Data Processing Inequality and the stability of the recovery map

Anna Vershynina

Department of Mathematics,
University of Houston

Joint work with
Eric A. Carlen
(Rutgers University)

Arizona School of Analysis and Mathematical Physics

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Outline of the talk

- Quantum Information Inequalties

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- Data Processing Inequality

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- DPI for Rényi entropy

Quantum Entropy

The *entropy* of a state ρ is

$$S(\rho) = -\mathrm{Tr}\{\rho \log \rho\}$$

Quantum Entropy

The *entropy* of a state ρ is for a state ρ^A on a system A

$$S(\rho) = -\mathrm{Tr}\{\rho \log \rho\} \quad = S(A) = S(A)_\rho$$

von Neumann entropy

quantum entropy

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Properties

Positivity For all ρ we have $S(\rho) \geq 0$

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For all $\rho = \sum_j p_j \rho_j$ we have

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Unitary Invariance

For any unitary U we have $S(\rho) = S(U\rho U^*)$

Joint/Marginal Entropy

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Classically $\quad H(X,Y) \geq H(X) \qquad H(X,Y) \geq H(Y)$

Theorem $\quad$ For a pure bi-partite state $|\psi\rangle^{AB}$ we have $\quad S(AB)_\psi = 0$

while *marginal* entropies are equal $\qquad S(A)_\psi = S(B)_\psi$

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$$|S(\rho^A) - S(\rho^B)| \leq S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B)$$

proved by E.H.Lieb & H.Araki in 1970

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The quantum relative entropy between two states ρ and σ is as follows

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let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = \epsilon|\psi\rangle\langle\psi| + (1-\epsilon)|\psi^{\perp}\rangle\langle\psi^{\perp}|$

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$$\begin{aligned} D(\rho\|\sigma) &= -H(\rho) - \mathrm{Tr}\{\rho\log\sigma\} \\ &= -\mathrm{Tr}\{|\psi\rangle\langle\psi|(\log\epsilon|\psi\rangle\langle\psi| + \log(1-\epsilon)|\psi^{\perp}\rangle\langle\psi^{\perp}|)\} \\ &= -\log\epsilon \underset{\epsilon\to 0}{\longrightarrow} +\infty \end{aligned}$$

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Completely Positive Trace Preserving (CPTP) map

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In particular, $S(\rho^{AB}\|\sigma^{AB}) \geq S(\rho^{A}\|\sigma^{A})$

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Strong Sub-additivity of quantum entropy

For a tri-partite state ρ^{ABC}

$$S(\rho^{ABC}) + S(\rho^{B}) \leq S(\rho^{AB}) + S(\rho^{BC})$$

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In the monotonicity of relative entropy $S(\rho^{ABC} \| \sigma^{ABC}) \geq S(\rho^{AB} \| \sigma^{AB})$

take $\sigma^{ABC} = \rho^{A} \otimes \rho^{BC}$

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Note that

$$\log(\rho^{A} \otimes \rho^{B}) = \log(\rho^{A}) \otimes I^{B} + I^{A} \otimes \log \rho^{B}$$

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Therefore,

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For $\rho = \sum_j p_j \rho_j$ and $\sigma = \sum_j p_j \sigma_j$

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$$S(\rho^{AB} \| \sigma^{AB}) \geq S(\rho^{A} \| \sigma^{A})$$

all "equivalent"

e.g. Monotonicity under partial traces holds for all states on ABC iff Strong sub-additivity relation holds

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see "Strong Subadditivity of Quantum Entropy" on Wikipedia

Case of equality

Theorem (Petz '88)

For states ρ, σ, and CPTP map $\mathcal{N}$

$$S(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) = S(\rho||\sigma)$$

if and only if, there exits a recovery map $\mathcal{R}_{\mathcal{N}}$ such that

$$\mathcal{R}_{\mathcal{N}}(\mathcal{N}(\rho)) = \rho \text{ and } \mathcal{R}_{\mathcal{N}}(\mathcal{N}(\sigma)) = \sigma$$

