Bounding Mixing Time of Glauber Dynamics

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Outline

Introduction

- Consider the space $\mathcal{X} = \{0,1\}^n$ as the space of vertex weight configurations on n vertices and for $X \in \mathcal{X}$, $X(i)$ the *spin* at vertex i.
- Vertex spins i.i.d. where $X(i) = 1$ with prob. p and 0 with prob. $(1-p)$.
- H_1: number of edges in X and H_2: number of triangles in X.
- Define a Gibbs probability measure on $\mathcal{X}$ by

$$\pi(X) = Z^{-1} \exp\left(H(X)\right) p^{f(X)} (1-p)^{n-f(X)} \tag{1}$$

where $H = (\alpha_1/n)H_1 + (\alpha_2/n^2)H_2$.

Markov Chain

- Given this Gibbs measure, we consider single-site Glauber Dynamics of the Markov chain $(X_t)_{t=0}^{\infty}$ on $\mathcal{X}$.
- For a spin configuration X, define $c(X) = (1/n)\sum_{i=1}^{n} X(i)$. $(c_t)_{t=0}^{\infty}$ is a projection of $(X_t)_{t=0}^{\infty}$.
- If $c_0 = c$, after one Glauber update, c_1 could be $c - 1/n$, c, or $c + 1/n$.
- The asymptotic probability that a chosen vertex is updated to spin 1 is

$$\lambda(c) = \frac{p\exp\left(\alpha_1 c + \frac{\alpha_2}{2}c^2\right)}{p\exp\left(\alpha_1 c + \frac{\alpha_2}{2}c^2\right) + (1-p)} \tag{2}$$

High Temperature Phase

- The magnetization chain is a deciding factor in the convergence of the Glauber dynamics for (X_t).
- $0 \leq c_t \leq 1$ and λ smooth and increasing.
- $\lambda(0) > 0$ and $\lambda(1) < 1$, so $\lambda(c) = c$ admits at least one solution in $(0, 1)$.
- The *high temperature phase* occurs when the fixed point is unique and not an inflection point.

Lambda Plot

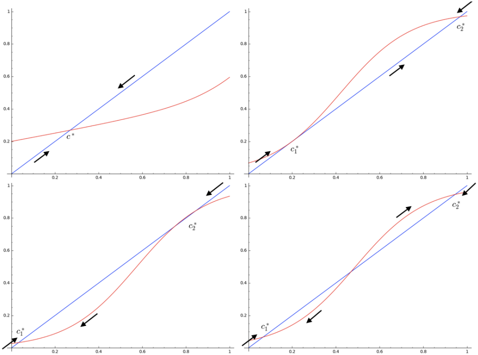

Figure: Behavior of the λ function in different regions of the parameter space with arrows indicating whether the fixed point is an attractor or a repellor.

Mixing Time

Definition 1

Given $\epsilon > 0$, the *mixing time*, $t_{\text{mix}}(\varepsilon)$, for the Markov chain $\{X_t\}_{t=0}^{\infty}$ is defined as

$$t_{\text{mix}}(\varepsilon) := \min\{t : d(t) \leq \varepsilon\} \tag{3}$$

where

$$d(t) = \max_{X \in \mathcal{X}} \left\| P^t(X, \cdot) - \pi \right\|_{\text{TV}} \tag{4}$$

and

$$\|\mu - \nu\|_{\text{TV}} = \max_{A \subseteq \mathcal{X}} |\mu(A) - \nu(A)|. \tag{5}$$

By convention, $t_{\text{mix}} := t_{\text{mix}}(1/4)$.

Lower Bound on Mixing Time

Theorem 2

In the high temperature phase, the mixing time for the Glauber Dynamics is $O(n \log n)$.

Theorem 3

In the high temperature phase, the mixing time for the Glauber dynamics is $\Omega(n \log n)$.

- Let $f(X)$ count the number of spin 1 vertices for $X \in \mathcal{X}$ and γ be the spectral gap for the Glauber dynamics transition matrix, $P(X, Y)$.

