On the Non-Relativistic Limit of Quantum Electrodynamics

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joint work with David Hasler

The Hilbert Space of QED

Hilbert space:

$$\mathcal{H} = \mathcal{F}_D \otimes \mathcal{F}_{ph} = \left(\bigoplus_{n=0}^{\infty} \otimes_a^n L^2(\mathbb{R}^3; \mathbb{C}^4)\right) \otimes \left(\bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2)\right)$$

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annihilation / creation operators: a/a^* on $\mathcal{F}_{ph}$ and F/F^* on $\mathcal{F}_D$

$$a(f) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} a_\lambda(k)\overline{f_\lambda(k)}\mathrm{d}k \qquad a^*(f) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} a_\lambda^*(k) f_\lambda(k)\mathrm{d}k$$

$$F(g) = \sum_{a=1}^{4} \int_{\mathbb{R}^3} F_a(p)\overline{g_a(p)}\mathrm{d}p \qquad F^*(g) = \sum_{a=1}^{4} \int_{\mathbb{R}^3} F_a^*(p) g_a(p)\mathrm{d}p$$

$$[a_\lambda(k), a_\mu(k')] = [a_\lambda^*(k), a_\mu^*(k')] = 0 \qquad [a_\lambda(k), a_\mu^*(k')] = \delta_{\lambda\mu}\delta(k-k')$$

$$\{F_a(p), F_b(q)\} = \{F_a^*(p), F_b^*(q)\} = 0 \quad \{F_a(p), F_b^*(q)\} = \delta_{ab}\delta(p-q).$$

Vacuum: $a(f)\Omega = F(g)\Omega = 0$

span$\{a^*(f_1)\cdots a^*(f_n)F^*(g_1)\cdots F^*(g_m)\Omega\}$ is dense in $\mathcal{H}$

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From now: $b(f) = F((f,0))$, $c(f) = F((0,f))$ for $f \in L^2(\mathbb{R};\mathbb{C}^2)$.

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Free Hamiltonian: $H_0 = \mathbb{1} \otimes H_{ph} + H_D \otimes \mathbb{1}$, where

$$H_{ph} = \int_{\mathbb{R}^3} |k| a^*(k) a(k) \mathrm{d}k$$

$$H_D = \int_{\mathbb{R}^3} E_M(p) \big(b^*(p) b(p) + c^*(p) c(p) \big) \mathrm{d}p, \quad E_M(p) = \sqrt{M^2 + p^2}$$

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Interaction Terms: H_1 photon fermion interaction, H_2 Coulomb interaction

$$H_1 = \int_{\mathbb{R}^3} \psi^\dagger(x) (\alpha \cdot A(x)) \psi(x) \, \chi_1(x) \mathrm{d}x$$

$$H_2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y)}{|x-y|} \, \chi_2(x)\chi_2(y) \mathrm{d}x \mathrm{d}y$$

Dirac matrices: $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

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Field Operators: A photon field on $\mathcal{F}_{ph}$, ψ Dirac field on $\mathcal{F}_D$

$$\begin{aligned} A(x) &= a(f_x) + a^*(f_x) & \qquad f_x(k) &= \chi_{ph}(k)(2|k|)^{-1/2}\mathrm{e}^{-\mathrm{i}kx}\varepsilon(k) \\ \psi(x) &= b(g_x) + c^*(h_x) & \qquad g_x(p) &= \chi_D(p)(2E_M(p))^{-1/2}\mathrm{e}^{-\mathrm{i}px}u_+^\dagger(p) \\ & & h_x(p) &= \chi_D(p)(2E_M(p))^{-1/2}\mathrm{e}^{-\mathrm{i}px}u_-(p) \\ & & (\pm\alpha\cdot p + \beta M)u_\pm(p) &= \pm E_M(p)u_\pm(p) \end{aligned}$$

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$$(\pm\alpha \cdot p + \beta M)u_\pm(p) = \pm E_M(p)u_\pm(p)$$

For simplicity: $\chi_{1,2}$ compactly supported, $\chi_{ph/D} = \chi_{\{|p|<\Lambda_{ph/D}\}}$.

