Geometric/Bernoulli Growth Process from Schur-Processes

Axel Saenz

University of Virginia

Arizona School of Analysis and Mathematical Physics

In 1986 [2], M. Kadar G. Parisi and Y. Zhang (KPZ) introduced a Stochastic Partial Differential Equation

$$\frac{\partial}{\partial t}h(x,t) = \mu \frac{\partial^2}{\partial x^2}h(x,t) + \nu \left(\frac{\partial}{\partial x}h(x,t)\right)^2 + W.$$

Solutions to the KPZ locally look like Brownian motion, meaning that it locally has a $1/2$ space-time scaling, but globally it has a $2/3$ space-time scaling. This appears to be a general phenomenon for a certain class of stochastic processes, which are said to belong to the KPZ Universality class. At the current state of affairs, the KPZ Universality class is only understood via specific examples and there is no rigorous characterization of it.
The Asymmetric Simple Exclusion Process (ASEP) model on the line is a specific example that lies in the KPZ Universality class. We consider four models (mixed geometric/Bernoulli TASEP, coupled geometric/Bernoulli TASEP, DFPP, HSS6V) and show that their asymptotic behavior is consistent with other models in the KPZ universality class.

We consider two totally asymmetric version of the ASEP (i.e. TASEP) model: *geometric* and *Bernoulli*. The configuration space for both processes with N particles is

$$\mathcal{X}_N = \{(x_1, \dots, x_N) \in \mathbb{Z}^N | x_N < \cdots < x_1\},$$

and we denote the configuration at time $t \in \mathbb{Z}_{\geq 0}$ by $\mathbf{x}(t)$ where we use the wedge initial conditions with $x_j(0) = -j$.

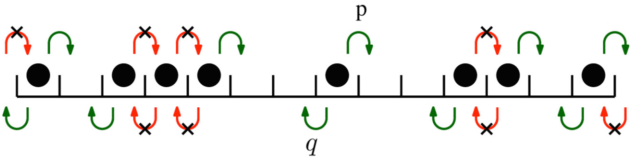

Figure: ASEP on a line.

The Height Function

The height function is a (convenient) representation of the location of the particles on the line. Simply put, the height function is the integral of the indicator function of the location of the particles:

$$h_{ASEP}(x,t) := \int_o^x 1 - 2\mathbb{1}(s)ds.$$

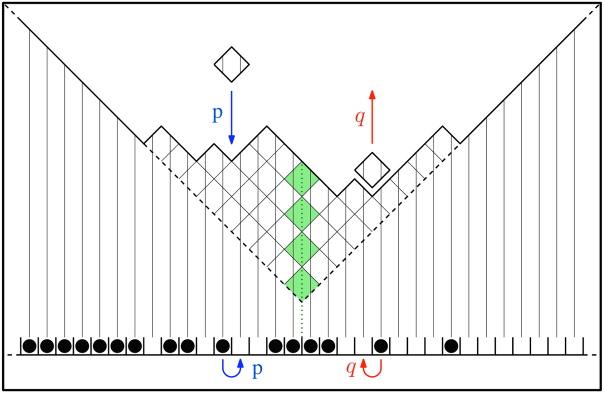

Figure: Random Growth.

Moreover, in [1], G. Amir I. Corwin and J. Quastel showed that the limit of the height function for ASEP on a line

$$h(x,t) := \lim_{\epsilon \to 0} \lambda_\epsilon h_{ASEP}(\epsilon^{-1}x, \epsilon^{-3/2}t) - \nu_\epsilon \epsilon^{-1/2} t$$

with $p - q = \epsilon^{1/2}$ and

$$\nu_\epsilon = p + q - 2\sqrt{qp} = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O(\epsilon^3)$$
$$\lambda_\epsilon = \frac{1}{2}\log(q/p) = \epsilon^{1/2} + \frac{1}{3}\epsilon^{3/2} + O(\epsilon^{5/2})$$

is a solution to the KPZ equation.

