random matrices,
asymptotic analysis, and combinatorics.

Mostly a survey!
But I'll describe some work with
Nick Ercolani (Arizona), K.M.

Random Matrices

Consider the probability measure on $N \times N$ Hermitean matrices given by

$$\frac{1}{\hat{Z}_N} \exp\left\{ -N \operatorname{Tr}\left[\frac{1}{2} M^2 + \sum_{k=1}^{2\nu} t_k \, M^k \right] \right\} dM$$

$$dM = \prod_{j<k} dM_{jk}^{R} \, dM_{jk}^{I} \prod_{j=1}^{N} dM_{jj}$$

$$\hat{Z}_N = \int \exp\left\{ -N \operatorname{Tr}\left[\frac{1}{2} M^2 + \sum_{k=1}^{2\nu} t_k \, M^k \right] \right\} dM$$

Historically, the interest has been in the probabilistic description of the eigenvalues, as $N \to \infty$

Gaussian Unitary Ensemble: all the t_k's =0.

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Gaussian Unitary Ensemble: all the t_k's =0.

Typically interested in eigenvalues, whose induced probability density is:

$$\frac{1}{Z_N}\exp\left(-N\sum_{j=1}^{N}V(\lambda_j)\right)\prod_{1\le j<k\le N}\left(\lambda_k-\lambda_j\right)^2 ,$$

$$V(\lambda) = \left[\frac{1}{2}\lambda^2 + \sum_{k=1}^{2\nu} t_k\,\lambda^k\right]$$

Partition Function: $$Z_N = \int \exp\left(-N\sum_{j=1}^{N}V(\lambda_j)\right)\prod_{1\le j<k\le N}\left(\lambda_k-\lambda_j\right)^2 d^N\lambda$$

Closely related to polynomials $\left\{p_j(x)\right\}_{j\ge 0}$ orthogonal wrt $e^{-NV(x)}dx$

A Physical Interpretation

- **Statistical Mechanics of a Log-gas**

$$Z_N = \int \exp\left(-N \sum_{j=1}^{N} V(\lambda_j) + \sum_{j \neq k} \log\left|\lambda_k - \lambda_j\right| \right) d^N \lambda$$

  - Eigenvalues are a system of *particles on the line* with *logarithmic interaction potential*, log $|\lambda_i - \lambda_j|$, in the presence of an *external field* with potential

$$V(\lambda) = \frac{1}{2}\lambda^2 + \sum_{k=1}^{v} t_k \lambda^{2k}$$

  - The asymptotic behavior for $N \rightarrow \infty$ may then be interpreted as the limiting behavior of this statistical mechanical system in a *low temperature* and *many particle* asymptotic limit.

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$$V(\lambda) = \left[\frac{1}{2} \lambda^2 + \sum_{k=1}^{2\nu} t_k \, \lambda^k \right]$$

Partition Function: $$Z_N = \int \exp\left(-N \sum_{j=1}^{N} V(\lambda_j) \right) \prod_{1 \le j < k \le N} \left(\lambda_k - \lambda_j \right)^2 d^N \lambda$$

Closely related to polynomials $\left\{ p_j(x) \right\}_{j \ge 0}$ orthogonal wrt $e^{-NV(x)} dx$

Mean density of particles

Consider the random variable $\frac{1}{N}\#\{\text{evals} < x\}$

The mean density is defined via

$$\rho_1(x) = \frac{d}{dx}\left\langle \frac{1}{N}\#\{\text{evals} < x\}\right\rangle$$

The connection to orthogonal polynomials:

$$\rho_1(x) = K_N(x,x)$$

$$K_N(x,y) = e^{-\frac{N(V(x)+V(y))}{2}}\sum_{\ell=0}^{N-1} p_\ell(x)p_\ell(y)$$

All statistical properties can be expressed in terms of the orthogonal polynomials!

