)} dM$ $\quad$ *A*: 2 eigenvalues, $\pm a$, multiplicities n_1, n_2

Bleher and Kuijlaars: analysis of the associated 3x3 RHP
when $V=M^2/2$, $n_1=n_2=1/2$ for ALL $a>0$.

$$w^3 - zw^2 - \left(a^2 - 1\right) w + a^2 z = 0$$

The equilibrium measures are obtained from the solutions of this algebraic equation.

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$$w^3 - z^3 w^2 + \left(z^2 + \alpha\right) w + a^2 z^3 + \beta z = 0$$

with $\alpha=\alpha(a)$, $\beta= \beta(a)$, that allows for an analysis of the associated 3x3 RHP.

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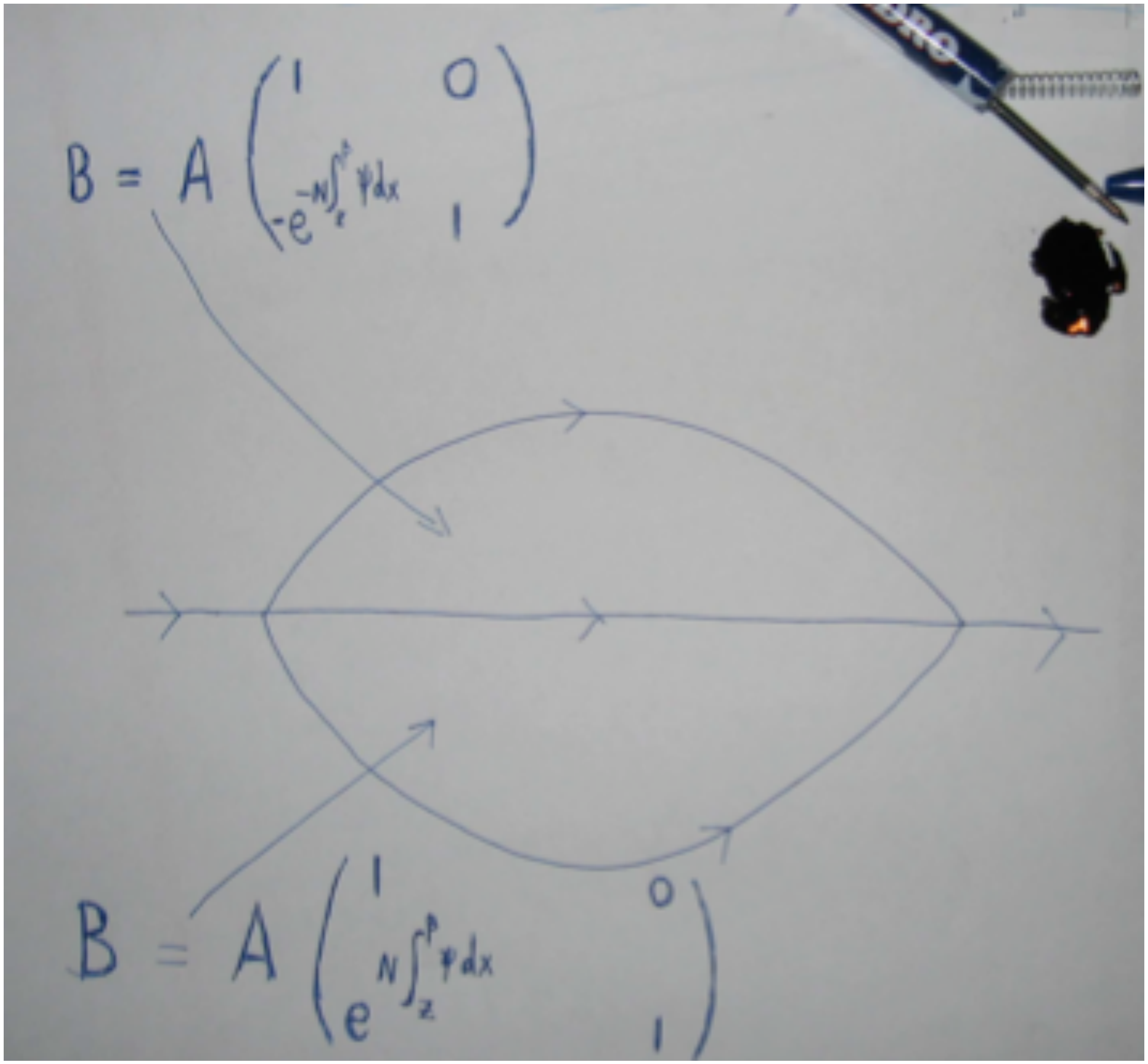
For general functions V: there is a family of algebraic curves:

$$w^3 - (V'(z))\, w^2 - \left[\mathcal{A}+a^2\right] w - \left[\left(\mathcal{A}-a^2\right) V'(z) - \mathcal{B}\right] = 0$$

Challenge: prove that within this family there exists a suitable choice yielding an equilibrium measure.

