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Exam 1, Math 254, Summer 2013

15 July 2013

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INSTRUCTIONS:

- This is a closed book, closed notes exam.
- You are not to give or receive help from any outside source during the exam.
- You have **1 hour**.

1. (20) Use separation of variables to solve the differential equation $y^3 \frac{dy}{dt} = t(y^4+1)$. Your answer should contain an arbitrary constant.

$$\int \frac{y^3}{y^4+1}\,dy = \int t\,dt \Rightarrow \frac{1}{4}\ln(y^4+1) = \frac{1}{2}t^2 + C$$

$$\ln(y^4+1) = 2t^2 + \tilde{C}, \text{ so } y^4+1 = C_1 e^{2t^2}$$

$$y^4 = C_1 e^{2t^2} - 1, \quad y = \pm\left(C_1 e^{2t^2} - 1\right)^{1/4}$$ ← This was sufficient.

or,

$$y = \left(C_1 e^{2t^2} - 1\right)^{1/4} e^{\frac{2\pi i n}{4}}, \quad n = 0, 1, 2, 3.$$

2. (10) Determine whether the function $u = \sin t + t^2$ is a solution to the differential equation $\frac{d^2u}{dt^2} + u = t^2 + 2$. (Show your work, of course.)

Yes, it is. (I won't show the work here, but everyone knows what to do.

5. (10) **For this question you should only FIND the differential equation. You do not need to solve it.** A nitric acid solution flows at a constant rate of 6 L/min into a large tank holding 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 6 L/min. If the solution entering the tank is 20% nitric acid, **determine the equation** representing $C(t)$, the nitric acid concentration of the tank, measured in kg/L.

$200 \times \frac{dC}{dt}$ = rate of change of amt. of Nitric acid.

20%

$6\left(\frac{L}{min}\right) \cdot \frac{20}{100}$: rate at which acid flows in.

$6\left(\frac{L}{min}\right) \cdot C(t)$ rate at which acid flows out

$$200\, C'(t) = 1.2 - 6C$$

$$C' = \frac{1.2 - 6C}{200}$$

Bonus (3 points): What would the differential equation be if the 20% solution flows in at the constant rate of 6 L/min, but the rate of flow out of the tank is 8 L/min?

$$(200-2t)\, C' - 2C = 1.2 - 8C$$

3. (20) Find the integrating factor, $\mu(t)$, for the following differential equation. (Note that you do not need to find the solution of the differential equation.)

$$\frac{dv}{dt} - \frac{3}{t}v = \sin(t).$$

$$\mu v' + \underbrace{\left(-\frac{3}{t}\right)\mu}_{\mu'} V = \mu \sin(t)$$

$$\mu' = -\frac{3}{t}\mu \leftarrow \text{solve this: } \int \frac{d\mu}{\mu} = \int -\frac{3}{t}\,dt$$

$$\ln \mu = -3\ln t \quad (c=0)$$

$$\mu = \frac{1}{t^3}$$

4. (20) Find an implicit solution to the following differential equation. Your answer should contain one arbitrary constant. Hint: This differential equation is exact.

$$(x-2y)\frac{dy}{dx} + (y-2x) = 0.$$

find F: $F_x = y - 2x$ and $F_y = x - 2y$

$F = xy - x^2 + C(y)$, so $F_y = x + C'(y) \overset{\text{must have}}{=} x - 2y$

This means $C'(y) = -2y$, and so $C(y) = -y^2$.

So $F(x,y) = xy - x^2 - y^2$ and the equation determining $y(x)$ is

$$xy - x^2 - y^2 = C_1.$$

6. (20) Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

Does the solution exist for all positive t? (Hint: to answer this question you should solve the initial value problem.)

$$y^{-2}\,dy = dt \implies \int y^{-2}\,dy = \int dt$$

$$-y^{-1} = t + c$$

$$y = \frac{-1}{t+c} \qquad y(0) = \frac{-1}{c} = \frac{1}{2}$$

$$c = -2.$$

So $$y(t) = \frac{-1}{t-2}$$

The solution exists for $t \in [0,2)$ but diverges to $+\infty$ as t approaches 2.