E.G.

$$\text{Prob}\left\{\text{no } \lambda_j\text{'s in } (a,b)\right\} = \det\left(1 - \mathbf{K_N}\right)_{L^2(a,b)}$$

$$\left(\mathbf{K_N} f\right)(x) = \int_a^b K_N(x,y) f(y) dy$$

In most applications, we are interested in the behavior for $N \rightarrow \infty$

Basic asymptotic result: Under rather weak assumptions on V, the following limit exists.

(Johansson, 1998)

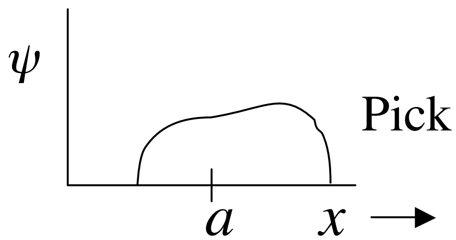
$$\lim_{N\to\infty} \rho_1^{(N)}(x) = \psi(x),$$ where $\psi \geq 0$ solves a well-known variational problem.

$$\sup_{\substack{0\leq d\mu, \\ \int d\mu = 1}} \left[-\int V d\mu + \iint \log|x-y| d\mu(x) d\mu(y) \right]$$

V real analytic with suitable growth $\Longrightarrow$ ψ is supported on finitely many intevals, and is analytic on the interior of each one.

(Deift, Kriecherbauer, McL ‘98)

Much more detailed asymptotic result:



Pick $a \in \mathrm{supp}(\psi)$ and consider $\mathrm{G_N(a,s)} = \mathrm{Prob}\left\{ \text{no } \lambda_j\text{'s in } \left(a, a + \frac{s}{N\psi(a)}\right) \right\}$.

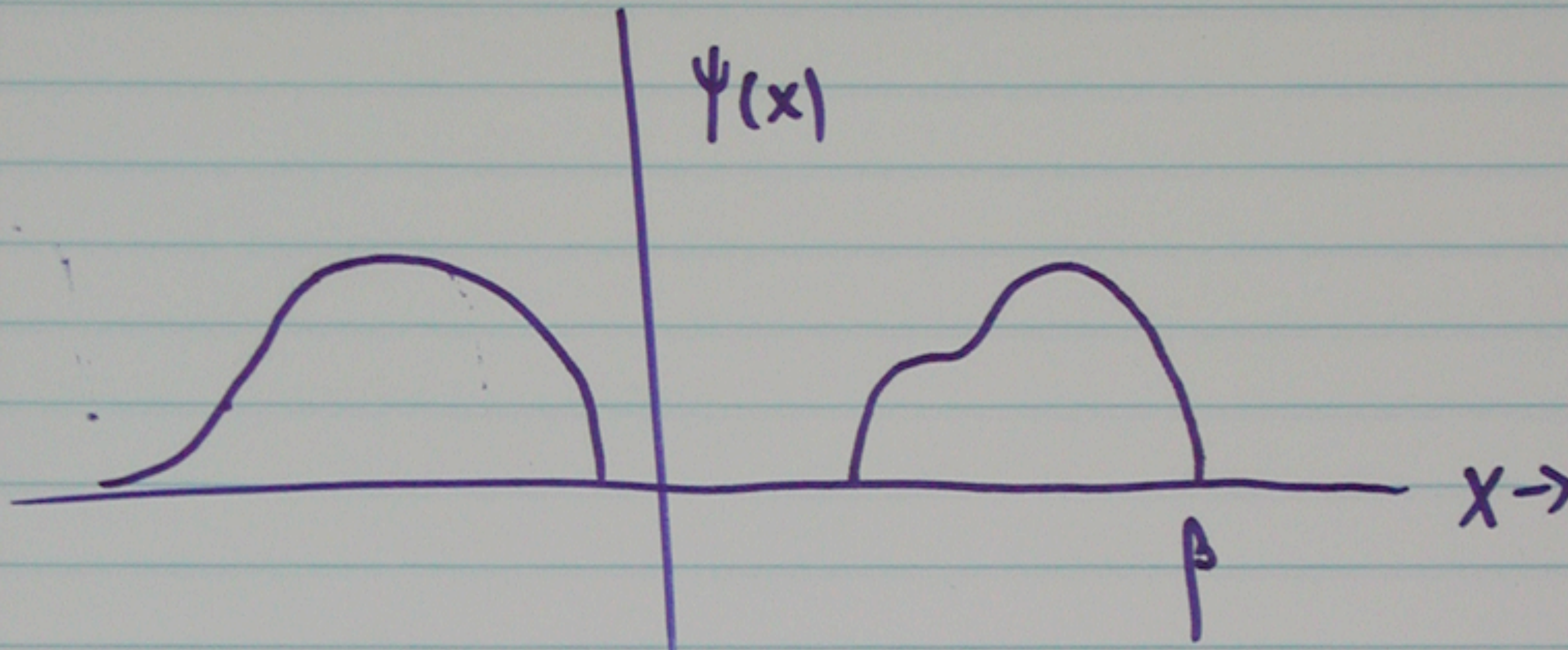
One version of universality result: for any real analytic V with suitable growth, and any such a,