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Some Previous Work on Relativistic QED

Chaix, Iracane 1989, Hainzl, Lewin, Solovej 2006
Hainzl, Siedentop 2003
Balaban 1985, Dimock 2015
Takaesu 2009

The Basic Idea

Fermion/Antifermion Number: $N_+ = \int b^*(p)b(p)\mathrm{d}p$, $N_- = \int c^*(p)c(p)\mathrm{d}p$
Charge Operator: $Q = N_+ - N_-$
Charge Conservancy: $[Q, H] = 0$, i.e., $\mathcal{H} = \bigotimes_{q\in\mathbb{Z}} \mathcal{F}_q$

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Pauli-Fierz Hamiltonian of non-relativistic QED: $L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes \mathcal{F}_{ph}$

$$H_{\mathrm{PF}} = H_{ph} + (p - eA(x))^2 + \sigma \cdot B(x), \quad \text{where } p = -\mathrm{i}\nabla,\ B = p \wedge A$$

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Previous results on related problems

Hunziker 1974
Arai 2001
Stockmeyer 2009

Main Result

We restrict to the charge $Q = 1$ sector and denote $:H_M: \upharpoonright \mathbb{1}_{Q=1}\mathcal{H}$ by H.
Further define $P = \mathbb{1}_{N_+=1,N_-=0}$, $\overline{P} = \mathbb{1}_{Q=1}(1-P)$.
There is a canonical unitary isomorphism $U : L^2(\mathbb{R}^3; \mathbb{C}^2) \to P\mathcal{F}_D$.
Let $A_\chi = \chi_1 A$ and $A^{(\mathrm{r})} = \chi_D \otimes A_\chi$.

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Theorem

Let $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z - M| < M$ and $|e| < \sqrt{M}$.
Then $(H - z)^{-1} = S_z \left(H_\infty + e^2(I_z + T_z) - z\right)^{-1} S_{\overline{z}}^* + (\overline{P}(H - z)\overline{P})^{-1}$,
where

$$S_z = (P - (\overline{P}H\overline{P} - z)^{-1}\overline{P}HP)U,$$

$$H_\infty = H_0 + \frac{e}{2M}(p \cdot A^{(\mathrm{r})}(x) + A^{(\mathrm{r})}(x) \cdot p + \sigma \cdot B^{(\mathrm{r})}(x)),$$

$$I_z(q', q) = \int_{\mathbb{R}^3} \widehat{A_\chi}(p + q)\frac{\chi_D(p)\chi_D(q)\chi_D(q')}{H_0 + E_M(p) + E_M(q) + E_M(q') - z}\widehat{A_\chi}(p + q')\mathrm{d}p,$$

$$T_z = \mathbb{1} \otimes t_z + \mathcal{O}(M^{-2}).$$

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- With $\tilde{P} = \mathbb{1}_{H_{ph} \le M}$ we find
  $\tilde{P} I_z \tilde{P} = \dfrac{1}{2M} \displaystyle\int_{\mathbb{R}^3 \times \mathbb{R}^3} A^{(\mathrm{r})}(x) \cdot A^{(\mathrm{r})}(y) \delta_\chi(x-y) \mathrm{d}x \mathrm{d}y + \mathcal{O}(M^{-2})$.

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- Here we imposed normal ordering.
  H_2 is usually not normal ordered, but has integral kernel

$$\frac{:\psi^\dagger(x)\psi(x)::\psi^\dagger(y)\psi(y):}{|x-y|}.$$

This yields a finite self-interaction term for the electron.

Idea of Proof

Decompose $A = \begin{pmatrix} PAP & PA\overline{P} \\ \overline{P}AP & \overline{P}A\overline{P} \end{pmatrix}$.

Feshbach Complement:

$$A^{-1} = \begin{pmatrix} P & 0 \\ -D\overline{P}AP & \overline{P} \end{pmatrix} \begin{pmatrix} F_P(A)^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} P & -PA\overline{P}D \\ 0 & \overline{P} \end{pmatrix},$$

where $F_P(A) = PAP - PA\overline{P}D\overline{P}AP$, $D = (\overline{P}A\overline{P})^{-1}$.

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Using this, the proof essentially consists of two steps.

1. Prove existence and upper bound for $(\overline{P}(H-z)\overline{P})^{-1}$.
2. Show $F_P(H-z) = H_\infty + T_z + I_z - z + \mathcal{O}(M^{-2})$.

Thank you for your attention!