Localization, and resonant delocalization, of a disordered polaron

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Localization of a Disordered Polaron

What is *many body localization*?

- Good question…
- "Fock space localization."
- Not many mathematical results
    - Droplet models
    - Imbry spin chain – conditional result
- **This talk does not directly address MBL**
    - But it is the context…
    - And the results have **Fock space localization**

(One Particle) Holstein Model

- A single particle on $\Lambda \subset \mathbb{Z}^d$
- A harmonic oscillator at each lattice site
- Particle interacts with the oscillator at the site it occupies

$$H_{\mathrm{Hol}}^{(\Lambda)} = \gamma \Delta^{(\Lambda)} + \omega (b_{\mathbf{X}}^\dagger - \beta^*)(b_{\mathbf{X}} - \beta) + \omega \sum_{\substack{x \in \Lambda \\ x \neq \mathbf{X}}} b_x^\dagger b_x$$

Holstein Model (1959)

$$H_{\mathrm{Hol}}^{(\Lambda)} = \gamma \Delta^{(\Lambda)} + \omega (b_{\mathbf{X}}^\dagger - \beta^*)(b_{\mathbf{X}} - \beta) + \omega \sum_{\substack{x \in \Lambda \\ x \neq \mathbf{X}}} b_x^\dagger b_x$$

- $\mathcal{H}_\Lambda \;=\; \{\psi : \Lambda \to \mathcal{F}_\Lambda\} \;=\; \ell^2(\Lambda; \mathcal{F}_\Lambda)$
- $\Delta^{(\Lambda)}\psi(x) \;=\; \sum_{\substack{x \sim y \\ y \in \Lambda}} \psi(x) - \psi(y)$
- $\boldsymbol{X}\psi(x) \;=\; x\psi(x)$
- ω large → "Polaron" with $m > 0$
- ω small: the picture is much less clear

Disordered Holstein

$$H_\gamma^{(\Lambda)} := H_{\mathrm{Hol}}^{(\Lambda)} + V^{(\Lambda)}$$

$$V^{(\Lambda)}\psi(x) = v_x \psi(x)$$

- $\{v_x\}_{x \in \mathbb{Z}^d}$ i.i.d. (uniform, say) in $[0, V_+]$
- $H_\gamma^{(\Lambda)}$ is non-negative definite.
- Spectrum is contained in

$$\bigcup_n [\omega n, \omega n + V_+ + 4d\gamma]$$

Theorem (Mavi & S., 2017)

For each n there is γ_n such that if $\gamma < \gamma_n$, then matrix elements of the resolvent with energies in the n-th band decay exponentially in a metric on the particle position and a basis for Fock space.

$$H_\gamma^{(\Lambda)} := H_{\mathrm{Hol}}^{(\Lambda)} + V^{(\Lambda)}$$

$$V^{(\Lambda)}\psi(x) = v_x \psi(x)$$

$$H_{\mathrm{Hol}}^{(\Lambda)} = \gamma\Delta^{(\Lambda)} + \omega(b_{\mathbf{X}}^\dagger - \beta^*)(b_{\mathbf{X}} - \beta) + \omega \sum_{\substack{x\in\Lambda \\ x\neq \mathbf{X}}} b_x^\dagger b_x$$

Displacement Operators

- Consider the Hilbert space for a single oscillator:

$$\mathcal{H} = \text{span}\{|n\rangle \mid |n = 0, 2, \ldots\}$$

- Let $D_\beta = e^{\beta b^\dagger - \beta^* b}$
- This operator is **unitary** and **intertwines** the eigenbasis for $b^\dagger b$ with that for $(b^\dagger - \beta^*)(b - \beta)$:

$$(b^\dagger - \beta^*)(b - \beta) D_\beta |m\rangle = D_\beta b^\dagger b |m\rangle = m D_\beta |m\rangle$$

Displacement Operator Bounds

Proposition: *Let* $\mu > 0$. *Then there is a finite constant* $A = A_{\mu,\beta}$ *such that*

$$|\langle m| D_\beta |n\rangle| \leqslant A e^{-\mu|\sqrt{n}-\sqrt{m}|}$$

- This result may be known, but we haven't found it in the literature.
- It follows from the following remarkable identity

$$\sum_m e^{2\mu(m-n)} |\langle m| D_\beta |n\rangle|^2 = e^{(e^{2\mu}-1)|\beta|^2} L_n\left(-|\beta|^2 (e^{\mu} - e^{-\mu})^2\right)$$

where L_n is the n-th order Laguerre polynomial.

