APPLICATIONS OF A COMPLETE EXPANSION FOR THE PARTITION FUNCTION OF RANDOM MATRIX THEORY

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We review some recent developments in random matrix theory, and establish a moderate deviation result for traces of powers of random matrices.

1. Random Matrix Theory: Unitary Ensemble

The unitary ensemble of random matrices refers to any probability distribution on the space of $N \times N$ Hermitian matrices of the form

$$d\mu = \hat{Z}_N^{-1} \exp\{-N \text{ Tr } [V(M)]\} dM, \tag{1}$$

where dM is Lebesgue measure on the matrix entries, i.e. $dM = \prod_{j<k} dM_{jk}^{\mathrm{R}} dM_{jk}^{\mathrm{I}} \prod_{j=1}^{N} dM_{jj}$, where M_{jk}^{R} denotes the real part of the matrix entry M_{jk}, and M_{jk}^{I} denotes the imaginary component of the matrix entry M_{jk}. In (1), there are many allowable choices for the function V. The simplest one is the quadratic $V(M) = M^2/2$; in this case the matrix entries (separately real and imaginary parts for off-diagonal entries) are independent Gaussians. But this is far from the only case of interest, and indeed the measure is equally well defined, for example, if $V(x)$ is a lower semicontinuous function of $x \in \mathbb{R}$, bounded from below, and growing faster than $(\log(1+x^2))^{1+\epsilon}$ as $|x| \to \infty$, for some $\epsilon > 0$. For any such function V, the measure (1) is invariant under arbitrary unitary transformations, and this is the origin of the term "unitary ensemble". (Note, however, that the matrix entries are no longer independent.) For more details, see [14].

In what follows we will assume the function V is a polynomial of degree ν with ν even:

$$V(x) = V_{\mathbf{t}}(x) = x^2/2 + \sum_{j=1}^{\nu} t_j x^j. \tag{2}$$

The vector of coefficients $\mathbf{t}$ will live in a cone defined as follows. For any given $T > 0$ and $\gamma > 0$, set

$$\mathbb{T}(T,\gamma) = \{\mathbf{t} \in \mathbb{R}^{\nu} : \ |\mathbf{t}| \le T, \ t_{\nu} > \gamma \sum_{j=1}^{\nu-1} |t_j|\}. \tag{3}$$

It is a basic fact that the measure (1) induces a probability measure on the eigenvalues, with density

$$\frac{1}{Z_N} \exp\left\{ -N^2 \left[\frac{1}{N} \sum_{j=1}^{N} V_{\mathbf{t}}(\lambda_j) + \frac{1}{N^2} \sum_{j \neq \ell} \log|\lambda_j - \lambda_\ell| \right] \right\} d^N \lambda, \tag{4}$$

where $Z_N = Z_N(\mathbf{t})$ is the partition function:

$$Z_N(t_1, t_2, \dots, t_\nu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \exp\left\{ -N^2 \left[\frac{1}{N} \sum_{j=1}^{N} V_{\mathbf{t}}(\lambda_j) - \frac{1}{N^2} \sum_{j \neq \ell} \log|\lambda_j - \lambda_\ell| \right] \right\} d^N \lambda. \quad (5)$$

Most of the interest in random matrices from the unitary ensemble has concerned the statistical behavior of the eigenvalues, and their asymptotic behavior when N, the size of the matrices in question, grows to ∞. A great deal is known about the limiting statistical behavior of the eigenvalues, for a rather general class of functions V. This is due to the explicit induced measure on eigenvalues (4), a remarkable connection between most (if not all) statistical questions concerning the eigenvalues, and orthogonal polynomials, due to Gaudin and Mehta [11], as well as recent developments from integrable systems which permit a complete asymptotic analysis of the polynomials [4, 5, 6, 2].

Let $\{p_j(x;\ N,\ \mathbf{t})\}_{j=0}^{\infty}$ be the sequence of polynomials orthogonal with respect to the measure $\exp(-NV_{\mathbf{t}}(x))dx$. That is, $\{p_j(x;\ N,\ \mathbf{t})\}_{j=0}^{\infty}$ satisfies $\int_{-\infty}^{\infty} p_j p_k \exp(-NV) dx = \delta_{jk}$, and $p_j(x;\ N,\ \mathbf{t}) = \gamma_j^{(N)} x^j + \cdots$, $\gamma_k^{(N)} > 0$. (The leading coefficient $\gamma_k^{(N)}$ is of course dependent on the parameters $t_1, \dots, t_\nu$; however, we suppress this dependence for notational convenience.)

