The Maslov index and the spectrum of differential operators

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Maslov–Morse square

Consider a one-parameter family of self-adjoint operators with discrete spectra

$$\begin{cases} \mathcal{L}^s := -\mathrm{div} A^s \nabla + B^s \nabla - \nabla \cdot \overline{B^s} + q^s, \ s \in [\tau, 1], \ \text{in } L^2(\Omega), \\ s\text{-dependent self-adjoint boundary conditions} \end{cases}$$

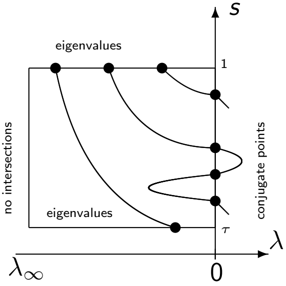

Figure: Maslov index = spectral flow (difference of Morse indices)

Applications include: Sturm Oscillation Theorem (for systems of ODE's); Explicit formula for the nodal deficiency for Dirichlet Laplacian – due to G. Cox, C. Jones, J. Marzuola; Eigenvalue Bracketing...

Minimal Operator, Green's Identity and Symplectic Form

Consider a formally self-adjoint differential expression

$$\mathcal{L} := -\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k + \sum_{j=1}^{n} a_j \partial_j - \partial_j \overline{a_j} + a.$$

- $L_{min} f := \mathcal{L} f,\ f \in \text{dom}(L) := H_0^2(\Omega)$ acting in $L^2(\Omega)$
- Green's Identity

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = \overline{\langle \gamma_N^{\mathcal{L}} v, \gamma_D u\rangle}_{-1/2} - \langle \gamma_N^{\mathcal{L}} u, \gamma_D v\rangle_{-1/2}$$

- $\text{Tr}_{\mathcal{L}} := (\gamma_D, \gamma_N^{\mathcal{L}}) : H^1_{\mathcal{L}}(\Omega) \to H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$
- Symplectic form

$$\omega : [H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)]^2 \to \mathbb{C},$$
$$\omega((f_1, g_1), (f_2, g_2)) = \overline{\langle g_2, f_1\rangle}_{-1/2} - \langle g_1, f_2\rangle_{-1/2}$$

- Green's identity

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = \omega\left(\text{Tr}_{\mathcal{L}}\, u, \text{Tr}_{\mathcal{L}}\, v\right),$$

S.A. Operators and Lagrangian Planes

Theorem (Latushkin, S.)

The **self-adjoint extensions** of $\mathcal{L}_{min}$ whose domains are contained in $H^1(\Omega)$ are in one-to-one correspondence with **Lagrangian planes** in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, that is, the following two assertions hold.

1. Let $\mathscr{D} \subset \mathcal{D}^1_{\mathcal{L}}(\Omega)$, and let $\mathcal{L}_{\mathscr{D}}$ be the linear operator acting in $L^2(\Omega)$ and given by

$$\mathcal{L}_{\mathscr{D}} f := \mathcal{L}_{max} f,\ f \in \mathrm{dom}(\mathcal{L}_{\mathscr{D}}) := \mathscr{D}.$$

If $\mathcal{L}_{\mathscr{D}}$ is **self-adjoint** then the set

$$\mathcal{G}_{\mathscr{D}} := \overline{\mathrm{Tr}_{\mathcal{L}}(\mathscr{D})}^{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}$$

is a **Lagrangian** plane in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, with respect to form ω.

2) If $\mathcal{G} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ is **Lagrangian** then

$$\mathcal{L}_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})} f := \mathcal{L}_{max} f,\ f \in \mathrm{dom}\left(\mathcal{L}_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})}\right) := \mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G}),$$

is essentially **self-adjoint**.

Family of Self-Adjoint extensions

Let us consider a family of differential expressions

$$\mathcal{L}^t := -\sum_{j,k=1}^{n} \partial_j a_{jk}^t \partial_k + \sum_{j=1}^{n} a_j^t \partial_j - \partial_j \overline{a_j^t} + a^t, \ t \in \mathcal{I},$$

$$\mathcal{L}_{min}^t f = \mathcal{L}^t f, \ f \in \mathrm{dom}(\mathcal{L}_{min}^t) := H_0^2(\Omega),$$

$$\mathcal{L}_{max}^t u := \mathcal{L}^t u, \ u \in \mathrm{dom}(\mathcal{L}_{max}^t) := \{u \in L^2(\Omega) : \mathcal{L}^t u \in L^2(\Omega)\}.$$

- Fix a family of self-adjoint extensions: $\mathcal{L}_{\mathscr{D}_t}^t u := \mathcal{L}^t u, \ u \in \mathrm{dom}\left(\mathcal{L}_{\mathscr{D}_t}^t\right) := \mathscr{D}_t$.

$$\mathcal{G}_t := \overline{\mathrm{Tr}_{\mathcal{L}^t}(\mathscr{D}_t)} \text{ is Lagrangian in } H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$$

$$\boxed{\lambda \in \mathrm{spec}(\mathcal{L}_{\mathscr{D}_t}^t) \iff \text{the planes have nonempty intersection}}$$

- the traces of weak solutions form Lagrangian plane in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$

$$\mathcal{K}_{\lambda,t} := \mathrm{Tr}_{\mathcal{L}^t}\left(\left\{w \in H^1(\Omega) : \mathcal{L}^t w - \lambda w = 0 \text{ weakly}\right\}\right)$$

- $\dim\left(\mathcal{K}_{\lambda,t} \cap \mathcal{G}_t\right) = \dim\ker\left(\mathcal{L}_{\mathscr{D}_t}^t - \lambda\right)$.

