



Data Processing Inequality and the stability of the recovery map

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*Joint work with
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Outline of the talk

- Quantum Information Inequalities

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- Data Processing Inequality

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- Strengthening of the DPI

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- DPI for Rényi entropy

Quantum Entropy

The *entropy* of a state ρ is

$$S(\rho) = -\text{Tr}\{\rho \log \rho\}$$

Quantum Entropy

The *entropy* of a state ρ is

for a state ρ^A on a system A

$$S(\rho) = -\text{Tr}\{\rho \log \rho\} = S(A) = S(A)_\rho$$

von Neumann entropy

quantum entropy

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Properties

Positivity

For all ρ we have $S(\rho) \geq 0$

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Concavity

For all $\rho = \sum_j p_j \rho_j$ we have

$$S(\rho) \geq \sum_j p_j S(\rho_j)$$

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Unitary Invariance

For any unitary U we have $S(\rho) = S(U\rho U^*)$

Joint/Marginal Entropy

The *joint quantum entropy* of a density operator ρ^{AB}

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Theorem For a pure bi-partite state $|\psi\rangle^{AB}$ we have $S(AB)_\psi = 0$
while *marginal* entropies are equal $S(A)_\psi = S(B)_\psi$

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The quantum entropy is additive for tensor product states

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$$|S(\rho^A) - S(\rho^B)| \leq S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B)$$

proved by E.H.Lieb & H.Araki in 1970

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The quantum relative entropy between two states ρ and σ is as follows

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$$\begin{aligned} D(\rho||\sigma) &= -H(\rho) - \text{Tr}\{\rho \log \sigma\} \\ &= -\text{Tr}\{|\psi\rangle\langle\psi|(\log \epsilon|\psi\rangle\langle\psi| + \log(1 - \epsilon)|\psi^\perp\rangle\langle\psi^\perp|)\} \\ &= -\log \epsilon \rightarrow +\infty \\ &\quad \epsilon \rightarrow 0 \end{aligned}$$

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↘ Completely Positive Trace Preserving (CPTP) map

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In particular, $S(\rho^{AB}||\sigma^{AB}) \geq S(\rho^A||\sigma^A)$

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Strong Sub-additivity of quantum entropy

For a tri-partite state ρ^{ABC}

$$S(\rho^{ABC}) + S(\rho^B) \leq S(\rho^{AB}) + S(\rho^{BC})$$

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In the monotonicity of relative entropy $S(\rho^{ABC} \parallel \sigma^{ABC}) \geq S(\rho^{AB} \parallel \sigma^{AB})$

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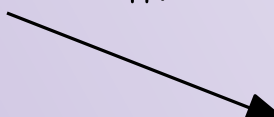
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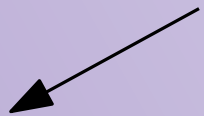
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Therefore,


$$S(\rho^A) + S(\rho^{BC}) - S(\rho^{ABC})$$


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Joint Convexity of Quantum Relative Entropy

For $\rho = \sum_j p_j \rho_j$ and $\sigma = \sum_j p_j \sigma_j$

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$$S(\rho \parallel \sigma) \geq S(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma))$$

Monotonicity under partial traces

$$S(\rho^{AB} \parallel \sigma^{AB}) \geq S(\rho^A \parallel \sigma^A)$$

all "equivalent"

e.g. Monotonicity under partial traces holds for all states on ABC iff Strong sub-additivity relation holds

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see "Strong Subadditivity of Quantum Entropy" on Wikipedia

Case of equality

Theorem (Petz '88)

For states ρ , σ , and CPTP map $\mathcal{N}$

$$S(\mathcal{N}(\rho) || \mathcal{N}(\sigma)) = S(\rho || \sigma)$$

if and only if, there exists a recovery map $\mathcal{R}_{\mathcal{N}}$ such that

$$\mathcal{R}_{\mathcal{N}}(\mathcal{N}(\rho)) = \rho \text{ and } \mathcal{R}_{\mathcal{N}}(\mathcal{N}(\sigma)) = \sigma$$