Investigations of the statistical properties of the eigenvalues $\{\lambda_j\}_{j=1}^N$ usually begin with the mean density of eigenvalues, defined as follows:

$$\rho_1^{(N)}(\lambda) = (d/d\lambda)\mathbb{E}\left(N^{-1}\ \#\{j\ :\ \lambda_j < \lambda\}\right), \quad (6)$$

where $\mathbb{E}(\cdot)$ denotes integration with respect to the measure (4). The connection to orthogonal polynomials is expressed by the following formula:

$$\rho_1^{(N)}(\lambda) = N^{-1} \exp(-NV(x)) \sum_{\ell=0}^{N-1} p_j(\lambda;\ N,\ \mathbf{t})^2. \quad (7)$$

There are many other remarkable formulae expressing the explicit connection between eigenvalues statistics and the orthogonal polynomials, but for our purposes, (7) will suffice.

The partition function Z_N is a generating function for moments of the eigenvalues, in the sense that by taking derivatives of Z_N with respect to any of the t_j's, one obtains moments of the fundamental random variables

$$\varphi_k = \sum_{n=1}^{N} \lambda_n^k. \quad (8)$$

Here are two examples that will prove extremely useful:

$$\partial_{t_k} \log Z_N = -N\mathbb{E}(\varphi_k), \quad (9)$$

$$\sigma_{k,N}^2 := N^{-2} \partial_{t_k}^2 \log Z_N = \mathrm{Var}(\varphi_k). \quad (10)$$

Let us define normalized random variables ϕ_k as follows:

$$\phi_k = (\varphi_k - \mathbb{E}(\varphi_k))/\sigma_{k,N}. \quad (11)$$

One may also use the partition function to express the moment generating function of ϕ_k as follows:

$$\mathbb{E}\left(e^{s\phi_k}\right) = \exp\left(\frac{s\partial_{t_k} \log Z_N}{N\sigma_{k,N}}\right) Z_N(\mathbf{t} - \tfrac{s}{N\sigma_{k,N}}\hat{\jmath}_k)/Z_N(\mathbf{t}), \quad (12)$$

where $\hat{\jmath}_k$ is the kth unit vector in $\mathbb{R}^\nu$, $\hat{\jmath}_k(i) = \delta_{ik}$. Moreover, the partition function is directly related to $\rho_1^{(N)}$, through the following formula:

$$(\partial/\partial t_j) \log Z_N = -N^2 \int_{\mathbb{R}} \lambda^j \rho_1^{(N)}(\lambda)\, d\lambda. \tag{13}$$

2. Asymptotic behavior

One of the results in [13] is a complete mathematical proof that the mean density of eigenvalues $\rho_1^{(N)}$ possesses a weak limit. (While this result was known in some form previously, [13] contains a complete proof, in great generality.) The weak limit is a known object, namely the equilibrium measure, which arises in the consideration of the following variational problem:

$$Q(\mathbf{t}) := \sup_{\mu \in \mathbb{A}} \left\{ -\int V_{\mathbf{t}}(\lambda) d\mu(\lambda) + \int\int \log|\lambda - \eta| d\mu(\lambda) d\mu(\eta) \right\}, \tag{14}$$

where $\mathbb{A}$ is the set of all positive Borel measures on the real axis, with unit mass. This variational problem has been considered extensively in the approximation theory literature (see [16]). It is well known that the supremum is achieved at a unique measure μ^* (a complete proof of this fact can be found in [16]). Moreover, for real analytic functions V, the equilibrium measure μ^* is supported on finitely many disjoint intervals, and on the interior of each interval, it has analytic density [3]. We will denote the density of the equilibrium measure by ψ, so that $d\mu^* = \psi(\lambda) d\lambda$.

The mean density $\rho_1^{(N)}$ is an analytic function for each N, and so it is natural to ask if it converges to ψ point-wise, or uniformly, and to determine the rate of convergence. This was carried out in [2], and in [4]. In [2], the authors considered the two parameter family of functions $V = t\lambda^2 + g\lambda^4$, and in [4], the authors considered the general case of real analytic functions V, with sufficient growth at ∞. In [4], the following result was stated (the proof was given in [6], see also [15]).

Theorem 1 [4]. *Suppose that V is a real analytic function, growing faster than $(\log(1+\lambda^2))^{1+\epsilon}$ for some $\epsilon > 0$. If A is a closed subset of the support of the equilibrium measure such that $\psi(\lambda)$ is strictly positive on A, then there is a positive constant C such that for all N sufficiently large*

$$\sup_{\lambda \in A} \left| \rho_1^{(N)}(\lambda) - \psi(\lambda) \right| \le C/N. \tag{15}$$

Ercolani and McLaughlin carried this analysis to higher order, under assumptions which guarantee, in particular, that the equilibrium measure is supported on a single interval. Their result is the following.