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Explicitly,

$$\mathcal{R}_{\mathcal{N}}(\omega) = \rho^{1/2}\mathcal{N}^*((\mathcal{N}(\rho))^{-1/2}\omega(\mathcal{N}(\rho))^{-1/2})\rho^{1/2}$$

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For the partial trace

$$\mathcal{R}_\rho(\omega_A) = \rho_{AB}^{1/2}\,\rho_A^{-1/2}\omega_A\rho_A^{-1/2}\rho_{AB}^{1/2}$$

Strengthening of the DPI

Data Processing Inequality

$$S(\rho^{AB}\|\sigma^{AB}) - S(\rho^{A}\|\sigma^{A}) \geq 0$$

Strengthening of the DPI

$$S(\rho^{AB}\|\sigma^{AB}) - S(\rho^{A}\|\sigma^{A}) \geq ???$$

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2015: Fawzi, Renner — near saturation of SSA results in near satisfaction of quantum Markov chain condition

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2015: Wilde

2016: Berta, Tomamichel

2016: Sutter, Fawzi, Renner

2016: Sutter, Tomamichel, Harrow

2016: Zhang

. . .

some sort of the recovery map,
not the Petz map

Strengthening of the DPI

Theorem Carlen, V '17

$$S(\rho^{AB}\|\sigma^{AB}) - S(\rho^{A}\|\sigma^{A}) \geq C\|\Delta_{\sigma,\rho}\|^{-2}\|\mathcal{R}_\rho(\sigma_A) - \sigma_{AB}\|_1^4$$

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independent of σ - "universal" bound

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exchanging ρ and σ

Consequently,

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Bounding the norm of states by one,

$$S(\rho^{AB}\|\sigma^{AB}) - S(\rho^{A}\|\sigma^{A}) \geq C\|\rho^{-1}\|^{-2}\|\sigma_A^{-1}\|^{-2}\mathcal{R}_{\sigma}(\rho_A) - \rho_{AB}\|_1^4$$

In fact, even stronger statement holds

Stability of the recovery map

The *relative modular operator* is defined as

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Hilbert-Schmidt norm

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$$\rho_A^{-1/2}\rho^{1/2} = \sigma_A^{-1/2}\sigma^{1/2} \longleftarrow$$ stronger? No

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symmetric

$S(\rho\|\sigma) = S(\rho_A\|\sigma_A)$ if and only if $S(\sigma\|\rho) = S(\sigma_A\|\rho_A)$

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$$\|\sigma_A^{1/2}\rho_A^{-1/2}\rho^{1/2} - \sigma^{1/2}\|_2 \geq \tfrac{1}{2}\|\mathcal{R}(\sigma_A) - \sigma\|_1$$

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Follows from for $\mathrm{Tr}(X^*X) = \mathrm{Tr}(Y^*Y) = 1$ we have $\|X^*X - Y^*Y\|_1 \leq 2\|X - Y\|_2$

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operator norm

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$$|\mathrm{Tr}[Z(X^*X - Y^*Y)]| \leq |\mathrm{Tr[Z(X^* - Y^*)X]}| + |\mathrm{Tr}[ZY^*(X - Y)]|$$

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$$\leq (\mathrm{Tr}(X^* - Y^*)(X - Y))^{1/2}\left\{(\mathrm{Tr}(X^*Z^*ZX))^{1/2} + (\mathrm{Tr}(Y^*Z^*ZY))^{1/2}\right\}$$

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Follows from for $\mathrm{Tr}(X^*X) = \mathrm{Tr}(Y^*Y) = 1$ we have $\|X^*X - Y^*Y\|_1 \leq 2\|X - Y\|_2$

$$|\mathrm{Tr}[Z(X^*X - Y^*Y)]| \leq |\mathrm{Tr}[\mathrm{Z}(\mathrm{X}^* - \mathrm{Y}^*)\mathrm{X}]| + |\mathrm{Tr}[ZY^*(X-Y)]|$$

$$\leq (\mathrm{Tr}(X^* - Y^*)(X-Y))^{1/2}\{(\mathrm{Tr}(X^*Z^*ZX))^{1/2} + (\mathrm{Tr}(Y^*Z^*ZY))^{1/2}\}$$

$$\leq 2\|X - Y\|_2$$

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Follows from

for $\mathrm{Tr}(X^*X) = \mathrm{Tr}(Y^*Y) = 1$ we have $\quad \|X^*X - Y^*Y\|_1 \leq 2\|X - Y\|_2$