Transforming the basis

$$H_0^{(\Lambda)} = \omega H_{\mathrm{ph}}^{(\Lambda)} + V^{(\Lambda)}$$

$$H_{\mathrm{ph}}^{(\Lambda)} = \sum_{x\in\Lambda} a_x^\dagger a_x \qquad a_x = b_x - \beta I[\boldsymbol{X} = x]$$

$$D_\beta^{(x)} |\boldsymbol{m}\rangle := \mathrm{e}^{\beta b_x^\dagger - \beta^* b_x} |\boldsymbol{m}\rangle$$

$$\boxed{\begin{aligned} H_0^{(\Lambda)} |x, \boldsymbol{m}\rangle &= (\omega|\boldsymbol{m}| + v_x) |x, \boldsymbol{m}\rangle \\ |x, \boldsymbol{m}\rangle &:= |x\rangle \otimes D_\beta^{(x)} |\boldsymbol{m}\rangle \end{aligned}}$$

Kinetic Operator

$$\langle x, \boldsymbol{m}| \, \Delta^{(\Lambda)} \, |y, \boldsymbol{\xi}\rangle$$

$$= \begin{cases} 2d & x = y \,\&\, \boldsymbol{m} = \boldsymbol{\xi} \\ \langle \boldsymbol{m}(x)| \, D_{-\beta} \, |\boldsymbol{\xi}(\boldsymbol{x})\rangle \, \langle \boldsymbol{m}(y)| \, D_{\beta} \, |\boldsymbol{\xi}(\boldsymbol{y})\rangle & x \sim y \,\&\, \boldsymbol{m}(u) = \boldsymbol{\xi}(u) \text{ for } u \neq x, y \\ 0 & \text{otherwise.} \end{cases}$$

- Matrix elements decay off the diagonal

This leads to Combes-Thomas type estimate for

$$H_\gamma^{(\Lambda)} = H_0^{(\Lambda)} + \gamma \Delta^{(\Lambda)}$$

in this basis.

Theorem (Mavi & S., 2017)

For each n there is γ_n such that if $\gamma < \gamma_n$, then

$$\mathbb{E}\left(|G(x,\boldsymbol{m},y,\boldsymbol{\xi})|^s\right) \leqslant A\mathrm{e}^{-s\nu D(x,\boldsymbol{m},y,\boldsymbol{\xi})}\mathrm{e}^{-\mu s|\sqrt{\boldsymbol{m}}-\sqrt{\boldsymbol{\xi}}|}$$

for energies in the n-th band where $\upsilon, \mu > 0$, $s < 1$, and $D(x, \mathbf{m}; y, \boldsymbol{\xi})$ is the larger of

1. $|x - y|$,
2. The distance from x to a site u where $\mathbf{m}(u) > 0$ and $\mathbf{m}(u) \neq \boldsymbol{\xi}(u)$,
3. The above with $(x, \mathbf{m}) \leftrightarrow (y, \boldsymbol{\xi})$.

Sketch of the proof

- Consider first the lowest band ($n = 0$).
- A state $|x, \boldsymbol{m}\rangle$ has on-site energy above the band, unless the oscillators are all in their ground state.
    - The oscillator at x must be in its **deformed** ground state.
- We use a fractional moment method.
- The Combes-Thomas bound is used to control contributions from the higher bands.

Sketch of the Proof (2)

- Now consider the second band.
- In this band, one of the oscillators can be in it's first excited state.
- In order for this excited oscillator to move, the particle must visit the excited site. **This leads to extra decay if the oscillator states differ in the Green's function.**
- Higher bands are handled inductively by a similar, but more complicated, argument.

Perspectives and Comments

- This is **NOT** a fully many body problem.
- But the Hilbert space is a many body Hilbert space.
- Dynamical localization **for the particle** follows from known methods

Dynamical Localization

- In the localized part of the spectrum, the particle does not propagate:

$$\sup_t \left\langle P_n\psi_t \ , \ \mathrm{e}^{\mu|X|} P_n\psi_t \right\rangle \ < \ \infty$$

  - P_n = projection onto the first n bands
  - $\psi_t = \mathrm{e}^{-\mathrm{i}tH_\gamma^{(\Lambda)}}\psi_0$

We do not prove (and don't believe?) that oscillator excitations are localized.