Theorem 2 [9]. *Suppose that V is a polynomial of the form (2), with coefficients satisfying (3). Then the equilibrium measure is supported on a single interval (α, β). On (α, β), the density ψ satisfies $\psi(\lambda) = \sqrt{(\lambda - \alpha)(\beta - \lambda)}\, h(\lambda)$ where h is a polynomial which is strictly positive on $\mathbb{R}$. If A is a closed subset of (α, β), then there is a complete asymptotic* *expansion of the form*

$$\rho_N^{(1)}(\lambda) = \psi(\lambda) + \frac{1}{4\pi N}\left(\frac{1}{\lambda-\beta} - \frac{1}{\lambda-\alpha}\right)\cos\left\{N\int_\lambda^\beta \psi(s)ds\right\} \tag{16}$$
$$+\frac{1}{N^2}\left[H_2(\lambda) + G_2(\lambda)\sin\left\{N\int_\lambda^\beta \psi(s)ds\right\}\right] + \cdots$$

in which $H_j(\lambda)$ and $G_j(\lambda)$ are locally analytic functions which are explicitly computable in terms $V(\lambda)$.

In [9] Ercolani and McLaughlin also obtained a complete asymptotic expansion for $\rho_1^{(N)}$ valid in neighborhoods of endpoints of the support of ψ. This was used, together with formula (13) to obtain the main result of that paper, which is a complete asymptotic expansion for $\log Z_N$. The result is the following:

Theorem 3 [9]. *There is $T > 0$ and $\gamma > 0$ so that for $\mathbf{t} \in \mathbb{T}(\mathbf{T}, \gamma)$, one has the $N \to \infty$ asymptotic expansion*

$$\log\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right) = N^2 e_0(\mathbf{t}) + e_1(\mathbf{t}) + \frac{1}{N^2}e_2(\mathbf{t}) + \cdots . \tag{17}$$

The meaning of this expansion is: if you keep terms up to order N^{-2k}, the error term is bounded by CN^{-2k-2}, where the constant C is independent of $\mathbf{t}$ for all $\mathbf{t} \in \mathbb{T}(\mathbf{T}, \gamma)$. For each j, the function $e_j(\mathbf{t})$ is an analytic function of the (complex) vector $\mathbf{t}$, in a neighborhood of $\mathbf{0}$. Moreover, the asymptotic expansion of derivatives of $\log(Z_N)$ may be calculated via term-by-term differentiation of the above series.

3. Application: Central limit theorem and moderate deviation result

Another beautiful result in [13] is a proof that finite collections of the random variables ϕ_k defined in Section 1 converge to finite collections of independent Gaussian random variables. (It is worth mentioning that in [13] similar results are stated for other values of β as well.) In this Section we will present a new proof of this result, using the asymptotic expansion for Z_N described in Section 2, and we will investigate moderate deviations of ϕ_k.

3.1. *Central Limit Theorem*

Consider the moment generating function

$$F_N(s) := \mathbb{E}\left(e^{s\phi_k}\right) \tag{18}$$

as defined in (12). It is quite straightforward to verify that $\sigma_k^2 := \lim_{N\to\infty}\sigma_{k,N}^2 = \partial_{t_k}^2 e_0(\mathbf{t})$ using Theorem 3. It is well known that $e_0(\mathbf{t}) = Q(\mathbf{t})$, with Q defined in (14) (see, for example, [13] or [3]). Straightforward calculations show that

$$\partial_{t_k}^2 e_0(\mathbf{t}) = -2\int_\alpha^\beta\int_\alpha^\beta \log|x-y|(\partial_{t_k}\psi)(x)(\partial_{t_k}\psi)(y)dxdy =: \mathbb{Q}\left((\partial_{t_k}\psi), (\partial_{t_k}\psi)\right) > 0. \tag{19}$$

The last inequality is because the quadratic form $\mathbb{Q}$ is positive definite on mean–zero functions. Similarly, one may verify that $N^{-1}\partial_{t_k}\log Z_N = N\partial_{t_k}e_0(\mathbf{t}) + O(N^{-1})$ as $N \to \infty$.