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Explicitly,

$$\mathcal{R}_{\mathcal{N}}(\omega) = \rho^{1/2} \mathcal{N}^*((\mathcal{N}(\rho))^{-1/2} \omega (\mathcal{N}(\rho))^{-1/2}) \rho^{1/2}$$

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For the partial trace

$$\mathcal{R}_{\rho}(\omega_A) = \rho_{AB}^{1/2} \rho_A^{-1/2} \omega_A \rho_A^{-1/2} \rho_{AB}^{1/2}$$

Strengthening of the DPI

Data Processing Inequality

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq 0$$

Strengthening of the DPI

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq ???$$

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some sort of the recovery map,
not the Petz map

2016: Berta, Tomamichel

2016: Sutter, Fawzi, Renner

2016: Sutter, Tomamichel, Harrow

2016: Zhang • • •

Strengthening of the DPI

Theorem Carlen, V '17

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq C \|\Delta_{\sigma, \rho}\|^{-2} \|\mathcal{R}_\rho(\sigma_A) - \sigma_{AB}\|_1^4$$

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$$\Delta_{\sigma, \rho}(X) = \sigma X \rho^{-1} \quad \text{“modular operator”} \quad \leftarrow$$

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independent of σ - “universal” bound

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$$\mathcal{R}_\rho(\omega_A) = \rho_{AB}^{1/2} \rho_A^{-1/2} \omega_A \rho_A^{-1/2} \rho_{AB}^{1/2}$$

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↪ Petz's recovery map
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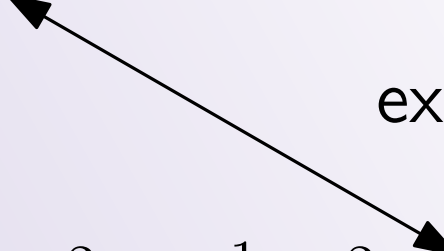
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Consequently,

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq C \|\Delta_{\sigma, \rho}\|^{-2} \|\rho_A\|^{-2} \|\sigma_A^{-1}\|^{-2} \|\mathcal{R}_\sigma(\rho_A) - \rho_{AB}\|_1^4$$

exchanging ρ and σ



Strengthening of the DPI

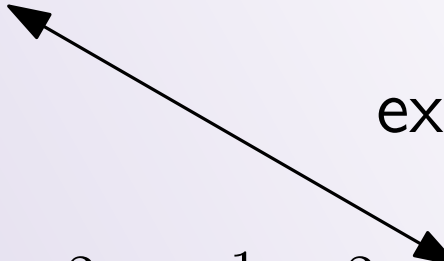
Theorem Carlen, V '17

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq C \|\Delta_{\sigma, \rho}\|^{-2} \|\mathcal{R}_\rho(\sigma_A) - \sigma_{AB}\|_1^4$$

Consequently,

$$S(\rho^{AB} \parallel \sigma^{AB}) - S(\rho^A \parallel \sigma^A) \geq C \|\Delta_{\sigma, \rho}\|^{-2} \|\rho_A\|^{-2} \|\sigma_A^{-1}\|^{-2} \|\mathcal{R}_\sigma(\rho_A) - \rho_{AB}\|_1^4$$

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Bounding the norm of states by one,

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In fact, even stronger statement holds

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The *relative modular operator* is defined as