Main Result

Theorem (Latushkin, S.)

Let $\mathscr{D}_t \subset \mathcal{D}^1_{\mathcal{L}^t}(\Omega)$, $t \in \mathcal{I} := [\alpha, \beta]$, and assume that the linear operator $\mathcal{L}^t_{\mathscr{D}_t}$ acting in $L^2(\Omega)$ and given by

$$\mathcal{L}^t_{\mathscr{D}_t} u := \mathcal{L}^t u, \;\; u \in \operatorname{dom}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) := \mathscr{D}_t,$$

is self-adjoint with the property $\operatorname{spec}_{ess}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) \cap (-\infty, 0] = \emptyset$, for all $t \in \mathcal{I}$. Assume further that there exists $\lambda_\infty < 0$, such that

$$\ker\left(\mathcal{L}^t_{\mathscr{D}_t} - \lambda\right) = \{0\}, \;\text{for all}\; \lambda \le \lambda_\infty, t \in \mathcal{I}.$$

Suppose, finally, that the path $t \mapsto \mathcal{G}_t := \overline{\operatorname{Tr}_{\mathcal{L}^t}\left(\mathscr{D}_t\right)}$, $t \in \mathcal{I}$, is contained in $C\left(\mathcal{I}, \Lambda\left(H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\right)\right)$.
Then

$$\operatorname{Mor}\left(\mathcal{L}^\alpha_{\mathscr{D}_\alpha}\right) - \operatorname{Mor}\left(\mathcal{L}^\beta_{\mathscr{D}_\beta}\right) = \operatorname{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t \in \mathcal{I}}\right),$$

where $\mathcal{K}_{0,t} := (\gamma_D, \gamma_N^{\mathcal{L}^t})\Big(\big\{w \in H^1(\Omega) : \mathcal{L}^t w = 0 \text{ weakly}\big\}\Big)$, $t \in \mathcal{I}$.

Nodal Deficiency via Maslov index

Let $V \in L^\infty(\Omega)$ be a real-valued function, and denote $L := -\Delta_D + V$. Suppose φ_k is the $k-$th eigenfunction of L, with eigenvalue λ_k. The nodal domains of φ_k are the connected components of the set $\{\varphi_k \neq 0\}$. Denote the total number of nodal domains by $n(\varphi_k)$, and define the nodal deficiency $\delta(\varphi_k) = k - n(\varphi_k)$.

Theorem (G. Cox, C. Jones, J. Marzuola, 2017)

Suppose λ_k is a simple eigenvalue of L. Define $L(\epsilon) := L - (\lambda_k + \epsilon)$ and let $\Lambda_\pm(\epsilon)$ denote the corresponding Dirichlet-to-Neumann maps on $\Omega_\pm := \{\pm\varphi_k > 0\}$ If $\epsilon > 0$ is sufficiently small, then

$$\delta(\varphi_k) = \mathrm{Mor}(\Lambda_+(\epsilon) + \Lambda_-(\epsilon)).$$

In particular $\delta(\varphi_k) \geq 0$.

Sketch of proof of Cox–Jones–Marzuola Theorem

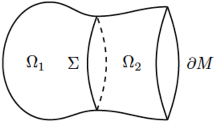

Illustration is taken from Cox–Jones–Marzuola's IUMJ paper

$$\mathrm{Mor}(L) = \mathrm{Mor}(L_1^D) + \mathrm{Mor}(L_2^D) + \mathrm{Mor}(\Lambda_1 + \Lambda_2)$$

Consider the Lagrangian planes

$$\mu(\lambda) := \left\{ \left(u_1, \frac{\partial u_1}{\partial \nu_1}, u_2, \frac{\partial u_2}{\partial \nu_2} \right) \restriction_\Sigma : u_i \text{ is weak solution to } Lu = \lambda u \text{ on } \Omega_i, i = 1, 2 \right\}$$

$$\beta(t) := \{(x, t\phi, tx, \phi) : x \in H^{1/2}(\Sigma), \phi \in H^{-1/2}(\Sigma)\}, \ t \in [0, 1]$$

Maslov Square:

$$t = 1 \ \rightsquigarrow \ \mathrm{Mor}(L);$$

$$t = 0 \ \rightsquigarrow \ \mathrm{Mor}(L_1^N) + \mathrm{Mor}(L_2^D)$$