Furthermore, the uniformity of the asymptotic expansion in Theorem 3, both with regards to the domain of allowable times, as well as the term–by–term differentiability of the series, allows one to establish

$$\frac{Z_N(\mathbf{t}-\frac{s\hat{\jmath}_k}{N\sigma_{k,N}})}{Z_N(\mathbf{t})} = \exp\left\{N^2\left[e_0(\mathbf{t}-\tfrac{s\hat{\jmath}_k}{N\sigma_{k,N}})-e_0(\mathbf{t})\right]+\left[e_1(\mathbf{t}-\tfrac{s\hat{\jmath}_k}{N\sigma_{k,N}})-e_1(\mathbf{t})\right]+O(N^{-2})\right\}$$
$$= \exp\left\{-\frac{Ns}{\sigma_k}\partial_{t_k}e_0(\mathbf{t})+\frac{s^2}{2\sigma_k^2}\partial_{t_k}^2 e_0(\mathbf{t})+O(N^{-1})\right\}.$$

Now combining our various asymptotic results, we find

$$\lim_{N\to\infty} F_N(s) = e^{s^2/2}, \tag{20}$$

from which it follows that ϕ_k converges to a Gaussian random variable with mean 0 and variance 1. It is similarly possible to consider $F_N(s_1, s_2, \ldots, s_k) := \mathbb{E}\left(\exp\left(\sum_{j=1}^k s_j\phi_j\right)\right)$, and show that this converges to $\prod_{j=1}^k e^{s_j^2/2}$, which implies that $\{\phi_j\}_{j=1}^k$ converge to a collection of independent Gaussians, as desired.

3.2. *Moderate Deviation Result*

As the reader may have noticed, our proof of the central limit theorem does not make full use of Theorem 3. Indeed, returning to the asymptotic calculation of $F_N(s)$, we may compute corrections to this, to any order we want, provided we place some assumptions on the domain of allowable values of s. We have the following result.

Theorem 4. *let T and γ be from Theorem 3, and suppose that $\mathbf{t} \in \mathbb{T}(T,\gamma)$. Then for any $0<\gamma<1$ and any $a>0$ we have*

$$\lim_{N\to\infty} N^{-2\gamma}\log\mathbb{P}\left(\phi_k \ge aN^\gamma\right) = -a^2/2. \tag{21}$$

The proof of this Theorem was motivated by the proof of Cramér's Theorem as discussed in [12, pp. 5-7]. We emphasize that ϕ_k is a sum of eigenvalues, and hence is NOT a sum of independent, identically distributed random variables.

Proof. We return to the asymptotic expansion of $F_N(s)$ considered in the previous subsection, and carry these calculations out to higher order, using Theorem 3. This yields the following expansion:

$$F_N(s) = \exp\left\{s^2/2 - \frac{s^3}{6N\sigma_{k,N}^3}\left(\partial_{t_k}^3 e_0 + N^{-2}\partial_{t_k}^3 e_1 + N^{-4}\partial_{t_k}^3 e_2+\cdots\right)\right. \tag{22}$$
$$\left. + \frac{s^4}{24N^2\sigma_{k,N}^4}\left(\partial_{t_k}^4 e_0 + N^{-2}\partial_{t_k}^4 e_1 + N^{-4}\partial_{t_k}^4 e_2+\cdots\right)+\cdots\right\}.$$

The above expansion holds true provided s/N tends to 0 when $N\to\infty$. For example, the condition $|s|\le N^\gamma$ for any $0<\gamma<1$ will suffice. Now having control on the error allows us to estimate the tail of the distribution for ϕ_k, as follows:

$$\mathbb{P}\left(\phi_k \ge A\right) = \mathbb{P}\left(e^{s(\phi_k - A)} \ge 1\right) \le F_N(s)e^{-As}, \tag{23}$$

provided $s > 0$. Now we may minimize $F_N(s)e^{-As}$. Analysis of the asymptotic expansion (22) shows that for $0 < A < N^\gamma$, $0 < \gamma < 1$, $F_N(s)e^{-As}$ achieves its local minimum for some s_* with $|s_* - A| \leq N^{-(1-2\gamma)}$, and hence we can conclude that for any $0 < \gamma < 1$ and any fixed $a > 0$

$$\lim_{N\to\infty} N^{-2\gamma} \log\left[\mathbb{P}\left(\phi_k \geq aN^\gamma\right)\right] \leq -a^2/2. \tag{24}$$

To establish the lower bound, let the density function for ϕ_k be denoted $f_n(\zeta)$. We define a new random variable, X_N, with density $Q_N(x) = e^{\tau x} f_N(x + aN^\gamma)/(e^{-aN^\gamma \tau} F_N(\tau))$, where τ is chosen to be the unique value of s minimizing $e^{-aN^\gamma s} F_N(s)$. By observing that $\mathbb{E}(e^{sX_N}) = e^{-(s+\tau)aN^\gamma} F_N(s+\tau)/(e^{-\tau a N^\gamma} F_N(\tau)) \to e^{s^2/2}$ (as $N \to \infty$), we know that X_N converges to a normalized Gaussian as $N \to \infty$.