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Hilbert-Schmidt norm



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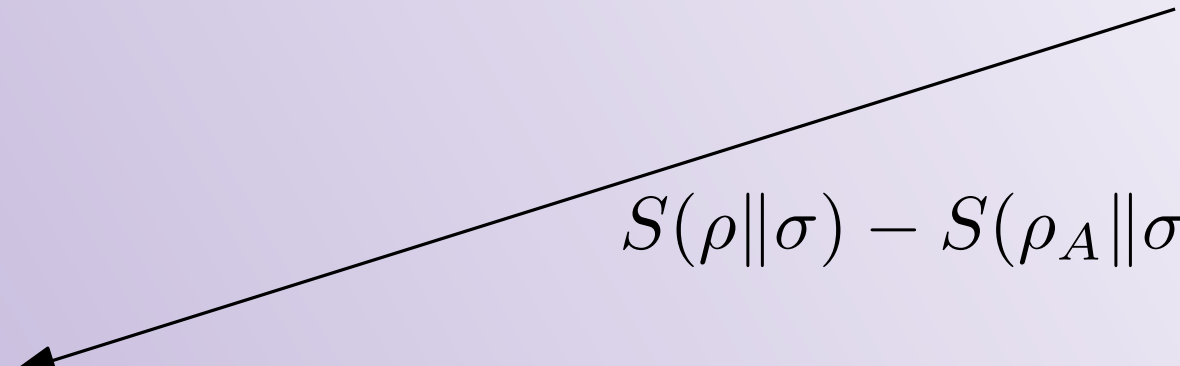
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$$\rho_A^{-1/2} \rho^{1/2} = \sigma_A^{-1/2} \sigma^{1/2} \longrightarrow \sigma_A^{1/2} \rho_A^{-1/2} = \sigma^{1/2} \rho^{-1/2}$$

