Math 129 – Exam 1 – Summer 2013 – Key

Print Name:

Signature:

(1a) **10 points** Using integration by parts, show every step of your calculation to find the antiderivative

$$\int z^2 \ln(z)\, dz$$

Hint: the answer is $\frac{1}{3}z^3\ln(z) - \frac{1}{9}z^3 + c$.

$u = \ln z \quad v' = z^2$

$u' = \frac{1}{z} \quad v = \frac{1}{3}z^3$

$$= \frac{1}{3}z^3 \ln z - \int \left(\frac{1}{z}\right)\left(\frac{1}{3}z^3\right) dz$$

$$= \frac{1}{3}z^3 \ln z - \frac{1}{3}\int z^2\, dz$$

$$= \frac{1}{3}z^3 \ln z - \frac{1}{9}z^3 + c$$

(1b) **10 points** If we have a function $f(x)$ for which we know that $\int_1^8 f(x)dx = 24$, find the value of

$$\int_0^{\ln(2)} e^{3t} f\left(e^{3t}\right) dt\,.$$

Hint: Try a substitution.

$w = e^{3t}$

$dw = 3e^{3t}\, dt$

$$= \frac{1}{3}\int_1^8 f(w)\, dw = \frac{24}{3} = 8$$

$w(t{=}0) = e^0 = 1$

$w(t{=}\ln 2) = e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8.$

(2) **10 points** Evaluate the integral. Be sure to show enough work so that I can see what you did!

$$\int \frac{y}{\sqrt{1+y^2}}dy = \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw$$

$w = 1+y^2$

$dw = 2y\,dy$

$$= \frac{1}{2}\,\frac{w^{1/2}}{1/2} + C$$

$$= w^{1/2} + C$$

$$= \sqrt{1+y^2} + C$$

(3) **15 points** A scientist drops his cellphone into a deep hole. The following table describes the velocity $V(t)$ at various times.

t (seconds)	0	250	500	750	1000
$V(t)$ (m/sec)	0	-5	-10	-20	-50

The distance traveled by the phone in 1000 seconds is $\int_0^{1000} V(t)dt$. Use the trapezoidal rule to estimate the distance travelled.

LEFT: $-250(0+5+10+20)$

RIGHT $-250(5+10+20+50)$

Trap: $\frac{-250}{2}(0+10+20+40+50)$

$= -\frac{250}{2}(120) = -250 \times 60$

$= -15{,}000$

phone traveled 15,000 meters down the hole

(4) **10 points** Use the trigonometric substitution $y = 2\sin(\theta)$ to rewrite the integral. **You do not need to evaluate the integral, but you should simplify the integrand.**

$$\int \frac{1}{y^2\sqrt{4-y^2}}\, dy.$$

$y = 2\sin\theta$

$dy = 2\cos\theta\, d\theta$

$$= \int \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int \frac{2\cos\theta}{4\sin^2\theta \cdot 2 \cdot \sqrt{1-\sin^2\theta}}\, d\theta = \int \frac{2\cos\theta}{8\sin^2\theta\sqrt{\cos^2\theta}}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$$

(5) **15 points** Use partial fractions to evaluate the following integral.

$$\int \frac{1}{(t^2-5t+6)}\, dt.$$

$$\frac{1}{t^2-5t+6} = \frac{1}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$$

some algebra shows that $A = 1$ and $B = -1$.

$$\int \frac{1}{t^2-5t+6}\, dt = \int \frac{1}{t-3} - \frac{1}{t-2}\, dt = \ln|t-3| + \ln|t-2| + C$$

(6) **15 points** Given the following formula from the Integral Tables,

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C\,,$$

evaluate the following integral:

$$\int \frac{1}{\sqrt{t^2 - 2t + 5}} dt\,.$$

complete the square $= \int \frac{1}{\sqrt{(t-1)^2 + 4}} dt$

$x = t-1$
$dx = dt$

$$= \int \frac{1}{\sqrt{x^2+4}} dx = \ln \left| x + \sqrt{x^2+4} \right| + C$$

$$= \ln \left| t-1 + \sqrt{(t-1)^2+4} \right| + C$$

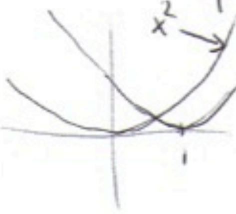
(7) **7 points** Does the following improper integral converge or diverge? You do not need to evaluate the integral, but you do need to justify your answer with intuition or an inequality.

$$\int_1^\infty \frac{(x-1)^2}{x^4 + x^2} dx$$

The guiding behavior: $\frac{x^2}{x^4} = \frac{1}{x^2}$ $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

So the intuition is that $\int_1^\infty \frac{(x-1)^2}{x^4+x^2} dx$ is convergent.

an inequality: $(x-1)^2 \le x^2$ if $x \geq 1$



$$\frac{1}{x^4+x^2} \le \frac{1}{x^4}\,. \text{ so, } \frac{(x-1)^2}{x^4+x^2} \le \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Therefore the integral in question is convergent.

(8) **8 points** Evaluate the following definite integral. Note: if the integral diverges, explain whether it diverges to $+\infty$ or $-\infty$.

$$\int_1^5 (x-1)^{-3/2}dx$$

Bad trouble! $x=1$.

$$\lim_{a\to 1^+}\int_a^5 (x-1)^{-3/2}dx = \lim_{a\to 1^+} \left.\frac{(x-1)^{-1/2}}{-1/2}\right|_a^5$$

$$= \lim_{a\to 1^+} \left.-2(x-1)^{-1/2}\right|_a^5 = \lim_{a\to 1^+}\left(-2(4)^{-1/2} + 2(a-1)^{-1/2}\right)$$

$$= \lim_{a\to 1^+}\left(\frac{-2}{2} + 2(a-1)^{-1/2}\right)$$

$$= \lim_{a\to 1^+}\left(-1 + \frac{2}{\sqrt{a-1}}\right) = +\infty$$

So the integral diverges to $+\infty$.