Pedro, Dan, Andre, Marcus, Carlos, and Jorge:
Thanks!!!

Asymptotic Analysis of nonlinear Schroedinger equation

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Initial Value Problem for the nonlinear Schroedinger equation

$$i\phi_t + \phi_{xx} - 2\left|\phi\right|^2 \phi = 0,$$
$$\phi(x,0) = q(x).$$

Goal: provide a complete description of solutions in asymptotic regimes.

Question 1: Does the equation regularize jump discontinuities?

How does it do so?

Simple example: Linear equation

$$i\phi_t + \phi_{xx} = 0$$

$$\phi(x,0) = h(x)$$

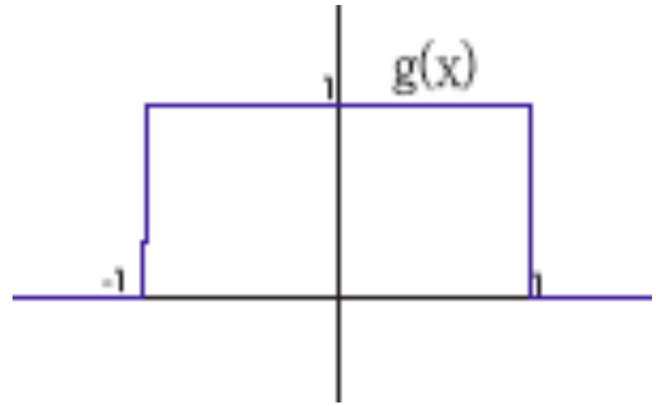
Solution:

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}} \int \widehat{h}(y) e^{-ity^2 + iyx} dy$$

Question 1 can be answered completely!

Gibbs phenomenon

$$i\phi_t + \phi_{xx} = 0$$
$$\phi(x,0) = g(x)$$



Take $t > 0$, and consider

$$\phi(x,t) \approx ?? \quad \text{as} \quad t \downarrow 0 ??$$

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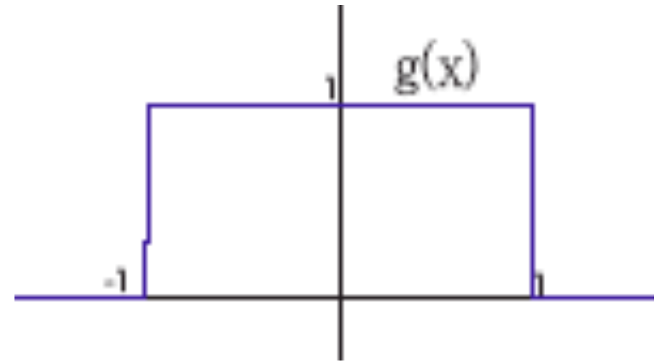