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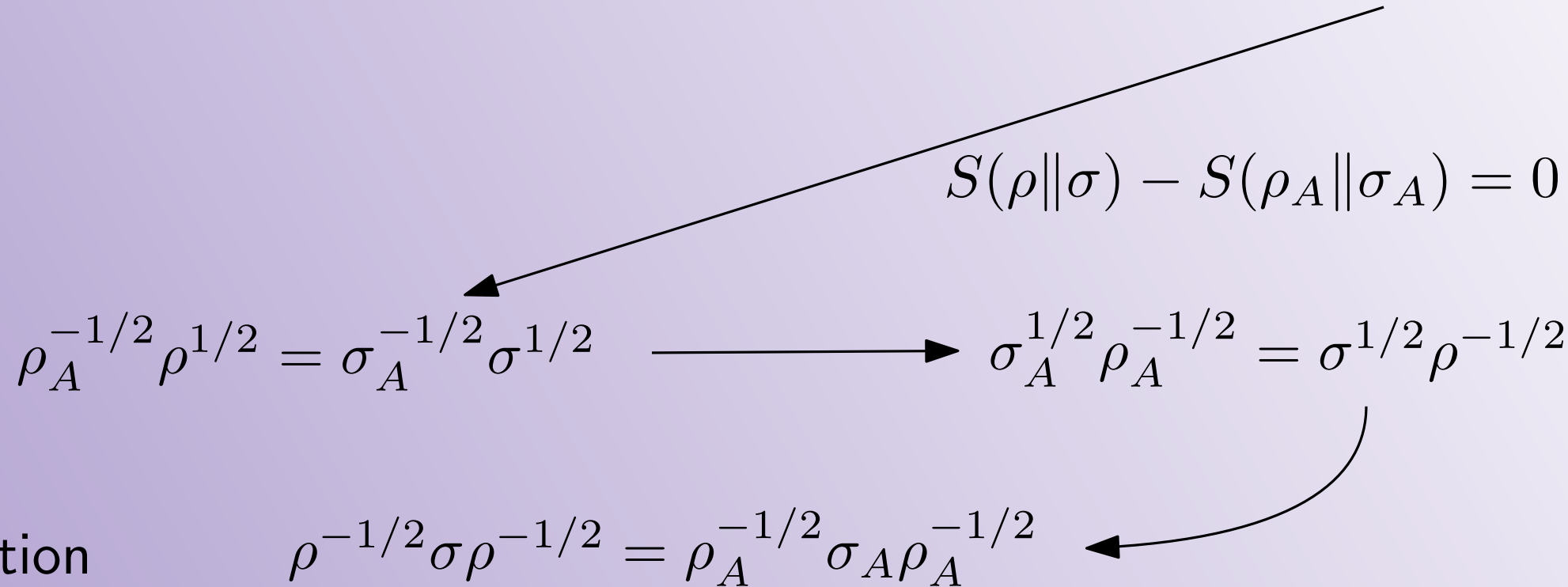
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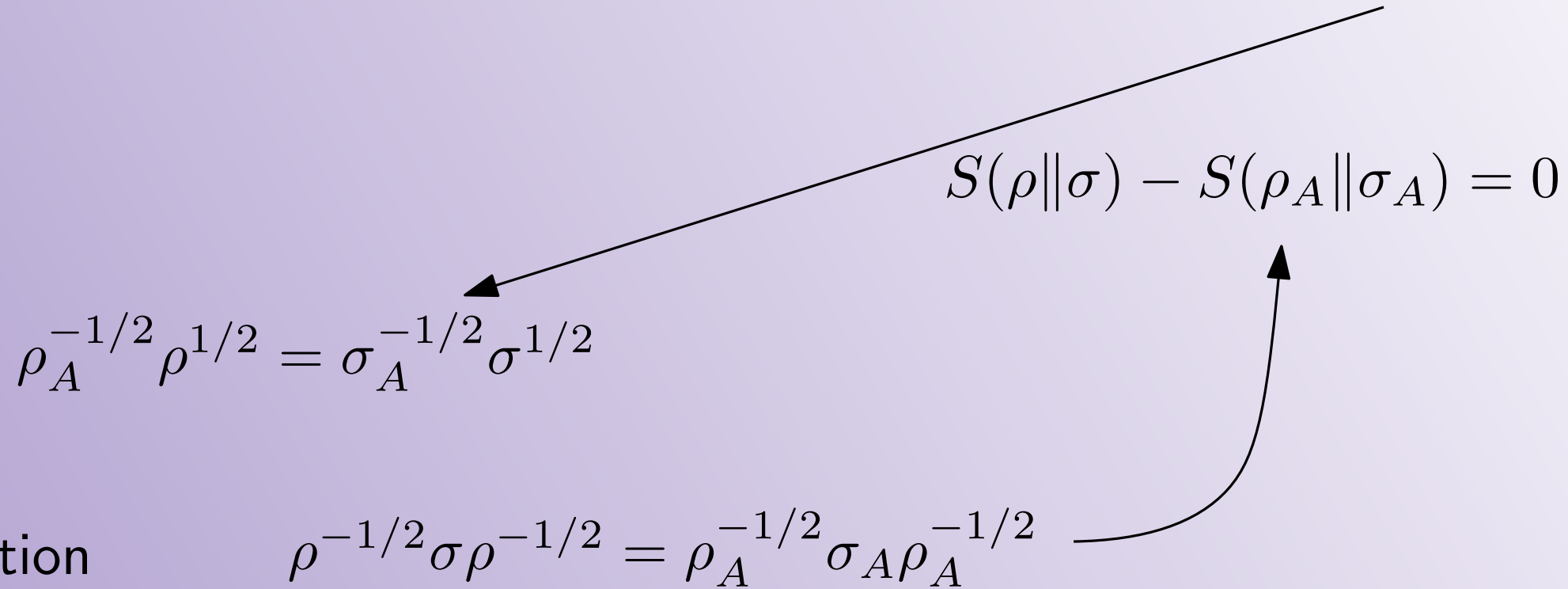
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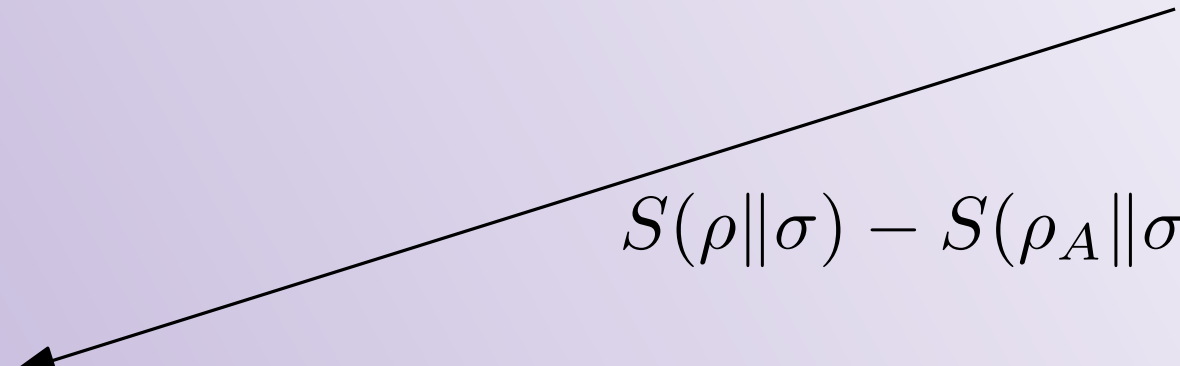
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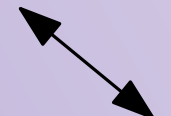
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symmetric