Applying with

$$X = \sigma_A^{1/2}\rho_A^{-1/2}\rho^{1/2} \qquad Y = \sigma^{1/2}$$

will get the lemma

Ideas of the proof

Quantum relative entropy $S(\rho\|\sigma) = \mathrm{Tr}\{\rho(\log\rho - \log\sigma)\}$

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Quasi-relative entropy

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Quasi-relative entropy Let $f : (0, \infty) \to \mathbb{R}$ be operator convex

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Quasi-relative entropy Let $f : (0,\infty) \to \mathbb{R}$ be operator convex for $n \in \mathbb{N}$

positive $n \times n$ matrices A and B

$$f(\tfrac{1}{2}A + \tfrac{1}{2}B) \leq \tfrac{1}{2}f(A) + \tfrac{1}{2}f(B)$$

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Quasi-relative entropy Let $f : (0,\infty) \to \mathbb{R}$ be operator convex

f-relative quasi-entropy is $S_f(\rho\|\sigma) = \mathrm{Tr}(f(\Delta_{\sigma,\rho})\rho) = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho})\rho^{1/2}\rangle$

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Choose $f(x) = -\log x$

$$-\log(\Delta_{\sigma,\rho})\rho^{1/2} = \rho^{1/2}\log\rho - \log\sigma\,\rho^{1/2}$$

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$$S_{-\log}(\rho\|\sigma) = S(\rho\|\sigma)$$

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$f_t(x) = (t+x)^{-1}$ is operator convex for each $t > 0$

$$S_{(t)}(\rho\|\sigma) = \mathrm{Tr}((t+\Delta_{\sigma,\rho})^{-1}\rho)$$

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Since

$$-\log A = \int_0^\infty \left(\frac{1}{t+A} - \frac{1}{1+t}\right) dt$$

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it follows

$$S(\rho\|\sigma) = \int_0^\infty \left(S_{(t)}(\rho\|\sigma) - \frac{1}{1+t}\right) dt$$

Ideas of the proof

Monotonicity of quasi-relative entropy

Define
$$U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$$

Ideas of the proof

Monotonicity of quasi-relative entropy

Define

$$U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$$

then

$$U^*(Y) = \mathrm{Tr}(Y \rho^{1/2}) \rho_A^{-1/2}$$

and $U^* U(X) = X$

Ideas of the proof

Monotonicity of quasi-relative entropy

Define $U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$ then $U^*(Y) = \mathrm{Tr}(Y\rho^{1/2})\rho_A^{-1/2}$ and $U^*U(X) = X$

After simple calculation $\Delta_{\sigma,\rho}^{1/2}(U(X_A)) = \sigma^{1/2} X_A \rho_A^{-1/2}$

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and hence $\langle \Delta_{\sigma,\rho}^{1/2}(U(X_A)), \Delta_{\sigma,\rho}^{1/2}(U(X_A)) \rangle = \mathrm{Tr}(\rho_A^{-1/2} X_A^* \sigma X_A \rho_A^{-1/2})$

Ideas of the proof

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and hence $$\langle \Delta_{\sigma,\rho}^{1/2}(U(X_A)), \Delta_{\sigma,\rho}^{1/2}(U(X_A))\rangle = \mathrm{Tr}(\rho_A^{-1/2} X_A^* \sigma X_A \rho_A^{-1/2}) = \mathrm{Tr}(\rho_A^{-1/2} X_A^* \sigma_A X_A \rho_A^{-1/2})$$
$$= \langle \Delta_{\sigma_A,\rho_A}^{1/2}(X_A), \Delta_{\sigma_A,\rho_A}^{1/2}(X_A)\rangle$$