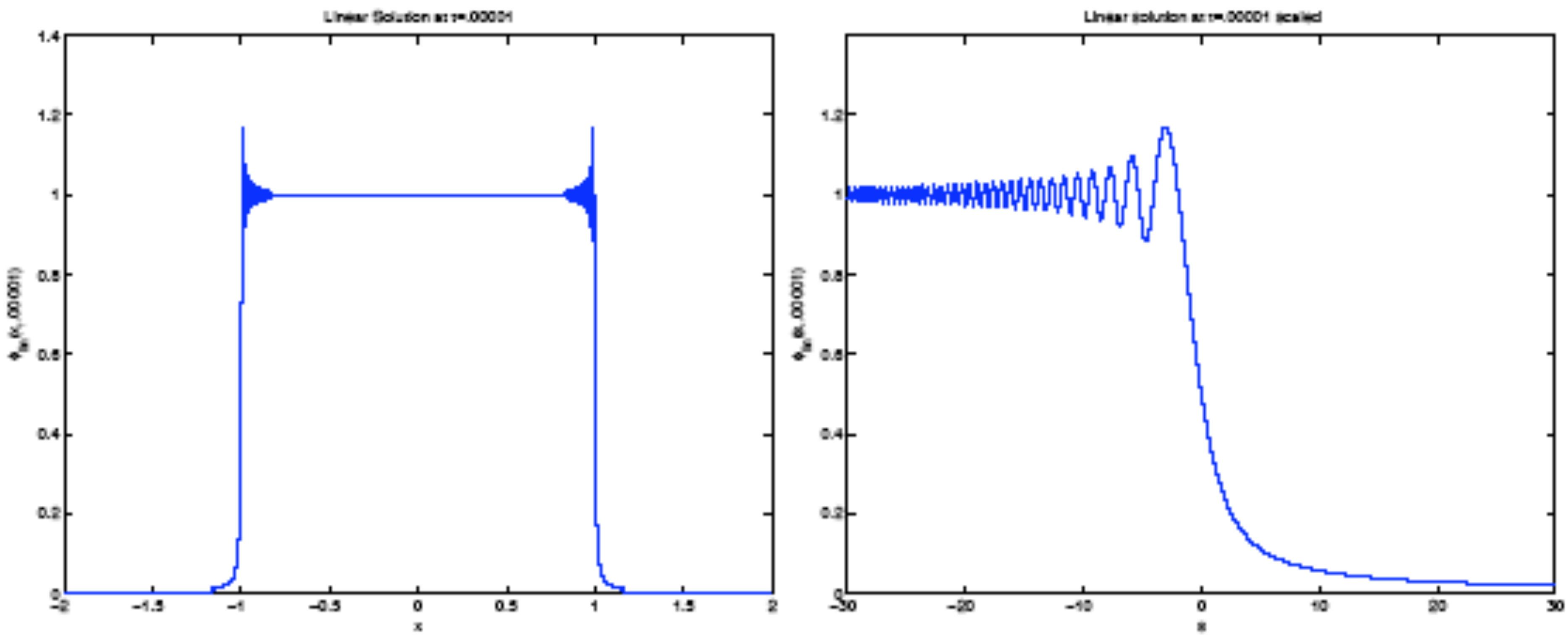

FIGURE 4. ϕ_{lin} at $t = .00001$

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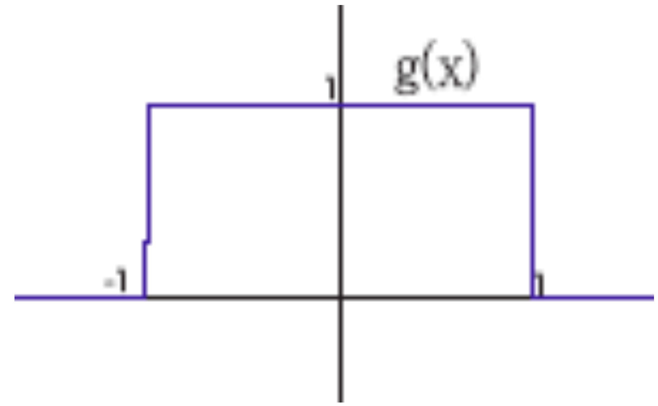


$$\phi_{\text{lin}}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(y)e^{-ity^2+ixy}}{y} dy.$$

Now let $t \downarrow 0$: straightforward calculations show

$$\phi(x,t)_{lin} = \frac{1}{2} - \frac{1}{2}Erf\left[\frac{se^{\frac{-i\pi}{4}}}{2}\right] + \mathcal{O}(t^{\frac{1}{2}})$$