$$\rho_A^{-1/2} \rho^{1/2} = \sigma_A^{-1/2} \sigma^{1/2}$$

$$S(\rho\|\sigma) = S(\rho_A\|\sigma_A) \quad \text{if and only if} \quad S(\sigma\|\rho) = S(\sigma_A\|\rho_A)$$

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 operator norm

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Applying with

$$X = \sigma_A^{1/2} \rho_A^{-1/2} \rho^{1/2} \quad Y = \sigma^{1/2}$$

will get the lemma

Ideas of the proof

Quantum relative entropy $S(\rho\|\sigma) = \text{Tr}\{\rho(\log \rho - \log \sigma)\}$

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Quasi-relative entropy Let $f : (0, \infty) \rightarrow \mathbb{R}$ be operator convex for $n \in \mathbb{N}$
positive $n \times n$ matrices A and B
$$f\left(\frac{1}{2}A + \frac{1}{2}B\right) \leq \frac{1}{2}f(A) + \frac{1}{2}f(B)$$

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$$S_{-\log}(\rho\|\sigma) = S(\rho\|\sigma)$$

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$$S(\rho\|\sigma) = \int_0^\infty \left(S_{(t)}(\rho\|\sigma) - \frac{1}{1+t} \right) dt$$

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Monotonicity of quasi-relative entropy

Define

$$U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$$

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then

$$U^*(Y) = \text{Tr}(Y \rho^{1/2}) \rho_A^{-1/2}$$

$$\text{and } U^*U(X) = X$$

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Monotonicity of quasi-relative entropy

Define $U(X) = X_A \rho_A^{-1/2} \rho^{1/2}$ then $U^*(Y) = \text{Tr}(Y \rho^{1/2}) \rho_A^{-1/2}$ and $U^*U(X) = X$

After simple calculation $\Delta_{\sigma, \rho}^{1/2}(U(X_A)) = \sigma^{1/2} X_A \rho_A^{-1/2}$

Ideas of the proof

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Ideas of the proof

Consider

$$S_{(t)}(\rho\|\sigma) - S_{(t)}(\rho_A\|\sigma_A) = \langle \rho^{1/2}, (t + \Delta_{\sigma,\rho})^{-1} \rho^{1/2} \rangle - \langle \rho_A^{1/2}, (t + \Delta_{\sigma_A,\rho_A})^{-1} \rho_A^{1/2} \rangle$$

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where

$$w_t := U(t + \Delta_{\sigma_A,\rho_A})^{-1} \rho_A^{1/2} - (t + \Delta_{\sigma,\rho})^{-1} \rho^{1/2}$$

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$$X^{1/2} = \frac{1}{\pi} \int_0^\infty t^{1/2} \left(\frac{1}{t} - \frac{1}{t+X} \right) dt$$

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A diagram showing two arrows originating from the integral $\int_0^\infty$ in the equation above. One arrow points down and to the left towards the integral $\int_0^T$, and the other arrow points down and to the right towards the integral $\int_T^\infty$.

$$\int_0^T \qquad \qquad \int_T^\infty$$

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Cauchy-Schwarz

$$\left(\int_0^T t^{1/2} \|w_t\|_2 dt \right)^2 \leq T \int_0^T t \|w_t\|_2^2 dt$$

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Cauchy-Schwarz

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Note that for any positive X

$$\int_T^\infty t^{1/2} \left(\frac{1}{t} - \frac{1}{t+X} \right) dt \leq \int_T^\infty \frac{\|X\|}{t^{1/2}(\|X\|+t)} I dt$$