$$\lim_{N\to\infty} G_N(a,s) = \det(1 - \mathbf{S})_{L^2(0,s)}$$

$$(\mathbf{S}h)(\sigma) = \int_0^s \frac{\sin(\pi(\sigma - \sigma'))}{\pi(\sigma - \sigma')} h(\sigma')\, d\sigma'$$

(Deift, Kriecherbauer, McL, Venakides, and Zhou '97)

Behavior of largest eval



$$\text{Prob}\left(|\lambda_N - \beta| > \epsilon\right) \xrightarrow{N \to \infty} 0$$

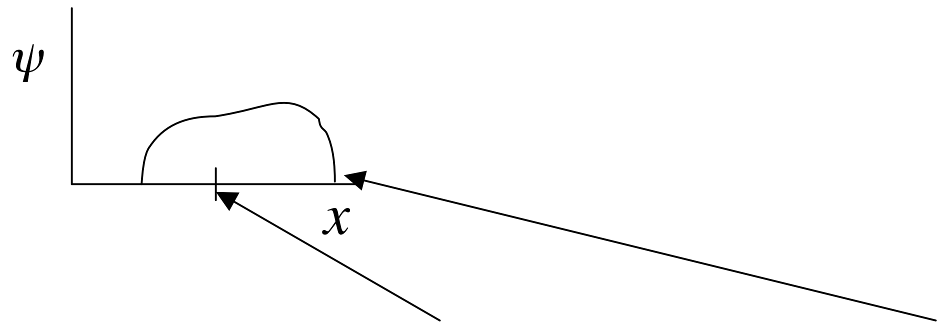
$$\text{Prob}\left[\sigma(\lambda_N - \beta) \cdot N^{2/3} < s \right] \longrightarrow F_{TW}(s)$$

- $V(x) = x^2$ Forrester, Tracy-Widom
- General V: DKMVZ

$$F_{TW}(s) = \exp - \int_s^{\infty} (x-s)\, q^2(x)\, dx$$

$$q'' = xq + 2q^3$$

Much more detailed asymptotic results:



Large N behavior of "Local" statistics are universal here (sine-kernel) and at the edge (Tracy-Widom distribution, aka Epps convergence theorem).

$$V(\lambda) = \frac{1}{2}\lambda^2 + \sum_{j=1}^{v} t_j \lambda^j \qquad \mathbb{T}(T,\gamma) = \{\mathbf{t} \in \mathbb{R}^v : \ |\mathbf{t}| \leq T, \ t_v \geq \gamma \sum_{j=1}^{v-1} |t_j|\}$$

Then

$$\psi \, d\lambda \ = \ \frac{1}{2\pi}\chi_{(\alpha,\beta)}(\lambda)\sqrt{(\lambda-\alpha)(\beta-\lambda)}\, h(\lambda) \ \ d\lambda,$$