$$s = (x-1)t^{-\frac{1}{2}}$$

$$Erf[s] := \frac{2}{\sqrt{\pi}}\int_0^s e^{-x^2} dx$$

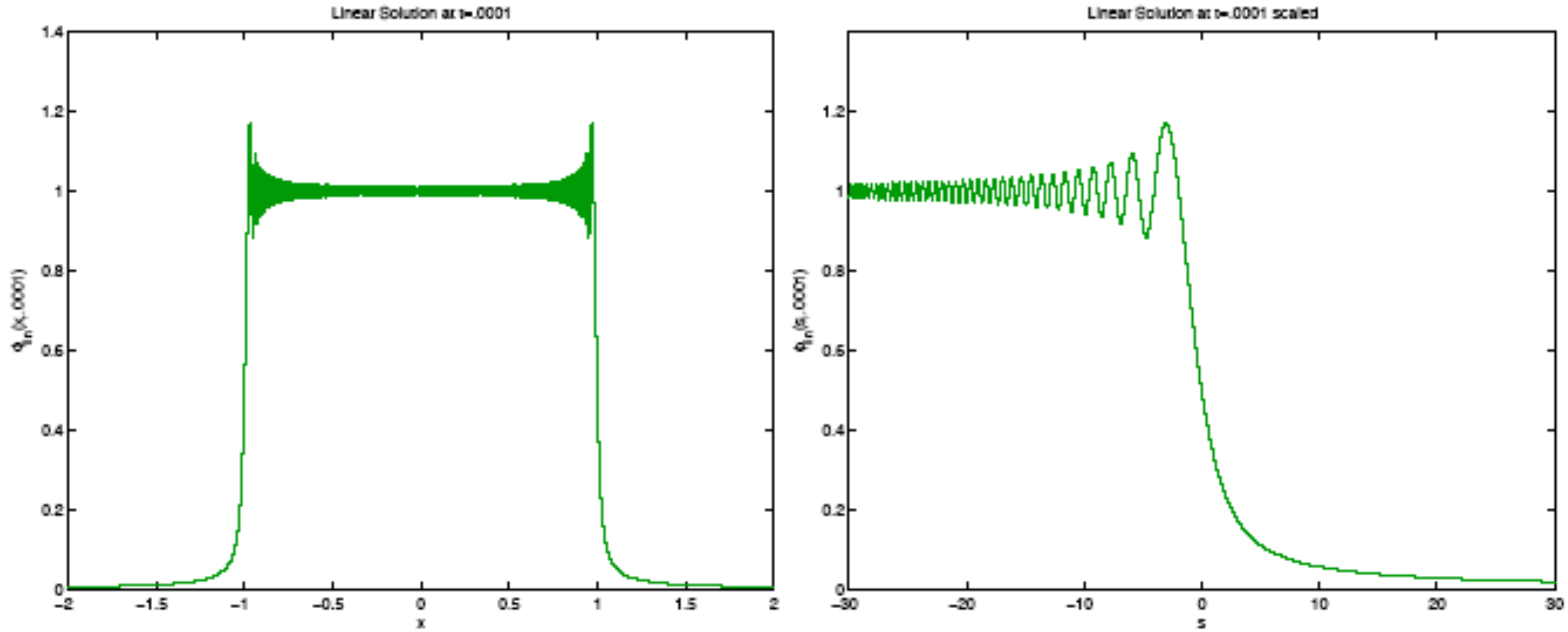
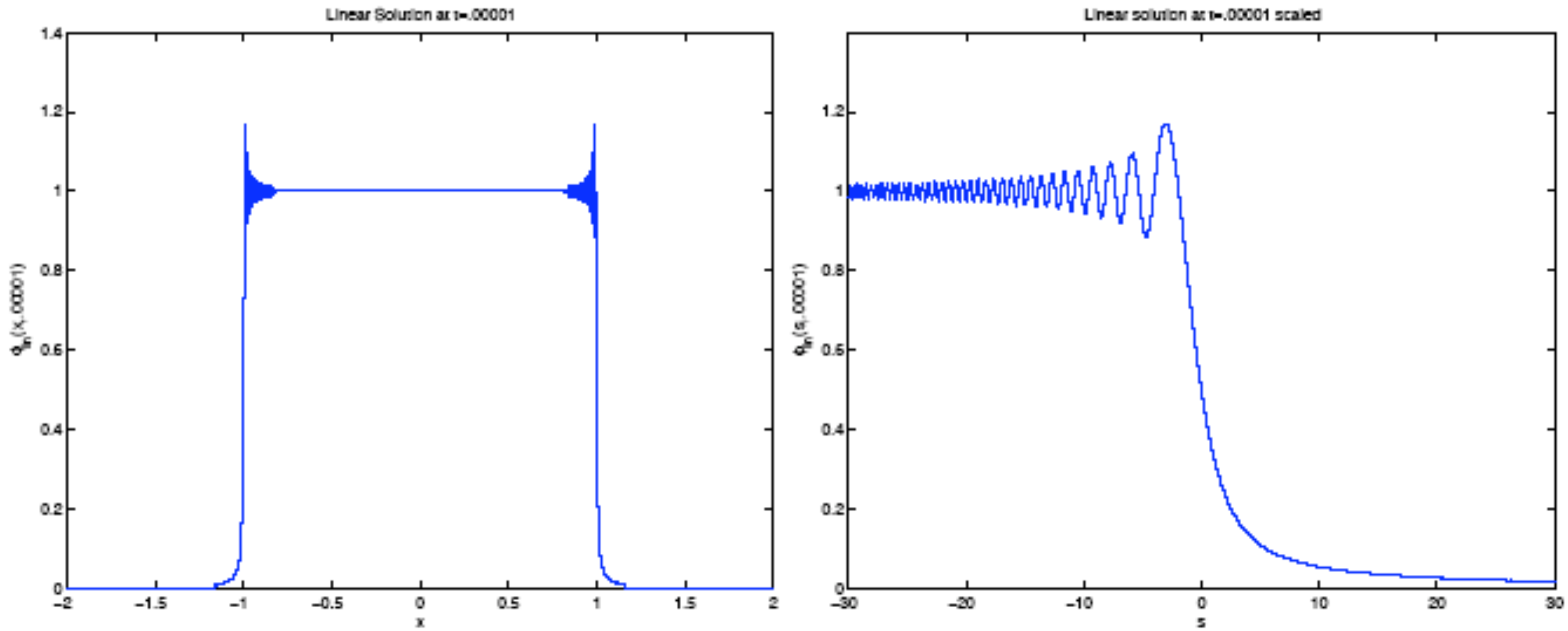


FIGURE 3. ϕ_{lin} at $t = .0001$



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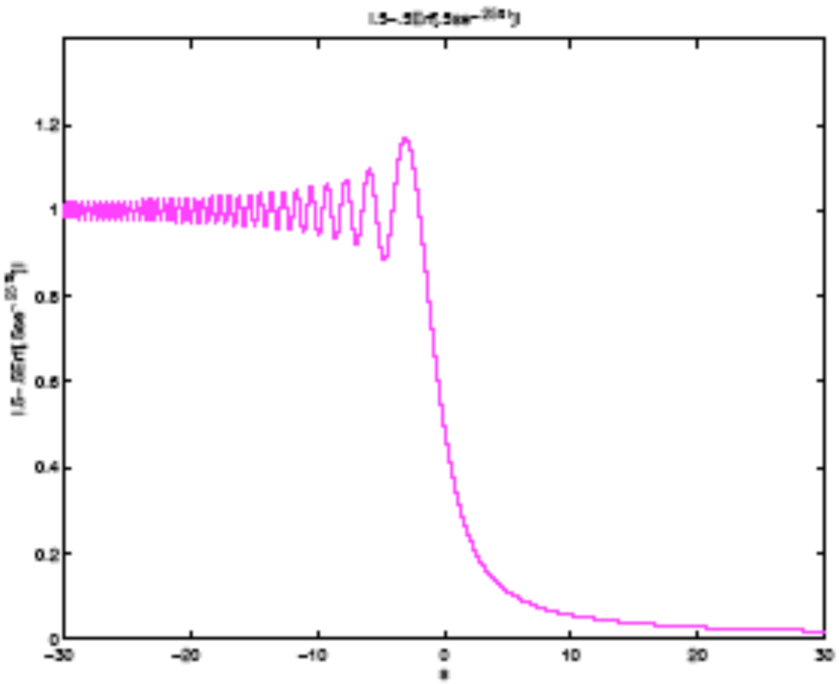


