Renormalization Analysis of second order split-up for Spin-Boson models

Contributed Talk
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- based on joint works with David Hasler (arXiv:1704.06966)

Quantum mechanical system

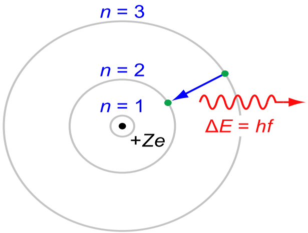

Figure: Bohr model of radiation
Source: Wikipedia

- describes charged particles and their interaction
- charged particles can absorb or emit photons
- the energy of the system is given by its Hamiltonian

Fock space

Fock space $\mathcal{F}$

Let $\mathfrak{h} = L^2(\mathbb{R}^3)$ and

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathfrak{h}^{\otimes_s n},$$

with vaccum vector $\Omega = (1, 0, \ldots)$.
Note that $\mathfrak{h}^{\otimes_s n}$ is the symmetrization of the n-fold tensor product.

Free field Hamiltonian (operator of energy)

$$H_f = \int_{\mathbb{R}^3} |k| a^*(k) a(k) d^3k$$

which acts by multiplication with $\sum_{j=1}^{n} |k_j|$ on $\mathfrak{h}^{\otimes_s n}$.

The Spin-Boson Hamiltonian

The Hilbert space of 'Matter' and Radiation

$$\mathcal{H} = \mathbb{C}^N \otimes \mathcal{F}$$

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Spin-Boson Hamiltonian

We are interested in the spectral behaviour of the following operator

$$H_g = H_{\mathrm{at}} + H_f + gW$$

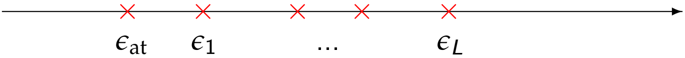
where g is the so called coupling constant and

$$W = a^*(|k|^{-1/2}G) + a(|k|^{-1/2}G)$$

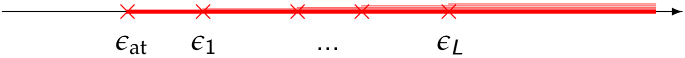
is a linear interaction between 'matter' and radiation that is infinitesimally bounded with respect to H_f.

Existence of ground state

- Spectrum of H_{at}:

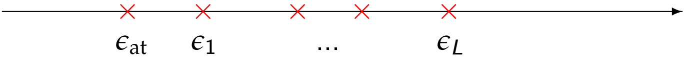


- Spectrum of H_0:

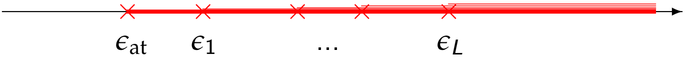


Existence of ground state

- Spectrum of H_{at}:



- Spectrum of H_0:



- Spectrum of H_g for $g \neq 0$:

Theorem (Bach-Fröhlich-Sigal '98, Griesemer-Lieb-Loss '01)

The number

$$E_g = \inf \sigma(H_g)$$

is an eigenvalue of H_g.

Degenerate case

Let the ground state eigenvalue be degenerate, i.e.

$$\dim(\mathrm{Ran} P_{\mathrm{at}}) > 1.$$

Assumption: The degeneracy is lifted at second order in formal perturbation theory.

Second order energy correction:

$$\begin{aligned} Z_{\mathrm{at}} &\widehat{\cong} -(P_{\mathrm{at}} \otimes P_{\Omega}) W (H_0 - \epsilon_{\mathrm{at}})^{-1} W (P_{\mathrm{at}} \otimes P_{\Omega})|_{\mathrm{Ran} P_{\mathrm{at}} \otimes P_{\Omega}} \\ &= -\int P_{\mathrm{at}} G^*(k) \left[\frac{P_{\mathrm{at}}}{|k|} + \frac{\overline{P}_{\mathrm{at}}}{H_{\mathrm{at}} - \epsilon_{\mathrm{at}} + |k|} \right] G(k) P_{\mathrm{at}} \frac{d^3 k}{|k|} \quad \upharpoonright \mathrm{Ran} P_{\mathrm{at}}, \end{aligned}$$

Note that $\overline{P}_{\mathrm{at}} = 1 - P_{\mathrm{at}}$.

Degeneracy lifted at 2nd order

Theorem (Hasler-L. '17)

Let $\mu > 0$*. Suppose* $G \in L^2_\mu(\mathbb{R}^3; \mathcal{L}(\mathcal{H}_{\mathrm{at}}))$ *and let* H_g *be given by*

$$H_g = H_{\mathrm{at}} + H_f + gW.$$

Let $\epsilon^{(2)}_{\mathrm{at}}$ *denote the smallest eigenvalue of* Z_{at}*. Assume that* $\epsilon^{(2)}_{\mathrm{at}}$ *is simple. Let* $0 \leq \delta_0 < \pi/2$*, and let*

$$S_{\delta_0} = \{z \in \mathbb{C} : |\arg(z)| < \delta_0 \text{ or } |\arg(-z)| < \delta_0\}.$$

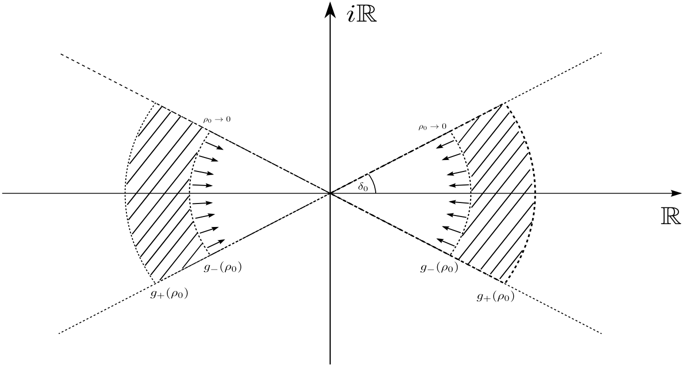
Then there exists a $g_0 > 0$ *such that for all* $g \in D_{g_0} \cap S_{\delta_0}$ *the operator* H_g *has an eigenvector* ψ_g *and a corresponding eigenvalue* E_g *such that*

$$E_g = \epsilon_{\mathrm{at}} + g^2 \epsilon^{(2)}_{\mathrm{at}} + o(|g|^2).$$

Degeneracy lifted at 2nd order

In addition:

- The eigenvalue E_g is continuous on $S_{\delta_0} \cap D_{g_0}$ and analytic in the interior of $S_{\delta_0} \cap D_{g_0}$.
- For real g the number E_g is the infimum of the spectrum of H_g.



Idea of the Proof

Based on operator-theoretic renormalization group method (RG) introduced by Bach-Fröhlich-Sigal '98.

Main steps of the proof

(1) use a first so-called Feshbach map to project on a suitable subspace of the full Hilbert space in order to focus on the ground-state eigenvalue $(P = P_{\mathrm{at}} \otimes \chi(r/\rho_0))$.

(2) apply a second Feshbach map in order to fully resolve the degeneracy of the ground-state eigenvalue.

(3) Use an analyticity result (Griesemer-Hasler '09) and consider the limit $\rho_0 \to 0$ to conclude the proof.

Thank you for your attention