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Then by definition of w_t

$$\left\| \int_T^\infty t^{1/2} w_t dt \right\|_2 \leq \frac{4\|\Delta_{\sigma,\rho}\|}{T^{1/2}}$$

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Combining two parts we obtain

$$\|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2 \leq \frac{1}{\pi} T^{1/2} (S(\rho||\sigma) - S(\rho_A||\sigma_A))^{1/2} + \frac{4\|\Delta_{\sigma,\rho}\|}{\pi T^{1/2}} \quad \text{for all } T > 0$$

Ideas of the proof

Consider

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$$\|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2 = \frac{1}{\pi} \left\| \int_0^\infty t^{1/2} w_t dt \right\|_2$$

Combining two parts we obtain

$$\|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2 \leq \frac{1}{\pi} T^{1/2} (S(\rho||\sigma) - S(\rho_A||\sigma_A))^{1/2} + \frac{4\|\Delta_{\sigma,\rho}\|}{\pi T^{1/2}} \quad \text{for all } T > 0$$

Optimizing in T

$$\|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2 \leq \frac{4}{\pi} \|\Delta_{\sigma,\rho}\|^{1/2} (S(\rho||\sigma) - S(\rho_A||\sigma_A))^{1/4}$$

□

DPI for f -relative quasi-entropies

Let $f : (0, \infty) \rightarrow \mathbb{R}$ be operator convex

f -relative quasi-entropy is $S_f(\rho\|\sigma) = \text{Tr}(f(\Delta_{\sigma,\rho})\rho) = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho})\rho^{1/2} \rangle$

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relative modular operator

$$\Delta_{\sigma,\rho}(X) = \sigma X \rho^{-1}$$

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$$S_f(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) \leq S_f(\rho||\sigma)$$

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DPI for f -relative quasi-entropies

Recall, for regular quantum relative entropy we already have

$$S(\rho\|\sigma) - S(\rho_A\|\sigma_A) \geq \left(\frac{\pi}{4}\right)^4 \|\Delta_{\sigma,\rho}\|^{-2} \|\sigma_A^{1/2} \rho_A^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2^4$$

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Theorem (Carlen, V. '18)

For a broad class of operator monotone decreasing functions f , and any $\beta \in (0, 1)$, there is a parameter $c > 0$ (that depends on f and β), such that

$$\text{for } \beta \leq 1/2 \quad K (S_f(\rho\|\sigma) - S_f(\rho_A\|\sigma_A))^{\frac{\beta(1-\beta)}{1+2c(1-\beta)}} \geq \|\sigma_A^\beta \rho_A^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{\frac{1}{2}-\beta}\|_2$$

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powers agree for $\beta = 1/2$

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Moreover, for $\beta = 1/2$ we have

$$\|\mathcal{R}_\sigma(\rho_A) - \rho\|_1, \|\mathcal{R}_\rho(\sigma_A) - \sigma\|_1 \leq C \|\sigma_A^{1/2} \rho_A^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2$$

DPI for f -relative quasi-entropies

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Moreover, for $\beta = 1/2$ we have

$$\|\mathcal{R}_\sigma(\rho_A) - \rho\|_1, \|\mathcal{R}_\rho(\sigma_A) - \sigma\|_1 \leq C \|\sigma_A^{1/2} \rho_A^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2$$

Then

$$\max\{\|\mathcal{R}_\sigma(\rho_A) - \rho\|_1, \|\mathcal{R}_\rho(\sigma_A) - \sigma\|_1\} \leq K (S_f(\rho\|\sigma) - S_f(\rho_A\|\sigma_A))^{\frac{1}{4} \frac{1}{1+c}}$$

DPI for f -relative quasi-entropies

Consequently,

$$S_f(\rho\|\sigma) = S_f(\rho_A\|\sigma_A) \quad \text{if and only if} \quad \begin{array}{l} \sigma = \mathcal{R}_\rho(\sigma_A) \\ \rho = \mathcal{R}_\sigma(\rho_A) \end{array}$$

both hold

DPI for f -relative quasi-entropies

Consequently,

$$S_f(\rho||\sigma) = S_f(\rho_A||\sigma_A) \quad \text{if and only if} \quad \begin{array}{l} \sigma = \mathcal{R}_\rho(\sigma_A) \\ \rho = \mathcal{R}_\sigma(\rho_A) \end{array}$$