Asymptotic behavior for large N: $\rho_N^{(1)} \stackrel{*}{\rightharpoonup} \psi$, but we also have a **complete expansion away from endpoints:** For $\lambda \in (\alpha, \beta)$,

$$\rho_N^{(1)}(\lambda) = \psi(\lambda) + \frac{1}{4\pi N}\left(\frac{1}{\lambda-\beta} - \frac{1}{\lambda-\alpha}\right)\cos\left\{N\int_\lambda^\beta \psi(s)ds\right\} \tag{4}$$
$$+\frac{1}{N^2}\left[H(\lambda) + G(\lambda)\sin\left\{N\int_\lambda^\beta \psi(s)ds\right\}\right] + \cdots$$

in which $H(\lambda)$ and $G(\lambda)$ are locally analytic functions which are explicitly computable in terms of the original external field $V(\lambda)$.

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Near an endpoint:

$$\rho_N^{(1,\beta)}(z) = N^{-1}\mathcal{A}(z)\left[2\, Ai\left(N^{2/3}\phi_\beta(z)\right)\, Ai'\left(N^{2/3}\phi_\beta(z)\right)\right] + N^{-1/3}\phi_\beta'(z)\left[\left(Ai'\left(N^{2/3}\phi_\beta(z)\right)\right)^2 - N^{2/3}\phi_\beta(z)\left(Ai\left(N^{2/3}\phi_\beta(z)\right)\right)^2\right] + \mathcal{E}_N(z)$$

$$\mathcal{E}_N(z) = \sum_{j \text{ even},\, j\geq 2} N^{-j+1/3}\tilde{a}_j(z)\left(Ai\left(N^{2/3}\phi_\beta(z)\right)\right)^2 + \sum_{j \text{ even},\, j\geq 2} N^{-j-1/3}\tilde{b}_j(z)\left(Ai'\left(N^{2/3}\phi_\beta(z)\right)\right)^2 + \sum_{j \text{ odd},\, j\geq 3}^{\infty} N^{-j}\tilde{c}_j(z)\left(Ai\left(N^{2/3}\phi_\beta(z)\right)\right)\left(Ai\left(N^{2/3}\phi_\beta(z)\right)\right)$$

THE WORKHORSE RESULT: Integrals with respect to $\rho_N^{(1)}$

Theorem 3. *[Ercolani and McLaughlin, [4]] There is $T > 0$ and $\gamma > 0$ so that for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$, the following expansion holds true:*

$$\int_{-\infty}^{\infty} f(\lambda)\rho_N^{(1)}(\lambda)d\lambda = f_0 + N^{-2}f_1 + N^{-4}f_2 + \cdots , \tag{7}$$

provided the function $f(\lambda)$ is C^∞ smooth, and grows no faster than a polynomial for $\lambda \to \infty$. The coefficients f_j depend analytically on $\mathbf{t}$ for $\mathbf{t} \in \mathbb{T}(T, \gamma)$, and the asymptotic expansion may be differentiated term by term.

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$$\hat{Z}_N = \int \exp\left\{-N \operatorname{Tr}\left[\frac{1}{2}M^2 + \sum_{k=1}^{2\nu} t_k M^k\right]\right\} dM = C_N \int \exp\left(-N\sum_{j=1}^{N} V(\lambda_j)\right) \prod_{1\le j<k\le N} (\lambda_k - \lambda_j)^2 d^N\lambda$$

APPLICATION 1: Complete expansion of the partition function

$$\frac{\partial}{\partial t_\ell} \log(Z_N) = -N^2 \mathbb{E}\left(\frac{1}{N}\operatorname{Tr} M^\ell\right) = -N^2 \int_{-\infty}^{\infty} \lambda^\ell \rho_N^{(1)}(\lambda)d\lambda$$

$$\hat{Z}_N = \int \exp\left\{-N \,\mathrm{Tr}\left[\frac{1}{2}M^2 + \sum_{k=1}^{2\nu} t_k\, M^k\right]\right\} dM \;= C_N \int \exp\left(-N\sum_{j=1}^{N} V(\lambda_j)\right) \prod_{1\le j<k\le N} \left(\lambda_k - \lambda_j\right)^2 d^N\lambda$$