And now for something completely different

Resonant Delocalization in Toy Models

Resolvent localization does not imply dynamical localization

- Define $H_S = \begin{pmatrix} H_A & 0 \\ 0 & H_A \end{pmatrix} + g \begin{pmatrix} 0 & |0\rangle\langle 0| \\ |0\rangle\langle 0| & 0 \end{pmatrix}$ with H_A the Anderson model on $\mathbb{Z}^d$.
    - The *same potential* on two copies of $\mathbb{Z}^d$.
- Fractional moments decay:
$$\mathbb{E}\left(\left|\left\langle x,\sigma \left| \frac{1}{H_S - E} \right| y,\tau \right\rangle\right|^s\right) \le C\mathrm{e}^{-\mu d_\Gamma(x,\sigma;y,\tau)}$$
with d_Γ the graph metric on $\mathbb{Z}^d \oplus \mathbb{Z}^d$ with a connection at the origin.
- However the best we can prove for dynamical localization is
$$\mathbb{E}\left(\left|\left\langle x,\sigma \middle| \mathrm{e}^{itH_S} \middle| y,\tau \right\rangle\right|\right) \le C\mathrm{e}^{-\mu|x-y|}$$

Theorem (Mavi & S., 2017)

With probability 1, there is a sequence of vectors $\varphi_k \in \ell^2(\mathbb{Z}^d)$ with localization centers x_k diverging to ∞ such that

1. For all $t < \mathrm{e}^{\mu|x_k|}$ we have
$$\left|\langle \varphi_k, +1 \middle| \mathrm{e}^{-itH_S} \middle| \varphi_k, +1 \rangle\right| \geq 1 - \mathrm{e}^{-\mu|x_k|}$$
2. There is a time t_1 such that
$$\left|\langle \varphi_k, -1 \middle| \mathrm{e}^{-it_1 H_S} \middle| \varphi_k, -1 \rangle\right| \geq 1 - \mathrm{e}^{-\mu|x_k|}$$
3. For all times t
$$\left\| 1_{\left\{|x| > \frac{|x_k|}{2}\right\}} \mathrm{e}^{-itH_S} |\varphi_k, +1\rangle \right\| \geq 1 - \mathrm{e}^{-\mu|x_k|}$$

A model with AC spectrum

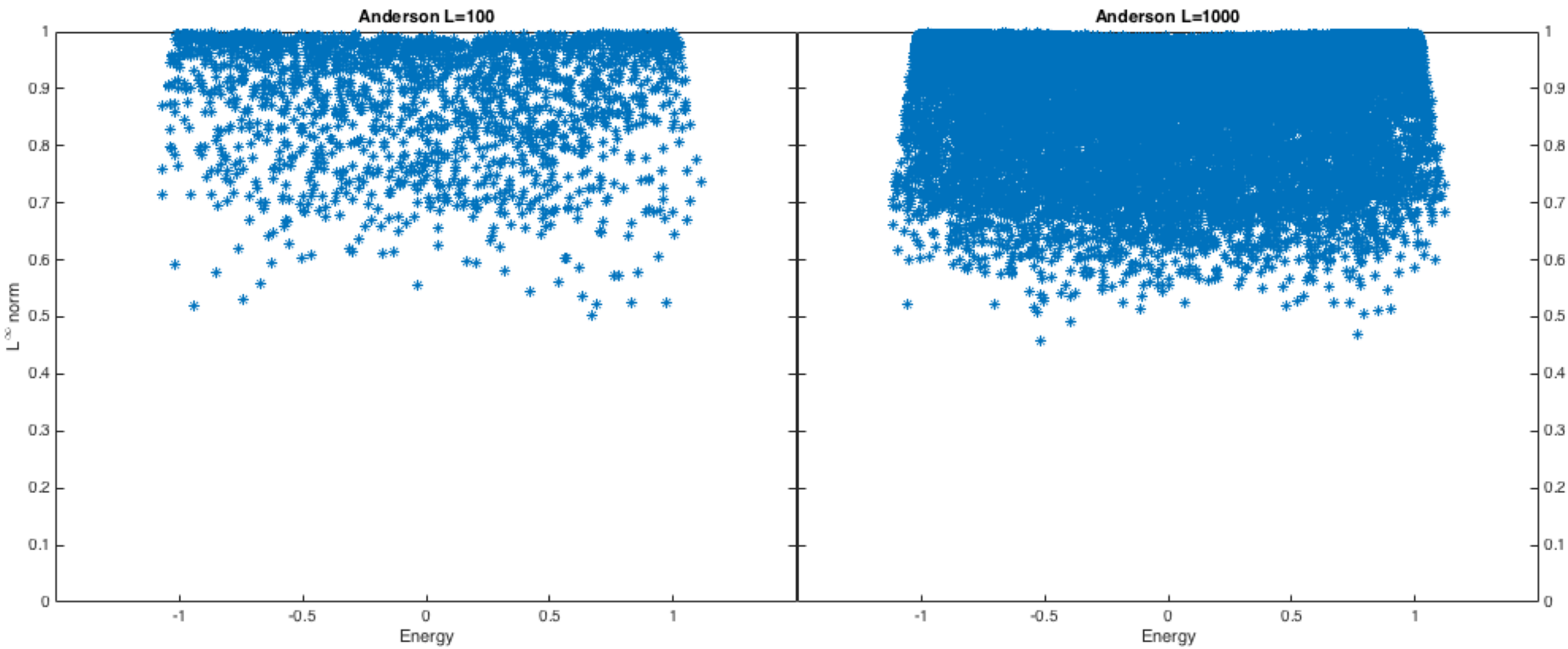
- Extend the above to infinitely many copies of $\mathbb{Z}^d$, all coupled at the origin.
- The resulting model has AC spectrum
- This is **resonant delocalization.**

An Ergodic Model

- Now connect the lines by connecting site n of the n-th line to site $n+1$ of the $n+1$-st line.
- **Thm.** (Matos, Mavi, S., in progress): positive Lyapunov exponent.
- Probably localized….?

