Derivation of the dipolar Gross-Pitaevskii energy

Arizona School of Analysis and Mathematical Physics
Arnaud Triay

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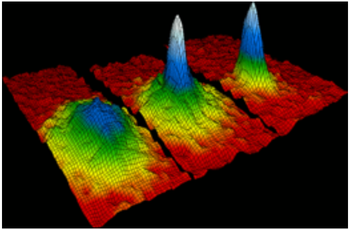
- Bosonic particles (He4, Rb, Cr,...)
- Ultra cold, dilute, *weak interaction*
- Condensation : "$\Psi(x_1, ..., x_N) \simeq \varphi(x_1)...\varphi(x_N)$"



(Cornell & Wieman 1995)

History

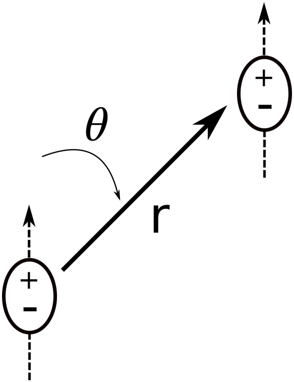
- Bose (1924) Einstein (1925)
- Cornell & Wieman (1995), Nobel prize (2001)

- A vast literature : **[LieYng-98 ; LieSeiYng-00] [NamRouSer-15][LewNamSerSol-13] [BocBreCenSch-18] [ErdSchYau-06] [Pickl-11] [ChePav-14]** ...

Dipolar (or dipole-dipole) interaction

$$K(x) = \frac{1 - 3\cos^2(\theta)}{r^3}$$

- Large distance approximation
- Dipoles aligned



Two aligned dipoles

New features

- Stable / unstable regime
- Roton-Maxon form of the Bogoliubov spectrum
- Can be self-trapped
- ...

The many body setting

$$H_N = \sum_{j=1}^{N} -\Delta_j + V(x_j) + \frac{1}{N} \sum_{1 \leq i < j \leq N} N^{3\beta} w(N^{\beta}(x_i - x_j)).$$

on $\bigotimes_s^N L^2(\mathbb{R}^3)$

$-\Delta$: kinetic energy, V : confining (trapping potential $V(x) \to \infty$ as $|x| \to \infty$)

$\boxed{w = w_0 + b\mathbb{1}_{|x|>R} K}$ with $w_0 \in L^1$ (pair interaction)

$\frac{1}{N}$: mean-field parameter

Interpolation between two regimes ($0 \leq \beta \leq 1$)

- $\beta = 0$ Hartree regime (weak collisions but often) (easy)
- $\beta = 1$ Gross-Pitaevskii regime (strong but rare collisions, TL limit) (difficult)

Effective range of interaction : $N^{-\beta}$
Mean distance between particles : $N^{-1/3}$

The (dipolar) Gross-Pitaevskii equation

Dipolar-GP functional

$a, b \in \mathbb{R}$, $K(x) = (1 - 3\cos^2(\theta))/r^3$, define

$$\mathcal{E}_{GP}^{a,b}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V|u|^2 + \frac{a}{2}\int_{\mathbb{R}^3} |u|^4 + \frac{b}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} (K \star |u|^2)|u|^2$$

- Ground state energy $e_{GP} := \inf_{\substack{u \in H^1(\mathbb{R}^3) \\ \|u\|_{L^2}=1}} \mathcal{E}_{GP}(u)$

Theorem (Bao-Cai-Wang, Carles-Hajaiej)

$e_{GP} > -\infty$ and has a unique minimizer if and only if

- $a > b4\pi/3$ *if* $b > 0$
- $a > -b8\pi/3$ *if* $b < 0$

Results

Theorem

Let $w = w_0 + b\mathbb{1}_{|x|>R}K$ *with* $w_0 \in L^1$, $V(x) \geq |x|^s$.
If $\widehat{w} \geq 0$ *and* $0 < \beta < 1/3 + s/(45+42s)$ *then (denoting* $a = \int w_0$*)*

$$\inf_{\|\Psi_N\|_2=1} \frac{\langle \Psi_N | H_N | \Psi_N \rangle}{N} \underset{N\to\infty}{\longrightarrow} \inf_{\substack{u\in H^1(\mathbb{R}^3)\\ \|u\|_{L^2}=1}} \mathcal{E}_{GP}^{a,b}(u),$$

$$\mathrm{tr}_{k+1\to N}\left(|\Psi_{N,0}\rangle\langle\Psi_{N,0}|\right) \underset{N\to\infty}{\longrightarrow} |u_0^{\otimes k}\rangle\langle u_0^{\otimes k}|, \quad \forall k \geq 1.$$

Proof

Techniques from [**NamRouSei-15** ; **LewNamRou-15b**],
Main tool : quantitative De Finetti theorem in finite dimension

$$\text{"}\ \mathrm{tr}_{k+1\to N}\left(|\Psi_N\rangle\langle\Psi_N|\right) \simeq \int_{SP\mathfrak{H}} |u^{\otimes k}\rangle\langle u^{\otimes k}|\, d\mu_\Psi(u)\text{"}$$

- difficulty : w has no sign, not compactly supported, non radial, not in L^1.
- $\beta_{\max} > 1/3$: stability relies on the kinetic energy

What remains to do ?

- Should be true up to $\beta < 1$
- For $\beta = 1$ the scattering process plays an important role $a \neq \int w_0$ but is the scattering length
- Study the next order (Bogoliubov excitation spectrum). Already done for $w \geq 0$ [**BocBreCenSch-18**].
- Time dependent problem $\Psi_{N,0} \simeq u_0^N \implies \Psi_{N,t} \simeq u_0^N$. Ok for $\beta < 1/6$ with Pickl's method.

Thank you for your attention !