We (Ercolani & McL) established:
There is $T > 0$, $\gamma > 0$ and $N_0 > 0$ so that the asymptotic expansion

$$\log\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right) = N^2 e_0(t_1,\ldots,t_{2\nu}) + e_1(t_1,\ldots,t_{2\nu}) + \frac{1}{N^2} e_2(t_1,\ldots,t_{2\nu}) + \cdots$$

is valid for $\mathbf{t}$ in $\mathsf{T}(T,\gamma)$ and $N > N_0$; where

$$\mathsf{T}(T,\gamma) = \left\{\mathbf{t}\in\mathbf{R}^{2\nu} : |\mathbf{t}| \le T,\; t_{2\nu} > \delta \sum_{j=1}^{2\nu-1} |t_j|\right\}.$$

- Each coefficient, $e_j(t_1,\ldots,t_{2\nu})$, is an analytic function of the (complex) vector $\boldsymbol{t}$ in a neighborhood of $\boldsymbol{0}$. The asymptotic expansion of derivatives of $\log(Z_N)$ may be calculated via term-by-term differentiation of the above series.

APPLICATION 2: enumerative geometry

A *map* is a graph which is embedded into a Riemann surface so that

1. the (images of the) edges do not intersect;
2. dissecting the surface along the edges decomposes it into a union of open cells; these cells are called the *faces* of the map.

A *g-map* is a labelled map that is embedded into a Riemann surface with genus g, for which the underlying graph is connected. The labelling ascribes a number to each vertex, along with a lexicographical ordering of the edges emanating from each vertex.

Theorem 5 (Ercolani and McLaughlin [4]). *The coefficients in the asymptotic expansion (9) satisfy the following relations. Let* g *be a nonnegative integer. Then*

$$e_g(t_1 \dots t_\nu) = \sum_{n_j \geq 1} \frac{1}{n_1! \dots n_\nu!} (-t_1)^{n_1} \dots (-t_\nu)^{n_\nu} \kappa_g(n_1, \dots, n_\nu) \tag{10}$$

in which each of the coefficients $\kappa_g(n_1, \dots, n_\nu)$ *is the number of g-maps with* n_j *j-valent vertices for* $j = 1, \dots, \nu$.

RMT and Combinatorics: Gaussian Unitary Ensemble: all the t_k's =0.

$$\frac{N^{N^2/2}}{2^{N/2}\pi^{N^2/2}} \exp\left\{-N \operatorname{Tr}\left[\frac{1}{2}M^2\right]\right\} dM$$

$\sqrt{N}\, m_{ij}, i \le j$, are independent complex standard normal random variables with $m_{ij} = \overline{m}_{ji}$.

Hence $\langle m_{ij} m_{ji} \rangle = N^{-1}$ and $\langle m_{ij} m_{kl} \rangle = 0$ if $(i,j) \ne (l,k)$.

This, together with Wick's formula yields a fundamental connection to combinatorics and the enumeration of maps.

$$\left.\frac{\partial^n}{\partial t_p^n}\hat{Z}(\vec{t})\right|_{\vec{t}=0} = (-N)^n \, \mathbf{E}_{\vec{t}=0}\left\{ \left[\operatorname{Tr}\left(M^p\right)\right]^n \right\} =$$

$$= (-1)^n \sum_{g} N^{2-2g} \; \#\{\text{things indexed by } n,\ p, \text{ and } g\}$$

Wick Rules. $l_1, \ldots, l_{2k}$ linear functions.

$$\langle l_1 l_2 \cdots l_{2k} \rangle = \sum \langle l_{r_1} l_{s_1} \rangle \langle l_{r_2} l_{s_2} \rangle \cdots \langle l_{r_k} l_{s_k} \rangle$$

Sum is over all $\{r_1, \ldots, r_k, s_1, \ldots, s_k\}$ such that

$1 \le r_j \le 2k-1$

$2 \le s_j \le 2k$

$r_1 < r_2 < \cdots < r_k$

$r_1 < s_1$ $\quad$ $s_1 \neq s_2$ etc.