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Conversely, if $\sigma = \mathcal{R}_\rho(\sigma_A)$ then by monotonicity theorem

$$S_f(\rho||\sigma) = S_f(\mathcal{R}_\rho(\rho_A)||\mathcal{R}_\rho(\sigma_A)) \leq S_f(\rho_A||\sigma_A) \leq S_f(\rho||\sigma)$$

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therefore

$$S_f(\rho||\sigma) = S_f(\rho_A||\sigma_A)$$

equality



DPI for f -relative quasi-entropies

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$$S_f(\rho||\sigma) = S_f(\mathcal{R}_\rho(\rho_A)||\mathcal{R}_\rho(\sigma_A)) \leq S_f(\rho_A||\sigma_A) \leq S_f(\rho||\sigma)$$

therefore

$$S_f(\rho||\sigma) = S_f(\rho_A||\sigma_A)$$

and therefore $\rho = \mathcal{R}_\sigma(\rho_A)$

equality

DPI for f -relative quasi-entropies

For $\beta \in (0, 1)$

$$\|\sigma_A^\beta \rho_A^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{\frac{1}{2}-\beta}\|_2 \geq \|\rho^{-1/2}\|^{-1} \|\sigma_A^\beta \rho_A^{-\beta} - \sigma^\beta \rho^{-\beta}\|_2$$

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Then

$$S_f(\rho\|\sigma) = S_f(\rho_A\|\sigma_A) \quad \text{if and only if} \quad \sigma_A^\beta \rho_A^{-\beta} = \sigma^\beta \rho^{-\beta}$$

Logarithmic function

Taking $f(x) = -\log(x)$, i.e. obtaining the Umegaki relative entropy

For $\beta \leq 1/2$

$$(S(\rho\|\sigma) - S(\rho_A\|\sigma_A)) \geq K_\beta^L(\rho, \sigma) \|\sigma_A^\beta \rho_A^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{\frac{1}{2}-\beta}\|_2^{\frac{1}{\beta(1-\beta)}}$$

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if $\beta = 1/2$ $\frac{1}{\beta(1-\beta)} = 4$

as before

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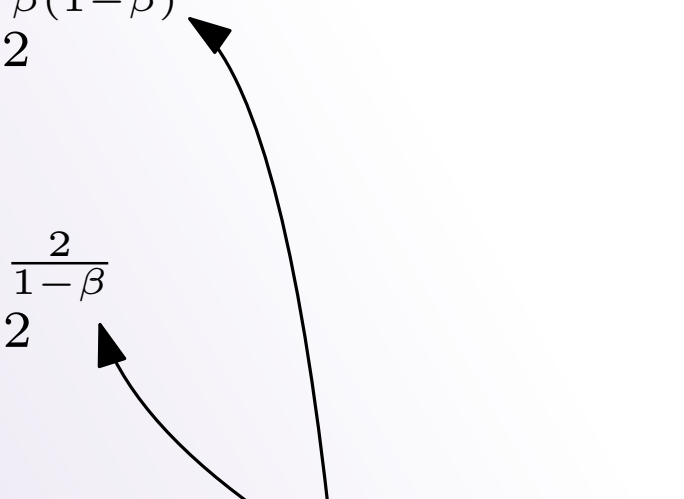
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equal to 4 for $\beta = 1/2$