FIGURE 5. $\left|\frac{1}{2} - \frac{1}{2} Erf[\frac{se^{-\frac{i\pi}{4}}}{2}]\right|$

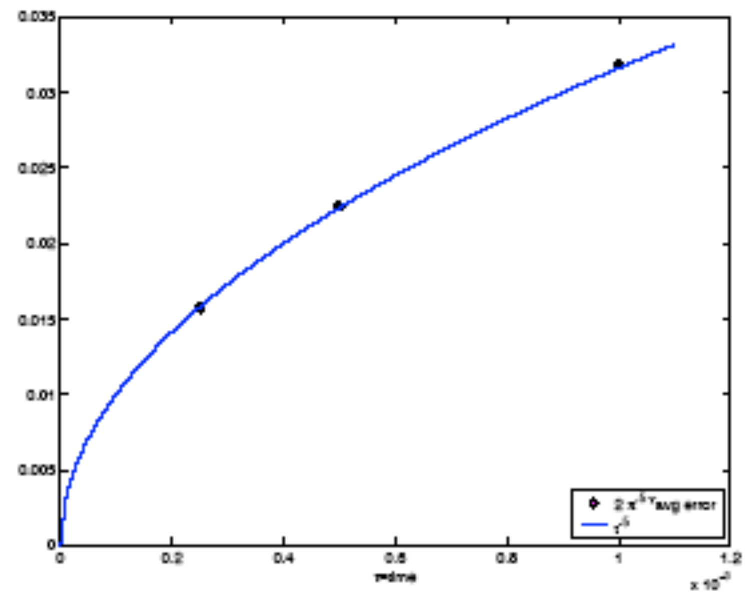
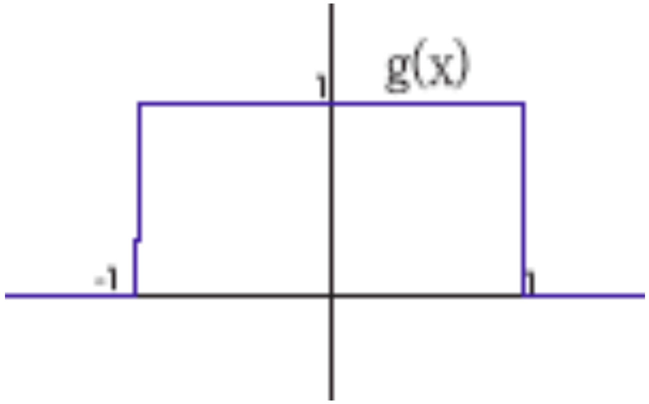


FIGURE 6. Linear Error

So…what about the nonlinear case?

$$i\phi_t + \phi_{xx} - 2\left|\phi\right|^2 \phi = 0,$$
$$\phi(x,0) = g(x)$$



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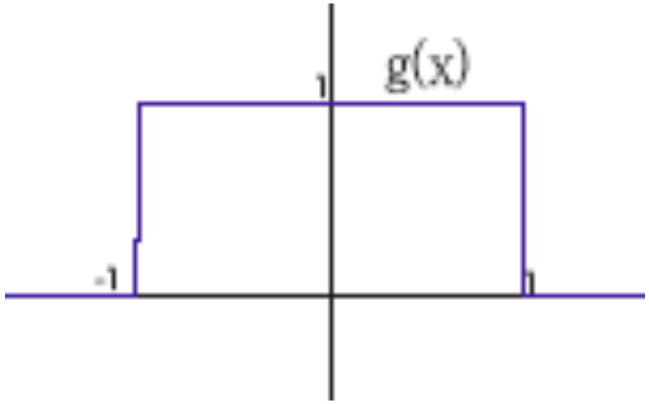

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YES! Given $q_o(x) \in L^2(\mathbb{R})$

$$\exists!\, \phi(x,t) \in C(\mathbb{R}_+, L^2(dx)) \cap (L^\infty(dx) \otimes L^4_{loc}(dt))$$

(Ginibre & Velo, ‘85, Cassenave & Weissler - ‘90, Chang, Shatah, Uhlenbeck, ‘00)