$r_2 < s_2$

$\vdots$

$r_k < s_k$

Wick Rules. $\ell_1, \dots, \ell_{2k}$ linear functions.

$$\langle \ell_1 \ell_2 \cdots \ell_{2k} \rangle = \sum \langle \ell_{r_1} \ell_{s_1} \rangle \langle \ell_{r_2} \ell_{s_2} \rangle \cdots \langle \ell_{r_k} \ell_{s_k} \rangle$$

Sum is over all $\{r_1, \dots, r_k, s_1, \dots, s_k\}$ such that

$1 \le r_j \le 2k-1$ $\qquad r_1 < r_2 < \cdots < r_k$

$2 \le s_j \le 2k$ $\qquad r_1 < s_1 \qquad s_1 \ne s_2$ etc.

$r_2 < s_2$

$\vdots$

$r_k < s_k$

There are $(2k-1)!!$ distinct Wick couplings.

$$\mathbf{E}_{\vec{t}=0}\left\{ \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\} = \sum_{\text{Configurations}} \mathbf{E}_{\vec{t}=0}\left\{ m_{i_1 j_1} m_{j_1 k_1} m_{k_1 l_1} m_{l_1 i_1} m_{i_2 j_2} m_{j_2 k_2} m_{k_2 l_2} m_{l_2 i_2} \right\}$$

$$= \sum_{\text{configs}} \mathbf{E}_{\vec{t}=0}\left\{ \ell_1 \ \ell_2 \ \ell_3 \ \ell_4 \ \ell_5 \ \ell_6 \ \ell_7 \ \ell_8 \right\}$$

$$= \sum_{(r,s)\in \mathbf{W}(\mathbf{8})} \sum \mathbf{E}_{\vec{t}=0}\left\{\ell_{r_1} \ell_{s_1}\right\} \mathbf{E}_{\vec{t}=0}\left\{\ell_{r_2} \ell_{s_2}\right\} \mathbf{E}_{\vec{t}=0}\left\{\ell_{r_3} \ell_{s_3}\right\} \mathbf{E}_{\vec{t}=0}\left\{\ell_{r_4} \ell_{s_4}\right\}$$

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For each of the 75 distinct Wick couplings, this is either 0 or N^{-4}.

A few of the Wick couplings are: $\{(1,2),(3,4),(5,6),(7,8)\},\{(1,4),(2,3),(6,8),(5,7)\}$

(r_1,s_1) (r_3,s_3)

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So we have:

$$= \sum \mathbf{E}_{\vec{t}=0}\left\{\ell_1\ell_2\right\}\mathbf{E}_{\vec{t}=0}\left\{\ell_3\ell_4\right\}\mathbf{E}_{\vec{t}=0}\left\{\ell_5\ell_6\right\}\mathbf{E}_{\vec{t}=0}\left\{\ell_7\ell_8\right\} +$$

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$$= \sum \mathsf{E}_{\vec{t}=0}\left\{m_{i_1 j_1} m_{j_1 k_1}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{k_1\ell_1} m_{\ell_1 i_1}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{i_2 j_2} m_{j_2 k_2}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{k_2\ell_2} m_{\ell_2 i_2}\right\} +$$

$$(i_1 = k_1)(i_2 = k_2)$$

$$+ \sum \mathsf{E}_{\vec{t}=0}\left\{m_{i_1 j_1} m_{\ell_1 i_1}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{j_1 k_1} m_{k_1\ell_1}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{\ell_2 i_2} m_{j_2 k_2}\right\}\mathsf{E}_{\vec{t}=0}\left\{m_{i_2 j_2} m_{k_2\ell_2}\right\} + \cdots$$

$$(j_1 = \ell_1)(i_2 = j_2 = k_2 = \ell_2)$$

$$= N^{6-4} \quad + \quad N^{4-4} + \quad \cdots$$

$$\mathbf{E}_{\vec{t}=0}\left\{ \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\} = \sum_{(r,s)\in\mathbf{W}(\mathbf{8})} N^{-4+F[(r,s)]}$$

Where $F\left[(r,s)\right]$ = number of free indices in the Wick coupling (r,s).

