Spectral Gaps in One Dimensional Quantum Spin Systems

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Objective

State precise lower bounds on uniform spectral gap for perturbations of frustration free, 1D quantum spin systems with local topological quantum order and "open" boundary conditions.

Periodic boundaries in $\mathbb{Z}^\nu$: Michalakis and Zwolak 2013, Young 2016

Notation

Sets:

$$P_f(\mathbb{Z}) = \{X \subset \mathbb{Z} : |X| < \infty\}$$

$\Lambda = [a, b]_{\mathbb{Z}}$ will always denote a finite discrete interval.

Algebras:

$$\mathfrak{A}_X \cong M_{d^{|X|}}(\mathbb{C})$$

$$\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\substack{X \in P_f(\mathbb{Z}) \\ X \uparrow \mathbb{Z}}} \mathfrak{A}_X \qquad\qquad \mathfrak{A}_{\mathbb{Z}} = \overline{\mathfrak{A}_{\mathrm{loc}}}^{\|\cdot\|_{op}}$$

Quasi-local property: If $Y \cap X = \emptyset$, then $\mathfrak{A}_Y \subset \mathfrak{A}'_X$.

Notation

Interactions: $\eta, \Phi : P_f(\mathbb{Z}) \to \mathfrak{A}_{\text{loc}}$ interactions:

$$\Phi(X) = \Phi(X)^* \in \mathfrak{A}_X \ \text{(resp. } \eta(X))$$

$$\Phi_\Lambda = \sum_{X \subset \Lambda} \Phi(X) \qquad \eta_\Lambda = H_\Lambda$$

Decay: Choose to describe spatial decay of interactions in terms of specific $\mathcal{F}$-function:

$$F_{K,s,\kappa}(x) = F(x) = e^{-Kx^s} \frac{1}{(1+x)^\kappa}, \qquad K > 0, s \in (0,1], \kappa > 2$$

F defines an extended norm on the $\mathbb{R}$-vector space of interactions:

$$\|\Phi\|_F = \sup_{x,y \in \mathbb{Z}} \left\{ \sum_{\substack{Z \in P_f(\mathbb{Z}) \\ x,y \in Z}} \frac{\|\Phi(Z)\|}{F(|x-y|)} \right\}$$

Equivalence of ground states in 1D

Perturbative set-up: $\eta + \varepsilon\Phi$, $\varepsilon \in [0,1]$

Definition (Bachmann, Michalakis, Nachtergaele, Sims 2011)

Ground states of two interactions Φ_0, Φ_1 are equivalent if $\exists$ differentiable curve $\Phi(\varepsilon)$, $\varepsilon \in [0,1]$, and F such that:

1. $\|\Phi\|_F + \|\partial\Phi\|_F < \infty$
2. $\Phi(0) = \Phi_0, \quad \Phi(1) = \Phi_1$
3. $\exists$ uniform lower bound $\gamma > 0$ s.t. for exhaustive sequence $\Lambda_n \uparrow \mathbb{Z}$, ground state spectral gap of $\Phi_{\Lambda_n}(\varepsilon)$ is at least γ on $\varepsilon \in [0,1]$.

$$\lambda_0 = \inf_{\|f\|=1} \langle f, \Phi_\Lambda(\varepsilon) f \rangle$$

$$\gamma(H_\Lambda(\varepsilon)) = \min\{\lambda(\varepsilon) : \lambda(0) > \lambda_0\} - \max\{\lambda(\varepsilon) : \lambda(0) = \lambda_0\}$$

Lower bounds on spectral gaps for spin chains

Assumptions on interactions:

$$\eta + \varepsilon\Phi, \quad \varepsilon \in [0, 1]$$

Φ, the perturbing interaction decays sufficiently fast:

$$\|\Phi\|_F < \infty$$

η, the unperturbed interaction:

- is finite range;
- has uniformly bounded, non-negative local Hamiltonians;
- uniformly locally gapped by $\gamma_0 > 0$;
- is frustration-free, i.e. $\forall \Lambda \in P_f(\mathbb{Z}), ker(H_\Lambda) \neq \{0\}$;

and η satisfies local topological quantum order (LTQO) for open boundary conditions and $\Omega(n) \leq n^{-\nu}$, $\nu > 4$.