Ideas of the proof

Monotonicity of quasi-relative entropy

Define $U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$ then $U^*(Y) = \mathrm{Tr}(Y \rho^{1/2}) \rho_A^{-1/2}$ and $U^* U(X) = X$

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Monotonicity of quasi-relative entropy

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Optimizing in T

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$\square$

DPI for f-relative quasi-entropies

Let $f : (0, \infty) \to \mathbb{R}$ be operator convex

f-relative quasi-entropy is $\quad S_f(\rho\|\sigma) = \mathrm{Tr}(f(\Delta_{\sigma,\rho})\rho) = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho})\rho^{1/2} \rangle$

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relative modular operator

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$$S_{f_\alpha}(\rho||\sigma) = -\mathrm{Tr}(\rho^{1-\alpha}\sigma^\alpha)$$

Rényi entropy

$$S_\alpha(\rho||\sigma) = \frac{1}{\alpha-1} \log \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha-1} \log(-S_{f_{1-\alpha}}(\rho||\sigma))$$

DPI for f-relative quasi-entropies

Recall, for regular quantum relative entropy we already have

$$S(\rho\|\sigma) - S(\rho_A\|\sigma_A) \geq \left(\tfrac{\pi}{4}\right)^4 \|\Delta_{\sigma,\rho}\|^{-2} \, \|\sigma_A^{1/2}\rho_A^{-1/2}\rho^{1/2} - \sigma^{1/2}\|_2^4$$

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Theorem (Carlen, V. '18)

For a broad class of operator monotone decreasing functions f, and any $\beta \in (0,1)$, there is a parameter $c > 0$ (that depends on f and β), such that

for $\beta \leq 1/2$ $$K\left(S_f(\rho\|\sigma) - S_f(\rho_A\|\sigma_A)\right)^{\frac{\beta(1-\beta)}{1+2c(1-\beta)}} \geq \|\sigma_A^{\beta}\rho_A^{-\beta}\rho^{1/2} - \sigma^{\beta}\rho^{\frac{1}{2}-\beta}\|_2$$

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$$\|\mathcal{R}_\sigma(\rho_A) - \rho\|_1, \; \|\mathcal{R}_\rho(\sigma_A) - \sigma\|_1 \leq C\|\sigma_A^{1/2}\rho_A^{-1/2}\rho^{1/2} - \sigma^{1/2}\|_2$$

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Then
$$\max\{\|\mathcal{R}_\sigma(\rho_A) - \rho\|_1, \|\mathcal{R}_\rho(\sigma_A) - \sigma\|_1\} \leq K\left(S_f(\rho\|\sigma) - S_f(\rho_A\|\sigma_A)\right)^{\frac{1}{4}\frac{1}{1+c}}$$

DPI for f-relative quasi-entropies

Consequently,

$$S_f(\rho\|\sigma) = S_f(\rho_A\|\sigma_A) \quad \text{if and only if} \quad \text{both hold} \quad \begin{array}{l} \sigma = \mathcal{R}_\rho(\sigma_A) \\ \rho = \mathcal{R}_\sigma(\rho_A) \end{array}$$

DPI for f-relative quasi-entropies

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Conversely, if $\sigma = \mathcal{R}_\rho(\sigma_A)$ then by monotonicity theorem

$$S_f(\rho||\sigma) = S_f(\mathrm{R}_\rho(\rho_A)||\mathrm{R}_\rho(\sigma_A)) \leq S_f(\rho_A||\sigma_A) \leq S_f(\rho||\sigma)$$

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equality

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DPI for f-relative quasi-entropies

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therefore

$$S_f(\rho\|\sigma) = S_f(\rho_A\|\sigma_A)$$

and therefore $\rho = \mathcal{R}_\sigma(\rho_A)$

DPI for f-relative quasi-entropies

For $\beta \in (0,1)$

$$\|\sigma_A^{\beta}\rho_A^{-\beta}\rho^{1/2} - \sigma^{\beta}\rho^{\frac{1}{2}-\beta}\|_2 \geq \|\rho^{-1/2}\|^{-1}\|\sigma_A^{\beta}\rho_A^{-\beta} - \sigma^{\beta}\rho^{-\beta}\|_2$$