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$$\left\langle \left(\mathrm{Tr}\, M^p\right)^n \right\rangle = \sum_{\text{configs}} \left\langle M_{i_1 j_1} M_{j_1 k_1} M_{k_1 l_1} \cdots \times M_{i_2 j_2} M_{j_2 k_2} \cdots \vdots \times M_{i_n j_n} M_{j_n k_n} \cdots \right\rangle = \sum_{\text{configs}} \left\langle f_1 f_2 \cdots f_{np} \right\rangle$$

$$= \sum_{W(np)} \sum_{\text{configs}} \left\langle f_{u_1} f_{v_1} \right\rangle \left\langle f_{u_2} f_{v_2} \right\rangle \cdots \left\langle f_{u_{\frac{np}{2}}} f_{v_{\frac{np}{2}}} \right\rangle$$

$$= \sum_{(r,s)\in\mathbf{W}(np)} N^{-\left(np/2\right)+F[(r,s)]} =$$

number of free indices in (r,s) amongst

$i_1, j_1, \ldots, \quad i_2, j_2, \ldots, \quad \ldots, \quad i_{np}, j_{np}, \ldots,$

$$\mathbf{E}_{\vec{t}=0}\left\{ \left[\mathrm{Tr}\left(M^4\right)\right]^2 \right\} = \sum_{(r,s)\in\mathbf{W}(\mathbf{8})} N^{4+F[(r,s)]}$$

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$$= \sum_{W(np)} \sum_{\text{configs}} \left\langle f_{u_1} f_{v_1} \right\rangle \left\langle f_{u_2} f_{v_2} \right\rangle \cdots \left\langle f_{u_{\frac{np}{2}}} f_{v_{\frac{np}{2}}} \right\rangle$$

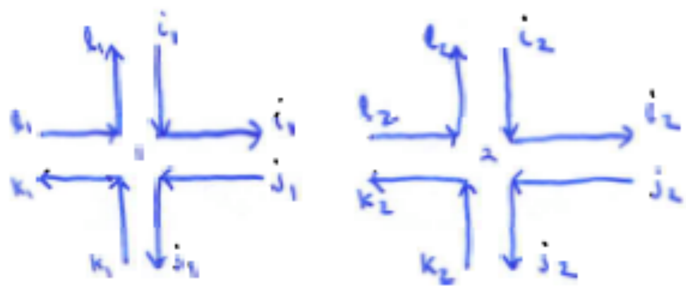
$$= \sum_{(r,s)\in\mathbf{W}(np)} N^{-\left(np/2\right)+F[(r,s)]} = N^{-\left(np/2\right)} \sum_f N^f \,\#\left\{(r,s)\in W(np) : F\left[(r,s)\right] = f\right\}$$

number of free indices in (r,s) amongst

$i_1, j_1, \ldots, \quad i_2, j_2, \ldots, \quad \ldots, \quad i_{np}, j_{np}, \ldots,$

Connection to labelled maps.

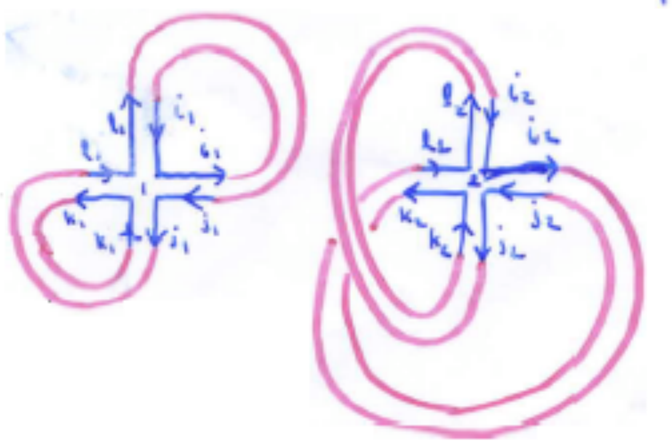
① draw n vertices with valence p.