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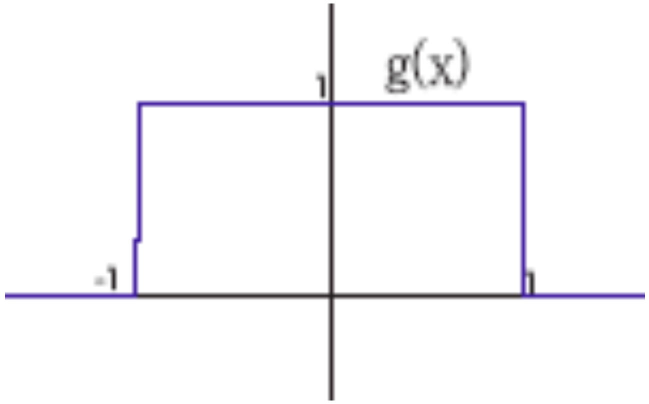

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Good! Now for t>0, can we describe $\phi(x,t)$ as $t \downarrow 0$?

Theorem (Jeff DiFranco, KM): There is K>0 s.t. for 0 < t < K, the following asymptotic expansion holds true for all x:

$$\phi(x,t) = \phi_{lin}(x,t) + \mathcal{O}(t).$$

$$\left(\phi(x,t)_{lin} = \frac{1}{2} - \frac{1}{2} Erf\left[\frac{se^{\frac{-i\pi}{4}}}{2}\right] + \mathcal{O}(t^{\frac{1}{2}}) \right)$$

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A complete asymptotic expansion is achievable.

For example, for $s = \frac{x-1}{\sqrt{t}} > -\#$:

$$\phi(x,t) = \phi_{lin}(x,t) + F(x,t) + O\left(t^{3/2}\right)$$

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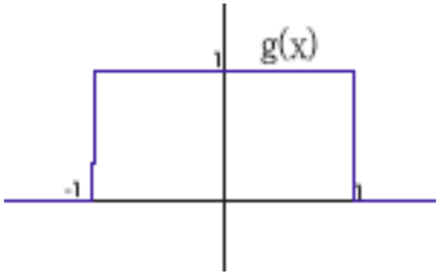
$$F(x,t) = -\frac{it}{\pi} \int_{\gamma_1 \cup \gamma_3} \frac{e^{-4i\lambda^2 - 2is\lambda}}{8\lambda^3} d\lambda$$

$$+ \frac{2it}{(2i\pi)^3} \int_{\gamma_1 \cup \gamma_2} \frac{e^{-4i\zeta^2 - 2is\zeta}}{2i\zeta} \int_{\gamma_1 \cup \gamma_2} \frac{e^{4i\omega^2 + 2is\omega}}{2i\omega(\omega-\zeta)_-} d\omega \left(\int_{\gamma_1 \cup \gamma_3} \frac{e^{-4i\rho^2 - 2is\rho}}{2i\rho(\rho-\zeta)_+} + \frac{e^{-4i\rho^2 - 2is\rho}}{2i\rho(\rho-\zeta)_-} d\rho \right) d\zeta$$

MAGIC! Scattering and Inverse Scattering!



$$\longmapsto \quad r_0(z) = \frac{e^{-2iz}\left[e^{-2i\left(\sqrt{z^2-1}\right)_-} - e^{2i\left(\sqrt{z^2-1}\right)_-}\right]}{\left(-iz - i\left(\sqrt{z^2-1}\right)_-\right)e^{-2i\left(\sqrt{z^2-1}\right)_-} + \left(iz - i\left(\sqrt{z^2-1}\right)_-\right)e^{2i\left(\sqrt{z^2-1}\right)_-}}$$

There is a nonlinear (and invertible!) mapping from spaces of initial data to “scattering data”. (More later (maybe)).

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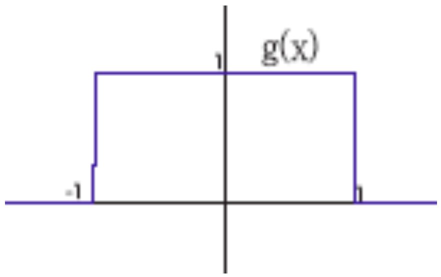

$$\mapsto \quad r(z,t) = \frac{e^{4iz^2t}e^{-2iz}\left[e^{-2i\left(\sqrt{z^2-1}\right)_-} - e^{2i\left(\sqrt{z^2-1}\right)_-}\right]}{\left(-iz - i\left(\sqrt{z^2-1}\right)_-\right)e^{-2i\left(\sqrt{z^2-1}\right)_-} + \left(iz - i\left(\sqrt{z^2-1}\right)_-\right)e^{2i\left(\sqrt{z^2-1}\right)_-}}$$

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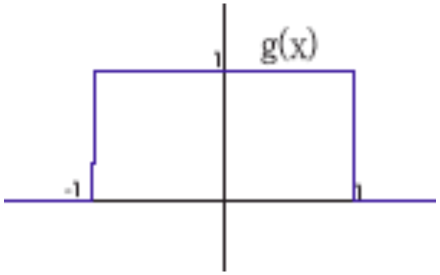

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Given $r_0(z)$, define $r_{x,t}(z) = e^{4itz^2 + 2izx} r_0(z)$

