Spectral gaps without frustration

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Quantum spin chains

We consider a chain of m quantum spins.

Hilbert space $\mathcal{H}_m := \bigotimes_{j=1}^{m} \mathbb{C}^d, \quad (d \geq 2)$

Hamiltonian Nearest-neighbor interaction described by a fixed projection $P \colon \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$.

$$H_m := \sum_{j=1}^{m-1} h_{j,j+1}, \qquad h_{j,j+1} = \mathbb{1}_{1,\ldots,j-1} \otimes P \otimes \mathbb{1}_{j+2,\ldots,n}$$

Spectral gap $\gamma_m := E_1(m) - E_0(m) > 0$,
where $\mathrm{spec}(H_m) = \{E_0(m) < E_1(m) < E_2(m) < \ldots\}$.

Why do we care about spectral gaps?

Fundamental dichotomy

(i) $\exists c > 0$ such that $\gamma_m \geq c$ for all m $\quad \rightarrow \quad$ "gapped"

(ii) $\liminf_{m \to \infty} \gamma_m = 0$ $\quad \rightarrow \quad$ "gapless"

Where a model falls in this dichotomy has far-reaching consequences:

- Ground states of gapped systems are well-controlled. (Exponential decay of correlations; area law; approximable in polynomial time)
- Hastings spectral flow requires a gap. (Closing of the gap, as a parameter is varied, indicates a phase transition.)

But the problem of determining (i) vs. (ii) is hard in general, so we make a simplifying assumption...

Frustration-free models

Assumption (FF)

For all $m \geq 2$,

$$\bigcap_{j=1}^{m-1} \ker h_{j,j+1} \neq \emptyset,$$

This is equivalent to $E_0(m) = 0$.

Intuition For the purpose of energy minimization, think of

$$H_m = \sum_{j=1}^{m-1} h_{j,j+1},$$

as a list of m non-commuting constraints (given by $\ker h_{j,j+1}$). Then FF means that all the non-commuting constraints can be simultaneously satisfied.

This assumption holds in several physically relevant examples (Heisenberg spin 1/2 ferromagnet; AKLT model; toric code).

Finite-size criteria

The basic idea of a finite-size criterion: If the (positive) spectral gap at a fixed system size (say, on 10 sites) is large enough, then the Hamiltonian is gapped.

Previous results involve periodic chains, $H_m^{\mathrm{per}} = H_m + h_{m,1}$, and their spectral gaps, γ_m^{per}.

Theorem (Knabe '88; Gosset-Mozgunov '16)

Assume H_m^{per} is FF. Let $3 \leq n \leq m/2$. There exists $c_n > 0$ such that

$$\gamma_m^{\mathrm{per}} \geq c_n \left(\gamma_n - \frac{6}{n^2} \right).$$

Main result

A finite-size criterion for spin chains with boundary.

Theorem (L-Mozgunov '18)

Assume H_m is FF. Let $3 \leq n \leq m/2$. There exists $c_n > 0$ such that

$$\gamma_m \geq c_n \left(\min_{n/2 \leq \ell \leq n} \gamma_\ell - \frac{4\sqrt{2}}{n^{3/2}} \right).$$

A similar result (again with $n^{-3/2}$ threshold) hold in 2D for arbitrary lattices and arbitrary finite-range interactions.

Corollary

If there exists $n_0 \geq 3$ such that $\min_{\ell \sim n_0} \gamma_\ell > \frac{4\sqrt{2}}{n_0^{3/2}}$, then H_m is gapped.

Corollary

If $\gamma_m = O(m^{-p})$, then $p \geq 3/2$. "Gap cannot close slowly"

Corollary

2D FF systems cannot exhibit gapless edge modes.

Thank you for your attention.