② form ribbons by connecting fattened edges, using Wick rules $(i_1 = i_1)(j_1 = \ell_1)(k_1 = k_1)(\ell_1 = j_1)\ (i_2 = \ell_2)(j_2 = i_2)(k_2 = j_2)(\ell_2 = k_2)$

Connection to labelled maps.

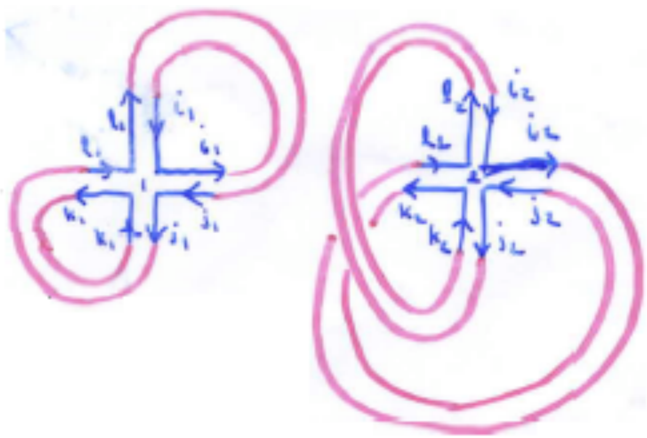
① draw n vertices with valence p.



② form ribbons by connecting fattened edges, using Wick rules $(i_1 = i_1)(j_1 = \ell_1)(k_1 = k_1)(\ell_1 = j_1)\ (i_2 = \ell_2)(j_2 = i_2)(k_2 = j_2)(\ell_2 = k_2)$

Connection to labelled maps.

① draw n vertices with valence p.



② form ribbons by connecting fattened edges, using Wick rules $(i_1 = i_1)(j_1 = \ell_1)(k_1 = k_1)(\ell_1 = k_1)\ (i_2 = j_2)(j_2 = k_2)(k_2 = \ell_2)(\ell_2 = i_2)$

These labelled "ribbon graphs" we call "diagrams."

If underlying graph is connected, the ribbon graph may be embedded in a Riemann surface, and we get a labelled map. Else embed each component separately.

For each Wick coupling (diagram) we can associate a labelled map (or a union of labelled maps).

# of faces is F(w).

# of vertices is n

# of edges is np/2

Euler characteristic tells us $2-2g = n - {}^{np}\!/_{2} + F$

We can use this to count diagrams according to # vertices, and genus, if we could evaluate these integrals. (This can be done.)

$$\frac{\partial^n}{\partial t_p^n}\hat{Z}(\vec{t})\Big|_{\vec{t}=0} = (-N)^n \,\mathbf{E}_{\vec{t}=0}\left\{ \left[\mathrm{Tr}\left(\mathrm{M}^p\right)\right]^n \right\} =$$

$$= (-1)^n \sum_{g} N^{2-2g} \;\#\left\{ w \in W(np) : F[w] = 2-2g + {}^{np}\!/_{2} - n \right\}$$

(2.) $$\frac{\partial^n}{\partial t_\varphi^n} e_g \Big|_{\vec{t}=0} = \# \left\{ W \in \mathcal{W}(n\varphi) : \begin{array}{l} F(W) = 2-2g+\frac{n\varphi}{2}-n \\ W \text{ is connected.} \end{array} \right\}$$

$$= \# \left\{ \underline{\text{diagrams}} \text{ with } \begin{array}{l} \bullet\ n \text{ vertices (valence } \varphi) \\ \bullet\ \text{genus } g \\ \bullet\ \text{connected.} \end{array} \right\}$$

(diagram: graph with *labelled vertices* $(1, 2, \ldots, n)$ and a labelling of the edges incident to each vertex $(i, j, k, \ldots)$)