Now solve the following Riemann-Hilbert problem:

Problem: Find 2x2 matrix M satisfying

$$M \text{ analytic for } z \notin \mathbb{R}$$

$$M_+(z,x,t) = M_-(z,x,t)V(z,x,t) \quad z \in \mathbb{R}$$

$$M = I + \mathcal{O}\left(\tfrac{1}{z}\right) \text{ as } z \to \infty$$

$$V(z,x,t) = \begin{pmatrix} 1 - |r(z)|^2 & -\overline{r(z)} \\ r(z) & 1 \end{pmatrix}$$

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And then…

$$\phi(x,t) = 2i \lim_{z\to\infty} \{z(M(z,x,t))_{12}\} = 2iM_{12}^{(1)}(x,t)$$
$$M(z,x,t) = I + \frac{1}{z}M^{(1)}(x,t) + o(z^{-1})$$

So if we had some way to unravel the asymptotic behavior of M(z,x,t) for t small, we would obtain asymptotics for $\phi(x,t)$.

But there is advanced machinery to do just this!!
Jeff DiFranco found

$\exists$ an explicit invertible transformation $M(z,x,t) \mapsto K(\lambda,x,t)$
so that $K(\lambda,x,t)$ solves

$$K \text{ analytic for } \lambda \notin \Gamma^*$$
$$K_+(\lambda,x,t) = K_-(\lambda,s,t)V_K(\lambda,x,t) \quad \lambda \in \Gamma^*$$
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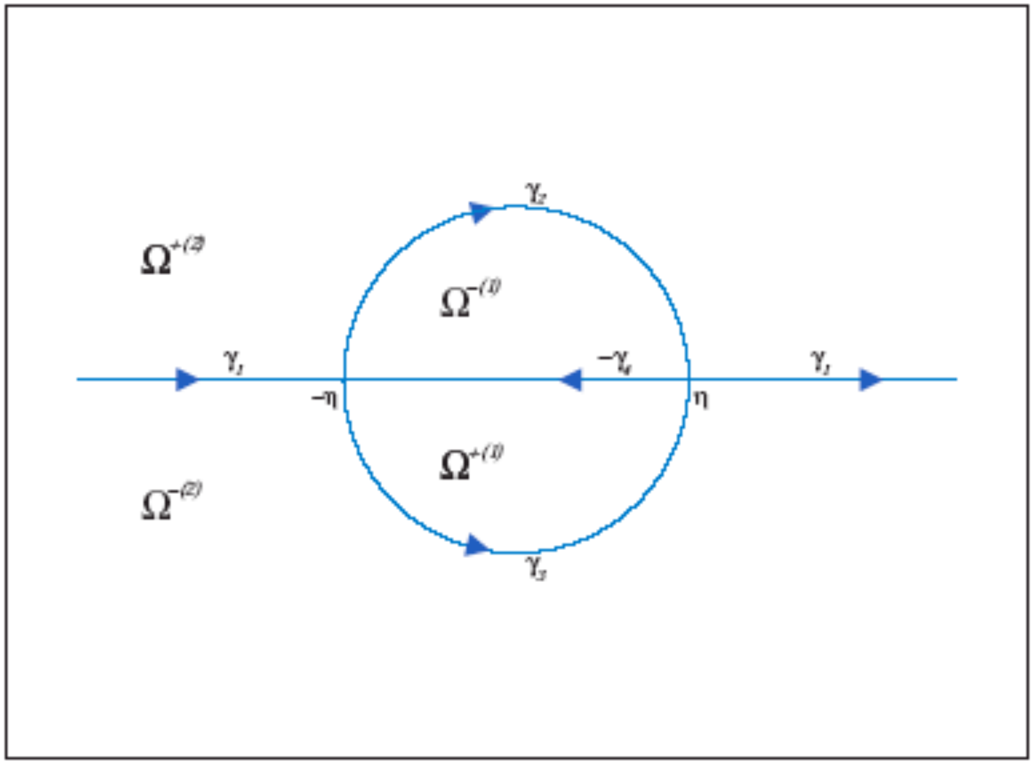