Local topological quantum order

Generalization of topological order introduced in (Bravyi, Hastings, Michalakis 2010).

$$P_X = \text{ orthog. projection onto } ker(H_X), \quad \omega_\Lambda(A) = \frac{1}{\mathrm{tr}(P_\Lambda)}\mathrm{tr}(P_\Lambda A)$$

$$r_x^{[a,b]} = \min\{x-a, b-x\}, \quad R_x^{[a,b]} = \max\{x-a, b-x\}$$

Definition

A frustration free interaction η has LTQO if there exists a monotone decreasing function $\Omega : [0,\infty) \to [0,\infty)$ such that $\forall \Lambda = [a,b] \in P_f(\mathbb{Z})$, $x \in \Lambda$, and $0 \leq k \leq r_x^\Lambda$, $k \leq n \leq R_x^\Lambda$,

$$\forall A \in \mathfrak{A}_{b_\Lambda(x,k)} : \|P_{b_\Lambda(x,n)}(A - \omega_\Lambda(A))P_{b_\Lambda(x,n)}\| \leq \Omega(z_x^\Lambda(n) - k)\|A\|$$

where $z_x^\Lambda : \mathbb{N} \to \mathbb{N}$ is the cut off function defined:

$$z_x^\Lambda(r) = \begin{cases} r & \text{if } r \leq r_x^\Lambda \\ r_x^\Lambda & \text{otherwise} \end{cases}$$

Lower bounds on spectral gaps for spin chains

Theorem (M, Nachtergaele)

Let $D \in \mathbb{N}$ be a distance parameter (distance from the boundary, uniform in Λ). There exists $\varepsilon_{\gamma_0} > 0$ and constant m_D such that $0 \leq \varepsilon < \varepsilon_{\gamma_0}$ and diam(Λ) sufficiently large implies:

$$\gamma(H_\Lambda(\varepsilon)) \geq \gamma_0 - m_D \varepsilon > 0$$

The constant m_D can be taken as the following expression:

$$m_D = [C_1 + C_2 D](\|\eta\|_F + \|\Phi\|_F)$$

$$C_1 \propto \sum_{n \in \mathbb{Z}} (3|n| + 2)\Big(\Omega((|n| - 1)/2)^{1/2} + F_0((|n| - 3)/2) + \Omega(|n|/2) + 2F_0(\lfloor |n|/2 \rfloor)\Big)$$

$$C_2 = C_2(F)$$

ε_{γ_0} is determined by m_D, F_0 is the base $\mathcal{F}$-function.

Application to fermion chains

Denote the Canonical Anticommutation Relation Algebra by:

$$\mathfrak{A}^f_\Lambda = C^*(a(x), a^*(x) : x \in \Lambda)$$

The Jordan-Wigner transformation (at Λ) $\vartheta_\Lambda : \mathfrak{A}^f_\Lambda \to \mathfrak{A}^s_\Lambda$ ($d = 2$) is a local map on even observables, i.e. if $\Lambda_0 \subset \Lambda$ is an interval and $X \subset \Lambda_0$, $A \in \mathfrak{A}^f_X$, then $\exp(i\pi N_\Lambda) A \exp(i\pi N_\Lambda) = A$ implies $\vartheta_\Lambda(A) \in \mathfrak{A}^s_{\Lambda_0}$.

Application to fermion chains

As a consequence, if we assume $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}_{\text{loc}}^{f,even}$ has LTQO with respect to even parity, and Ω and $\Phi : P_f(\mathbb{Z}) \to \mathfrak{A}_{\text{loc}}^{f,even}$ have the same decay assumptions as before:

Theorem (M, Nachtergaele)

There exist $\varepsilon'_{\gamma_0} > 0$ and constant m'_D such that $0 \leq \varepsilon < \varepsilon'_{\gamma_0}$ and diam(Λ) sufficiently large implies:

$$\gamma(H_\Lambda(\varepsilon)) \geq \gamma_0 - m'_D \varepsilon > 0$$

The constants m'_D and ε'_{γ_0} are determined by $m_D, \varepsilon_{\gamma_0}$ for a spin interaction related to η, Φ by the Jordan-Wigner transformation.