Now we have

$$\begin{aligned}\mathbb{P}\left(\phi_k \geq aN^\gamma\right) &= \int_{x \geq aN^\gamma} f_N(x)dx = e^{-a\tau N^\gamma} F_N(\tau) \int_{x\geq 0} e^{-\tau x} Q_N(x) dx \\ &\geq e^{-a\tau N^\gamma} F_N(\tau) \int_0^C e^{-\tau x} Q_N(x)dx \geq e^{-a\tau N^\gamma} F_N(\tau) e^{-\tau C}\, \mathbb{P}\left(X_N \in (0, C)\right).\end{aligned} \tag{25}$$

Now since X_N converges to a normalized Gaussian, we may chose C large enough (but fixed) so that $\mathbb{P}\left(X_N \in (0, C)\right) \geq 1/2$. Now we may take logs, divide by $N^{2\gamma}$, and let $N \to \infty$, to obtain

$$\lim_{N\to\infty} N^{-2\gamma} \log\left[\mathbb{P}\left(\phi_k \geq aN^\gamma\right)\right] \geq -a^2/2, \tag{26}$$

completing the proof of Theorem 4. □

4. Discussion

Were ϕ_k a standardized sum of independent, identically distributed random variables, then the correct scaling for a large deviation result would be $N^{1/2}$. In reality, Theorem 4 demonstrates that we can go much further into the tail of the distribution (N^γ for any $0 < \gamma < 1$), and still observe Gaussian behavior.

To really investigate the tail of the distribution, we should consider deviations of order N, rather than N^γ for $0 < \gamma < 1$. Such considerations will be undertaken in a later publication, the result of which will be as follows.

$$\lim_{N\to\infty} N^{-2} \log\left(\mathbb{P}\left(\phi_k \geq aN\right)\right) = I(a), \tag{27}$$

where

$$I(a) = \min_{x:\ \mathbf{t} - \frac{x}{\sigma}\hat{\jmath}_k \in \mathbb{T}(T,\gamma)} \left[e_0(\mathbf{t} - \frac{x}{\sigma}\hat{\jmath}_k) - e_0(\mathbf{t}) + \frac{x}{\sigma}\partial_{t_k} e_0(\mathbf{t}) - ax\right]. \tag{28}$$

The function appearing in the square brackets above is certainly convex for x in its domain of definition. It can be shown that there is $\tilde{a} > 0$ sufficiently small so that if $a \in (0, \tilde{a}]$, then the minimum in (28) is achieved at a value of x such that $\mathbf{t} - (x/\sigma)\hat{\jmath}_k \in \mathbb{T}(T, \gamma)$, and (27) holds for $a \in (0, \tilde{a}]$.

A rather different type of question has to do with corrections to the central limit theorem "in the bulk". In the case of sums of independent, identically distributed random variables, such results are known as Edgeworth expansions, or Berry-Eséen theorems (see, for example, [10]).

As one can see from the asymptotic expansion (22), a better approximation to the moment generating function $F_N(s)$ is

$$\tilde{F}_N(s) = e^{\frac{1}{2}s^2 - \frac{\partial_{t_k}^3 e_0(\mathbf{t})}{6N\sigma_k^3}s^3}. \tag{29}$$

Now $\tilde{F}_N(s)$ is the Laplace transform of a rather nice function: $\tilde{F}_N(s) = \int_{\mathbb{R}} e^{sx}\tilde{f}_N(x)dx$, where

$$\tilde{f}_N(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-z^2/2 + \frac{i\partial_{t_k}^3 e_0(\mathbf{t})}{6N\sigma_k^3}z^3 + izx}dz. \tag{30}$$

This is a classical special function, it can be directly related to the Airy integral. It is expected that the asymptotic expansion for the moment generating function of ϕ_k could yield a sequence of approximations to $f_N(x)$ for N large, which improve on the leading order approximation $(2\pi)^{-1/2}e^{-x^2/2}$ afforded by the central limit type result discussed above. However, obtaining f_N from F_N requires knowledge of $F_N(s)$ (and its asymptotic expansion) along the imaginary s-axis. Thus another interesting direction of research involves asymptotic analysis of the partition function (and hence the orthogonal polynomials) for values of $\mathbf{t}$ in the complex domain.

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