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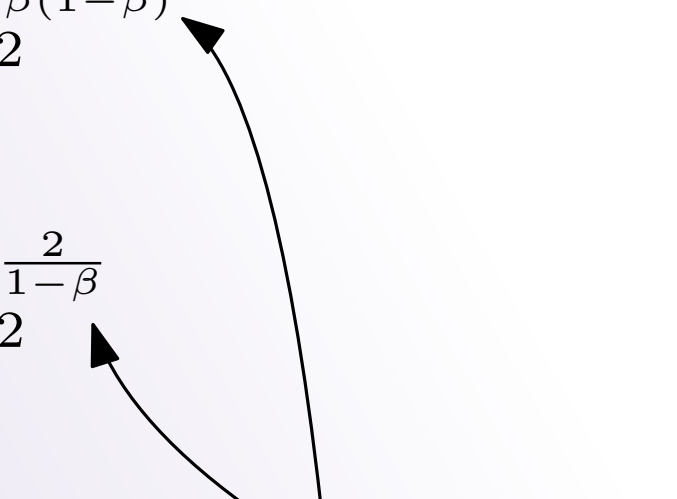
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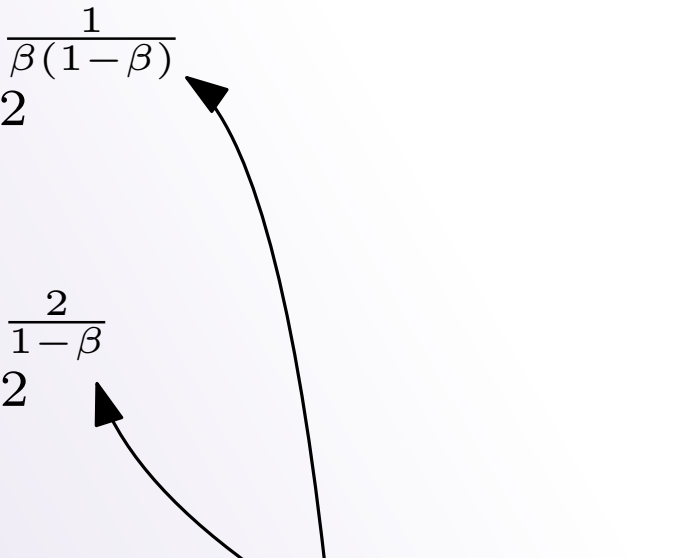
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Compare to the previous result

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Rényi entropy

For $\alpha \in (0, 1)$ Rényi entropy

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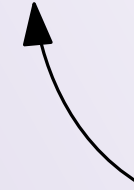
f -relative quasi-entropy for $f_{1-\alpha}(x) = x^{1-\alpha}$



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Power function

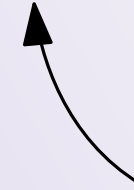
f -relative quasi-entropy for $f_{1-\alpha}(x) = x^{1-\alpha}$

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$$S_{f_\alpha}(\rho||\sigma) - S_{f_\alpha}(\rho_A||\sigma_A) \geq K_{\alpha,1/2}^U(\rho, \sigma) \|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2^{4+2\alpha}$$

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There are bounds for all $\beta \in (0, 1)$

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$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_A||\sigma_A) \geq \frac{1}{1-\alpha} \log \left(1 + K_{1-\alpha,1/2}^U(\rho, \sigma) \|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2^{4+2\alpha} \right)$$

Rényi entropy

If we know that the difference is small, say

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Then $e^{(1-\alpha)x} - 1 \leq \frac{e^{(1-\alpha)r} - 1}{r} x$

Theorem (Carlen, V. '18)

$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_A||\sigma_A) \geq \frac{1}{1-\alpha} \log \left(1 + K_{1-\alpha,1/2}^U(\rho, \sigma) \|\sigma_A^{1/2} \rho_A^{-1/2} \rho_{1/2} - \sigma^{1/2}\|_2^{4+2\alpha} \right)$$

Rényi entropy

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From theorem

$$\max\{\|\mathcal{R}_\rho(\sigma_{\mathcal{N}}) - \sigma\|_1, \|\mathcal{R}_\sigma(\rho_{\mathcal{N}}) - \rho\|_1\}^{6-2\alpha} \leq \frac{2\|\sigma^{-1}\|^{1/2}}{K_{1-\alpha}^U(\rho, \sigma)} (\exp[(1-\alpha)(S_\alpha(\rho||\sigma) - S_\alpha(\rho_{\mathcal{N}}||\sigma_{\mathcal{N}}))] - 1)$$

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Thank you!