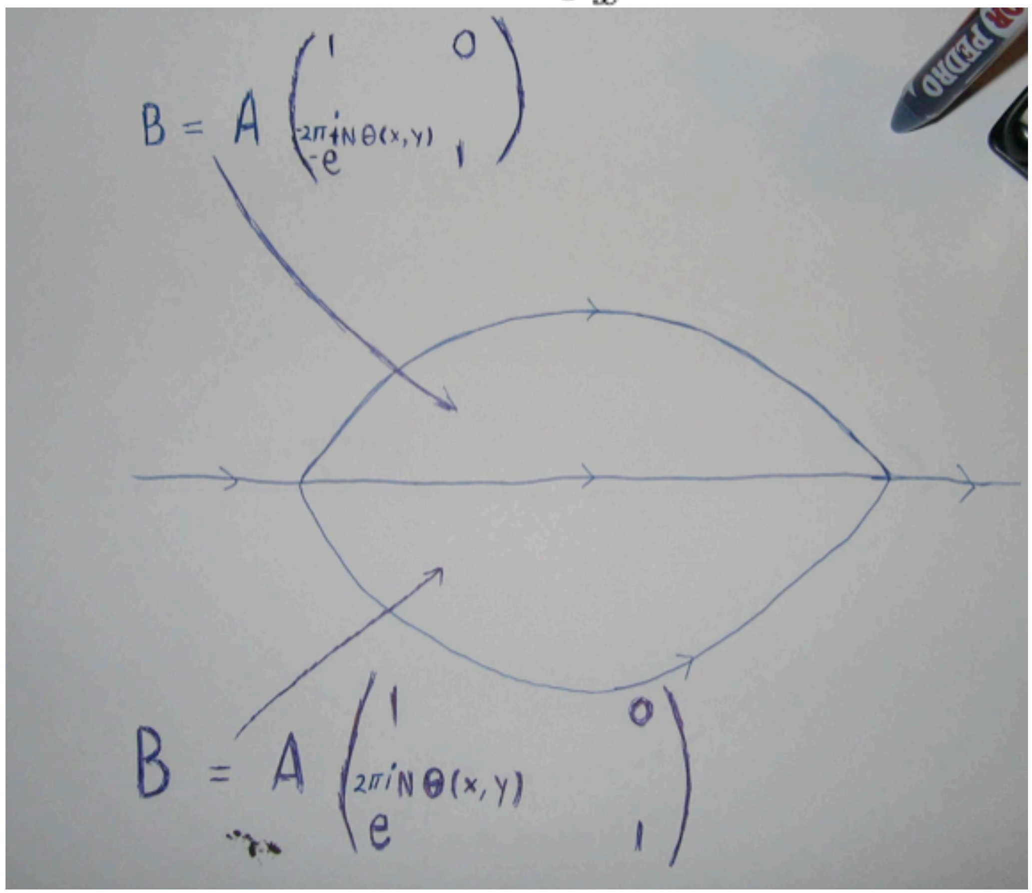
Lens opening without analyticity

$$A_+(z)\begin{pmatrix}1 & 0\\ -e^{-2\pi i N\int_z^{\beta_J}\psi(s)ds} & 1\end{pmatrix} = A_-(z)\begin{pmatrix}1 & 0\\ e^{2\pi i N\int_z^{\beta_J}\psi(s)ds} & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$



Lens opening without analyticity

Let $\theta(x, y)$ be an extension of $\int_x^\beta \psi(s)ds$ off $\mathbb{R}$, and then...



But B is not analytic! However,

$$\overline{\partial} B = B \begin{pmatrix} 0 & 0 \\ -Ne^{-N\theta(x,y)}\overline{\partial}\theta(x,y) & 0 \end{pmatrix}$$