DPI for f-relative quasi-entropies

For $\beta \in (0,1)$

$$\|\sigma_A^{\beta}\rho_A^{-\beta}\rho^{1/2} - \sigma^{\beta}\rho^{\frac{1}{2}-\beta}\|_2 \geq ||\rho^{-1/2}||^{-1}\|\sigma_A^{\beta}\rho_A^{-\beta} - \sigma^{\beta}\rho^{-\beta}\|_2$$

Then

$$S_f(\rho\|\sigma) = S_f(\rho_A\|\sigma_A) \qquad \text{if and only if} \qquad \sigma_A^{\beta}\rho_A^{-\beta} = \sigma^{\beta}\rho^{-\beta}$$

Logarithmic function

Taking $f(x) = -\log(x)$, i.e. obtaining the Umegaki relative entropy

For $\beta \leq 1/2$

$$(S(\rho\|\sigma) - S(\rho_A\|\sigma_A)) \geq K_\beta^L(\rho,\sigma) \|\sigma_A^\beta \rho_A^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{\frac{1}{2}-\beta}\|_2^{\frac{1}{\beta(1-\beta)}}$$

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if $\beta = 1/2$ $\quad \frac{1}{\beta(1-\beta)} = 4$

as before

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For $\beta > 1/2$

$$(S(\rho\|\sigma) - S(\rho_A\|\sigma_A)) \geq K_\beta^U(\rho,\sigma)\|\sigma_A^\beta \rho_A^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{\frac{1}{2}-\beta}\|_2^{\frac{2}{1-\beta}}$$

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equal to 4 for $\beta = 1/2$

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Compare to the previous result

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Rényi entropy

For $\alpha \in (0,1)$ Rényi entropy

$$S_\alpha(\rho||\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha-1}\log(-S_{f_{1-\alpha}}(\rho||\sigma))$$

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f-relative quasi-entropy for $f_{1-\alpha}(x) = x^{1-\alpha}$

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Power function

For $\beta = 1/2$ $\quad S_{f_\alpha}(\rho||\sigma) - S_{f_\alpha}(\rho_A||\sigma_A) \geq K^U_{\alpha,1/2}(\rho,\sigma)\|\sigma_A^{1/2}\rho_A^{-1/2}\rho_{1/2} - \sigma^{1/2}\|_2^{4+2\alpha}$

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Theorem (Carlen, V. '18)

$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_A||\sigma_A) \geq \tfrac{1}{1-\alpha}\log\left(1 + K^U_{1-\alpha,1/2}(\rho,\sigma)\|\sigma_A^{1/2}\rho_A^{-1/2}\rho_{1/2} - \sigma^{1/2}\|_2^{4+2\alpha}\right)$$

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If we know that the difference is small, say

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$$\max\{\|\mathcal{R}_\rho(\sigma_\mathcal{N}) - \sigma\|_1, \|\mathcal{R}_\sigma(\rho_\mathcal{N}) - \rho\|_1\}^{6-2\alpha} \leq \frac{2\|\sigma^{-1}\|^{1/2}}{K^U_{1-\alpha}(\rho,\sigma)} (\exp[(1-\alpha)(S_\alpha(\rho||\sigma) - S_\alpha(\rho_\mathcal{N}||\sigma_\mathcal{N}))] - 1)$$

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And then

$$\max\{\|\mathcal{R}_\rho(\sigma_\mathcal{N}) - \sigma\|_1, \|\mathcal{R}_\sigma(\rho_\mathcal{N}) - \rho\|_1\}^{6-2\alpha} \leq \frac{2\|\sigma^{-1}\|^{1/2}}{K^U_{1-\alpha}(\rho,\sigma)}\frac{e^{(1-\alpha)r}-1}{r}(S_\alpha(\rho||\sigma) - S_\alpha(\rho_\mathcal{N}||\sigma_\mathcal{N}))$$

Thank you!