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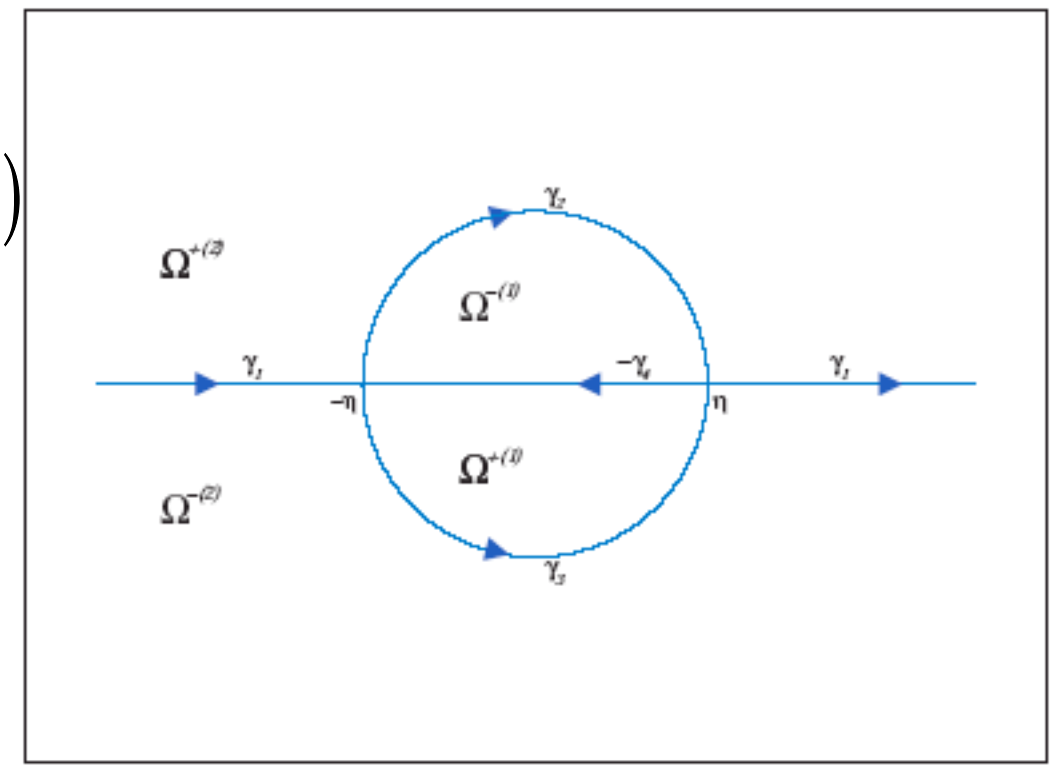
And everywhere on the contour,

$V_K(\lambda,x,t) - I$ **is small in** $Lip^{1/3}\left(\Gamma^*\right)$

$$\left\|V_K - I\right\|_{1/3} \le Ct^{1/4}$$



Conclusion: there is a unique solution, with explicit asymptotic expansion!
This is because of Macumba Number 2:

Riemann-Hilbert problems are equivalent to a system of singular integral equations whose invertibility is dictated by the size of $V_K(\lambda, x, t) - I$

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OK Hold on! That's 2 pieces of magic! The first one was

1. There is a nonlinear (and invertible!) mapping from spaces of initial data to "scattering data". And as the solution evolves According to the nonlinear Schroedinger equation, the scattering data just rolls around!

$$r_{x,t}(z) = e^{4itz^2 + 2izx} r_0(z)$$

Integrability for NLS. Freeze time, and consider

$$L\Psi = z\Psi,$$

$$L = i\sigma_3\partial_x - \begin{pmatrix} 0 & i\phi \\ -i\overline{\phi} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\phi(x,t) \in L^1(dx)$ and $z \in \mathbb{C}\backslash\mathbb{R}$, there is a unique normalized eigenfunction

$$\begin{cases} L\Psi = z\Psi \\ M \equiv \Psi e^{ixz\sigma_3} \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ as } x \to +\infty \\ M(x,z) \text{ is bounded as } x \to -\infty \end{cases}$$

(Beals, Coifmann, ‘84, Beals, Deift, Tomei, ‘88)

Picard iteration shows in addition that $\Psi(x,t,z)$ is analytic in z, and has boundary values for z real.

$$\Psi_{\pm}(x,t,z) := \lim_{\varepsilon \downarrow 0} \Psi(x,t,z \pm i\varepsilon)$$

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$$\Psi_{\pm}(x,t,z) := \lim_{\varepsilon \downarrow 0} \Psi(x,t,z \pm i\varepsilon)$$

and both solve the same equation $L\Psi = z\Psi$, so

$$\Psi_{+}(x,t,z) = \Psi_{-}(x,z,t)V(z;t)$$

Algebra now shows that $V(z,t) = \begin{pmatrix} 1-|r(z,t)|^2 & -\overline{r(z,t)} \\ r(z,t) & 1 \end{pmatrix}$, Picard iteration shows in addition that $\Psi(x, t, z)$ is analytic in z, and has boundary values for z real.

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The fact that $\phi(x,t)$ solves the NLS equation implies $r(z,t) = r_0(z)e^{4itz^2}$

Analysis then shows that the reflection coefficient enjoys regularity and decay properties depending on regularity and decay of the initial data.

