Existence of ground states of translation-invariant Pauli-Fierz models

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joint work with David Hasler

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FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

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Fock space

Bosonic Fock space:

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{s}}((\mathbb{R}^3 \times \{1,2\})^n)$$

Vacuum state:

$$\Omega = (1, 0, 0, 0, \dots)$$

Creation and annihilation operators, satisfying CCR, $k, k' \in \mathbb{R}^3$, $\lambda, \lambda' \in \{1,2\}$:

$$[a_\lambda(k), a^*_{\lambda'}(k')] = \delta_{\lambda\lambda'}\delta(k-k'),\; [a_\lambda(k), a_{\lambda'}(k')] = [a^*_\lambda(k), a^*_{\lambda'}(k')] = 0$$

$$a_\lambda(k)\Omega = 0$$

Field energy (with 'photon mass' $m \geq 0$):

$$(H_{\mathrm{f},m}\psi)_n(k_1,\lambda_1,\dots,k_n,\lambda_n) = \left(\sum_{i=1}^{n}\sqrt{k_i^2+m^2}\right)\psi_n(k_1,\lambda_1,\dots,k_n,\lambda_n)$$

The Pauli-Fierz model of non-relativistic QED

Hilbert space: $\mathcal{H} = L^2(\mathbb{R}^3 \times \{1,2\}) \otimes \mathcal{F}$, $H_\mathrm{f} := H_{\mathrm{f},0}$

Pauli-Fierz Hamiltonian

$$H = \frac{1}{2}(-\mathrm{i}\nabla + eA(x))^2 + e\sigma B(x) + \mathrm{Id} \otimes H_\mathrm{f},$$

- σ_i, $i = 1, \ldots, 3$, are the Pauli matrices .
- $e > 0$ is the coupling constant (charge of the electron).
- Quantized vector potential:

$$A(x) = \sum_{\lambda=1}^{2} \int \frac{\rho(k)}{\sqrt{2|k|}} \varepsilon_\lambda(k) \left(e^{ik\cdot x} a_\lambda(k) + e^{-ik\cdot x} a_\lambda^*(k) \right) dk$$

$$B(x) = \nabla \wedge A(x)$$

ρ cutoff function, e.g. choose $\rho(k) = \mathbb{1}_{B_\Lambda(0)}(|k|)$, $\Lambda > 0$ UV cutoff, vectors $\varepsilon_i(k)$, i=1,2, are constructed in a measurable way such that $(k/|k|, \varepsilon_1(k), \varepsilon_2(k))$ form an ONB of $\mathbb{R}^3$.

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The fiber Hamiltonian

Translation-invariance $\Longrightarrow$ direct-integral decomposition

Field momentum operator:

$$(P_{\mathrm{f}}\psi)_n(k_1,\lambda_1,\dots,k_n,\lambda_n) = \left(\sum_{i=1}^{n} k_i\right)\psi_n(k_1,\lambda_1,\dots,k_n,\lambda_n).$$

Let $W = e^{\mathrm{i}xP_{\mathrm{f}}}$. Then $W(p+P_{\mathrm{f}})W^* = p$, and

$$WHW^* = (p - P_{\mathrm{f}} + eA(0))^2 + e\sigma B(0) + H_{\mathrm{f}}.$$

Let F be the Fourier transform in x. Then

$$FWHW^*F^* = \int_{\mathbb{R}^3}^{\oplus} H(\xi)\,\mathrm{d}\xi,$$

where

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Result for zero total momentum

- $H(\xi)$ self-adjoint on $D(H_\mathrm{f}) \cap D(P_\mathrm{f}^2)$ for all $e > 0$ and $\xi \in \mathbb{R}^3$ (Hiroshima '00).
- Let $E(\xi) = \inf \sigma(H(\xi))$. Then $E(\cdot)$ attains its minimum at $\xi = 0$.

Theorem (Ground state for zero momentum)

For all values of the coupling constant e, $E(0)$ is an eigenvalue of $H(0)$.

- For e infinitesimally small already shown (Bach, Chen, Fröhlich, Sigal '07),
- Useful for proving enhanced binding (Chen, Vougalter, Vugalter '03).

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Now $\xi \neq 0$. Infrared problem (too many soft photons)

- Ground state exists in another representation inequivalent to the Fock representation. (Pizzo '05, Chen, Fröhlich '07).
- For non-charged atoms shown for all ξ small enough (Loss, Miyaoh, Spohn '07).
- For every ξ where $\nabla E(\xi) \neq 0$, there is no ground state (Hasler, Herbst '08).
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$$H_m(\xi) = \frac{1}{2}(\xi - P_\mathrm{f} + eA(0))^2 + e\sigma B(0) + H_{\mathrm{f},m},$$

and

$$U_m = \exp(\mathrm{i}(a(f_{m,\xi}) - a^*(f_{m,\xi}))).$$

for a specific function $f_{m,\xi} \in L^2(\mathbb{R}^3 \times \{1,2\})$.

Theorem (non-zero momentum)

For almost all $\xi \in \mathbb{R}^3$ with $|\xi| < \frac{1}{2}(\sqrt{3} - 1)$ and for arbitrary e the following holds:

(a) For all $m > 0$, the operators $U_m H_m(\xi) U_m^*$ are self-adjoint and $U_m H_m(\xi) U_m^*$ converges to a self-adjoint operator $\hat{H}(\xi)$ in the strong resolvent sense as m tends to zero.

(b) $\hat{H}(\xi)$ has a ground state, i.e., $E(\xi)$ is an eigenvalue of $\hat{H}(\xi)$.

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Thank you for your attention!