Proof

- It follows that for the function f,

$$\gamma \leq \frac{\varepsilon(f)}{\mathrm{Var}_\pi(f)}$$

where $\varepsilon(f)$ is the Dirichlet form

$$\varepsilon(f) = \frac{1}{2} \sum_{X,Y \in \mathcal{X}} (f(X) - f(Y))^2 \, \pi(X) P(X,Y)$$

and the variance under the stationary distribution

$$\mathrm{Var}_\pi(f) = \sum_{X \in \mathcal{X}} (f(X))^2 \, \pi(X) - \left(\sum_{X \in \mathcal{X}} f(X)\pi(X) \right)^2 .$$

Proof

- By the single site Glauber Dynamics, for configuration $X, Y \in \mathcal{X}$, $P(X,Y) = 0$ unless X and Y differ at at most one vertex. Thus

$$\varepsilon(f) \leq \frac{1}{2} \sum_{X,Y \in \mathcal{X}} \pi(X) P(X,Y) = \frac{1}{2} \sum_{X \in \mathcal{X}} \pi(X) = \frac{1}{2}.$$

- Thus $2\mathrm{Var}_\pi(f) \leq 1/\gamma$ and using spectral methods,

$$\log 2 \left(2\mathrm{Var}_\pi(f) - 1\right) \leq \log 2 \left(\frac{1}{\gamma} - 1\right) \leq t_{\mathrm{mix}}.$$

- Since $t_{\mathrm{mix}} = O(n \log n)$, $\mathrm{Var}_\pi(f) = O(n \log n)$.

Proof

- Let X be a configuration chosen according to π.

$$\pi\left(|f(X) - \mathbb{E}_\pi(f(X))| > n^{2/3}\right) \leq O(n^{-1/3}\log n)$$

$$\implies \pi\left(|f(X) - \mathbb{E}_\pi(f(X))| \leq n^{2/3}\right) \longrightarrow 1$$

- Let $X^+, X^- \in \mathcal{X}$ such that X^+ is all 1 and X^- is all 0, and that the two are coupled using monotone coupling. So $X_t^+ \geq X_t^-$ for all t.

Proof

- R_{t^*} be the number of vertices not yet selected to update by time $t^* = (1/4)n\log n$. By coupon collecting, $\mathbb{E}(R_{t^*}) \asymp n^{3/4}$ and $\mathrm{Var}(R_{t^*}) \leq \mathbb{E}(R_{t^*})$.
- Let $\varepsilon > 0$ and by Chebyshev, asymptotically,

$$\mathbb{P}\left(|R_{t^*} - \mathbb{E}\left(R_{t^*}\right)| > (1-\varepsilon)\mathbb{E}\left(R_{t^*}\right)\right) \leq O(n^{-3/4}).$$

- Thus

$$\mathbb{P}\left(R_{t^*} < \varepsilon\mathbb{E}\left(R_{t^*}\right)\right) \leq O(n^{-3/4})$$

and so $R_{t^*} = \Omega(n^{3/4})$ with probability 1 asymptotically.

Proof

- Since $f(X_{t^*}^+) - f(X_{t^*}^-) \geq R_{t^*}$ it follows that $f(X_{t^*}^+) - f(X_{t^*}^-) = \Omega(n^{3/4})$ with probability 1 asymp.
- Define the sets A and B such that

$$A := \left\{ \left| f(X_{t^*}^+) - \mathbb{E}_\pi(f(X)) \right| \leq n^{2/3} \right\}$$

$$B := \left\{ \left| f(X_{t^*}^-) - \mathbb{E}_\pi(f(X)) \right| \leq n^{2/3} \right\}$$

- By triangle inequality, $A \cap B$ satisfies $f(X_{t^*}^+) - f(X_{t^*}^-) = O(n^{2/3})$ if nonempty. Since $n^{3/4} > n^{2/3}$, this is a contradiction and so $A \cap B = \varnothing$ asymptotically.

Proof

- Since A and B asymp. disjoint,

$$\mathbb{P}\left(\left|f(X_{t^*}^{+}) - \mathbb{E}_{\pi}(f(X))\right| \leq n^{2/3}\right) \leq \frac{1}{2} - o(1).$$

- By definition of total variation distance,

$$\begin{aligned}\left\|P^{t^*}(X^+, \cdot) - \pi\right\|_{\mathrm{TV}} &\geq \left|P^{t^*}(X^+, A) - \pi(A)\right| \\ &\geq 1 - \frac{1}{2} + o(1)\end{aligned}$$

- Contradiction. Mixing time is asymptotically bigger than $t^* = (1/4)n\log n$.

Questions?

Thank you very much for having me!