Riemann-Hilbert problem for $M \equiv \Psi e^{ixz\sigma_3}$

$$M \text{ analytic for } z \notin \mathbb{R}$$

$$M_+(z,x,t) = M_-(z,x,t)V(z;x,t) \quad z \in \mathbb{R}$$

$$M \to I \text{ as } z \to \infty$$

Fundamental fact: There is a unique solution to this Riemann-Hilbert problem! (Beals, Deift, Tomei '88)

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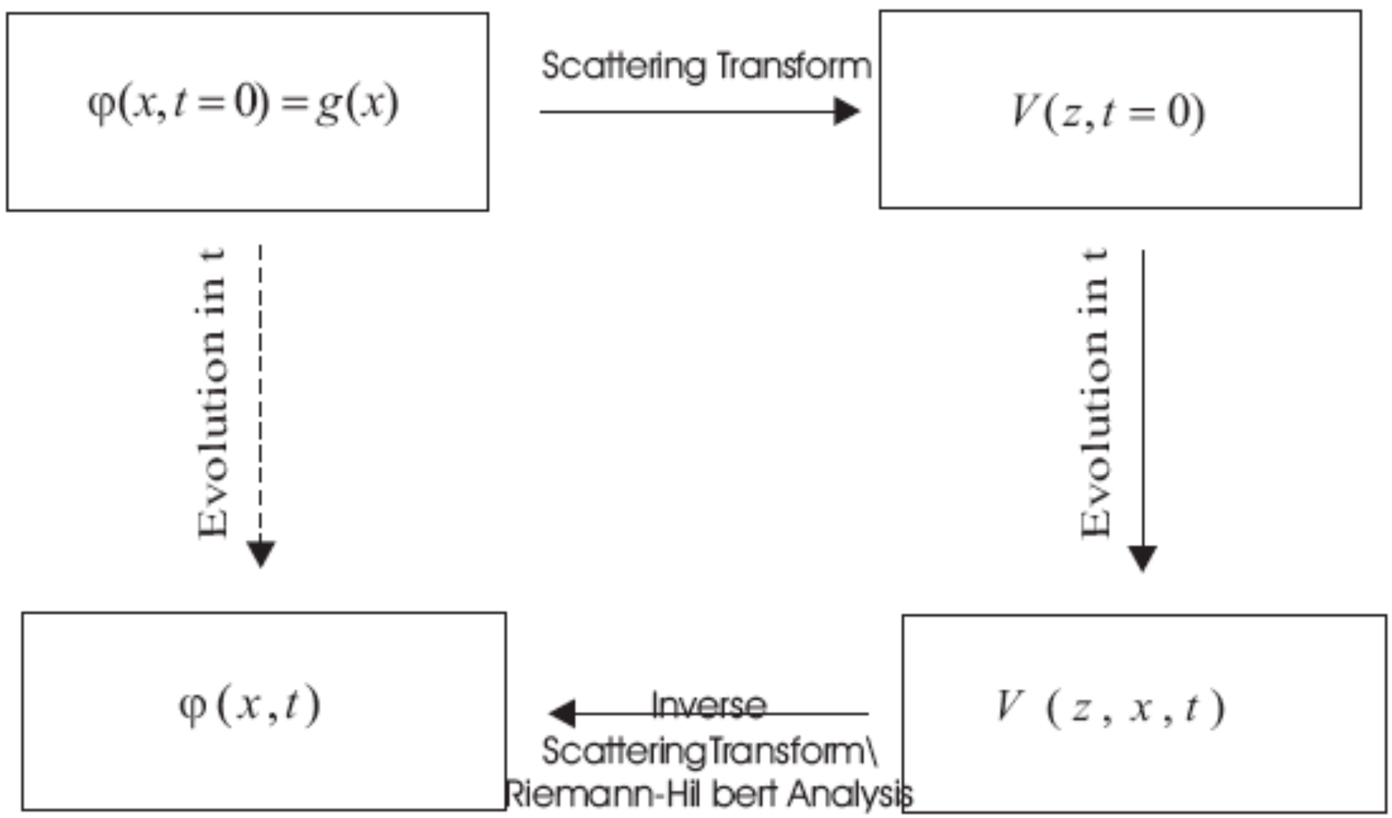

Riemann-Hilbert problems and singular integral equations

Given a contour Γ and a function f, define

$$f \mapsto C_\Gamma(f)(z) = \frac{1}{2i\pi}\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$C_\Gamma^\pm(f)(z) = \lim_{z' \to z, z' \in \Omega^\pm} C_\Gamma(f)(z')$$

$$C_{V,\Gamma}(f) = C_\Gamma^-\left(f\left(V - I\right)\right)$$

Solve this equation!!!

$$\left(1 - C_{V,\Gamma}\right)\gamma = C_{V,\Gamma}(I)$$

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Then…

$$M(z) = I + C_{\Gamma}\Big(\big(I + \gamma\big)\big(V - I\big)\Big)(z)$$

Solves the Riemann-Hilbert problem

It makes some sense that the size of V-I controls norm of the integral operator

$$\left\|C_{V,\Gamma}(f)\right\| \le K\left\|f\left(V - I\right)\right\| \le K\left\|f\right\|\left\|V - I\right\|$$

Question 2:

$$i\varepsilon\psi_t + \frac{\varepsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0,$$

$$\psi(x,0) = \psi_0(x).$$

Simple example: Linear equation

$$i\varepsilon\psi_t + \frac{\varepsilon^2}{2}\psi_{xx} = 0,$$
$$\psi(x,0) = \psi_0(x).$$

Solution:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int \hat{\psi}_0(y) e^{-i\frac{t}{2\varepsilon}y^2 + i\frac{x}{\varepsilon}y} dy$$

Question 2 can be answered completely!