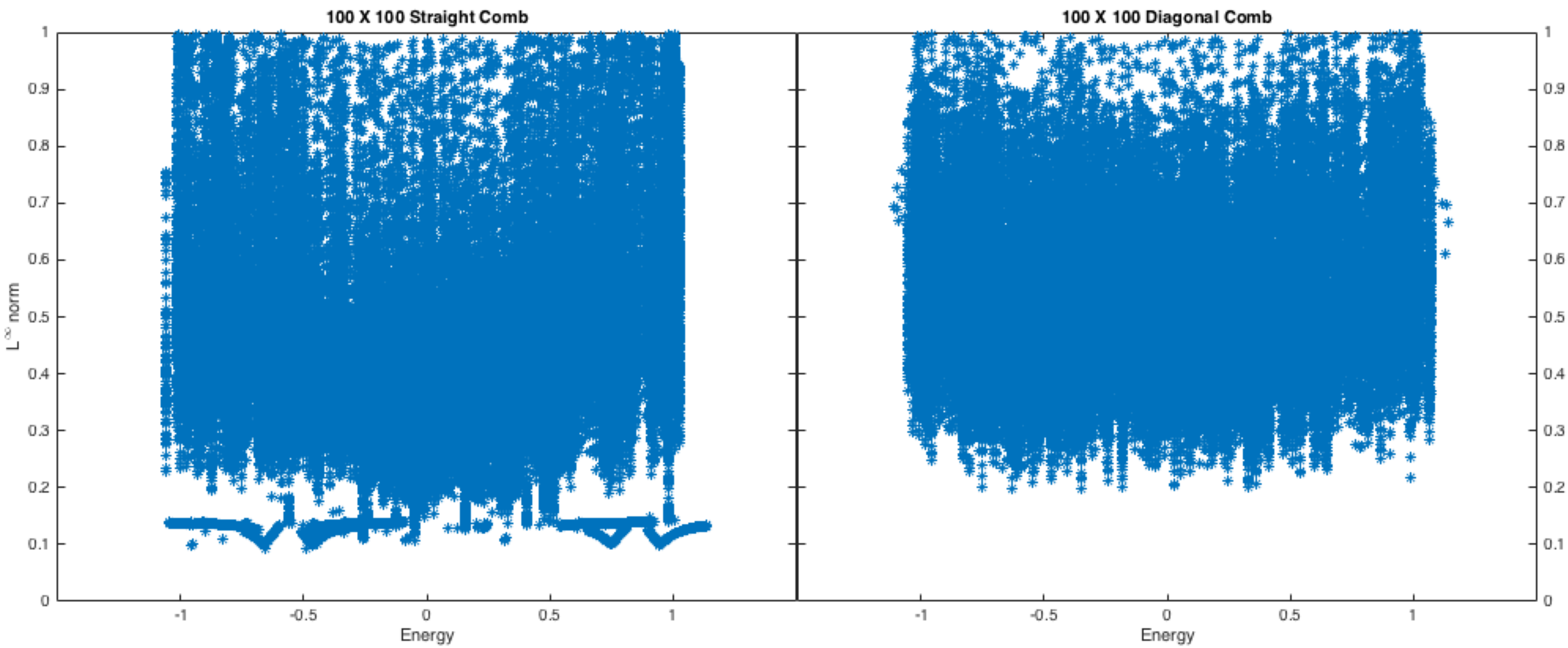
ℓ^∞ norm of Anderson model eigenvectors

disorder strength=10, L=100, L=1000



ℓ^∞ norm of Anderson Comb eigenvectors

disorder strength=10, L=100



Open Problems

- Do the oscillators localize?
- Howe does critical hopping strength on band number?
    - Current estimates are super exponential, but certainly not optimal
- Does randomizing the oscillator frequencies lead to full dynamical localization?
- Does complete localization hold in 1D or for weak hopping?

- What about positive energy density?
  - Do we even need on-site randomness?
- What about a multi particle Holstein model?
  - Finitely many particles could be doable *a la* Aizenman, Warzel or Sukhov, Chulaensky
- What about the *many particle Holstein model?*

THANK YOU!