The geometric TASEP is a Markov process whose Markov matrix we denote by $\mathbf{G}_\alpha$ where $\alpha \in (0,1)$. The transition probabilities are give as follows

$$\begin{aligned}\mathbb{P}(x_i(t+1) = x_i(t) + k|\mathbf{x}(t)) \\ &= \alpha^k(1-\alpha)\mathbb{1}(k < x_{i-1}(t) - x_i(t) - 1) \\ &+ \alpha^{x_{i-1}(t)-x_i(t)-1}\mathbb{1}(k = x_{i-1}(t) - x_i(t) - 1).\end{aligned}$$

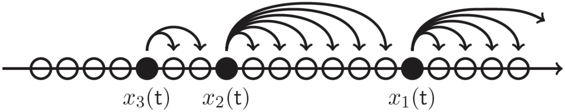

Figure: Update Rule

The geometric TASEP is a Markov process whose Markov matrix we denote by $\mathbf{B}_\beta$ where $\beta \in (0, \infty)$. The transition probabilities are give as follows

$$\mathbb{P}(x_i(t+1) = x_i(t) + 1 | \mathbf{x}(t)) = \frac{\beta}{1+\beta} \mathbb{1}(x_{i-1}(t+1) - x_i(t) > 1),$$

$$\mathbb{P}(x_i(t+1) = x_i(t) | \mathbf{x}(t)) = 1 - \frac{\beta}{1+\beta} \mathbb{1}(x_{i-1}(t+1) - x_i(t) > 1).$$

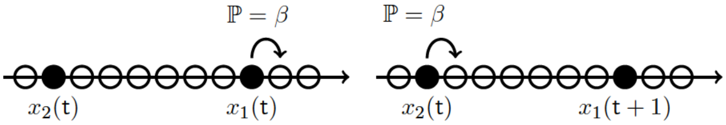

Figure: Update Rule

Lemma (Ref. [4])

For any $\alpha, \beta \in (0,1)$, we have that the Markov (transfer) matrices of the Geometric and Bernoulli TASEPS commute: $\mathbf{G}_\alpha \mathbf{B}_\beta = \mathbf{B}_\beta \mathbf{G}_\alpha$.

Remark

One can show that the matrices $\mathbf{G}_\alpha$ and $\mathbf{B}_\beta$ are diagonalized in the same basis. Ultimately it follows from the Yang-Baxter equation for the inhomogeneous stochastic higher spin six vertex model.

Definition

Let $\alpha, \beta/(1+\beta) \in (0,1)$. The *mixed geometric/Bernoulli* TASEP with parameter (α, β) is a Markov process on particle configurations in the integer lattice $\mathbb{Z}$ where the distribution of a particle configuration $\mathbf{x}(M, T)$ is given by applying the geometric TASEP with parameter α update rule M times and applying the Bernoulli TASEP with parameter β update rue T times.

Proposition (A. Knizel, L. Petrov, A.S.)

For fixed κ, η, τ, there exist a χ (which depends on the parameters algebraically) such that $x_{\kappa t}(\eta t, \tau t)/t \to \chi$ in probability as $t \to \infty$.

$$\lim_{t\to\infty} \mathbb{P}\left(\left|\frac{x_{\kappa t}(\eta t, \tau t) - \chi t}{t}\right| < \epsilon\right) = 1$$

Moreover, the fluctuation of $x_{\kappa t}(\eta t, \tau t)$ around it's limiting location obeys the Tracy-Widom distribution in the infinite time limit

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_{\kappa t}(\eta t, \tau t) - \chi t}{\sigma \gamma^{2/3} t^{1/3}} < s\right) = \det(I - K_{Airy})_{L_2(s,\infty)} = F_2(s).$$

with σ and γ constants depending on the parametrs α and β.

(a) $\beta = 1/10$, $N = 1000$

(b) $\beta = 25$, $N = 1000$

Figure: Bernoulli TASEP Simulations

Figure: Geometric TASEP, $\alpha = 3/4$, $N = 600$

Figure: $\alpha = 3/4, \beta = 1$, $T = 500$, $M = 500$, $N = 800$

We can give a rational parametrization $(x(z,t), h(x(z,t),t))$ of the limit shape for all of the models where $z \in (0,1)$. In particular, we have

$$\frac{x(z,t)}{t} = \frac{-1}{(\alpha z-1)^2(\beta z+1)^2}\Big(\alpha\eta + \beta\tau + \alpha^2\beta^2\eta z^4 - \alpha^2\beta^2\tau z^4 + 2\alpha^2\beta\eta z^3 - 2\alpha^2\beta\tau z^3 - 2\alpha\beta^2\eta z^3 + 2\alpha\beta^2\tau z^3 + \alpha^2\beta\tau z^2 + \alpha^2\eta z^2 + \alpha\beta^2\eta z^2 - 4\alpha\beta\eta z^2 + 4\alpha\beta\tau z^2 - \beta^2\tau z^2 + 2\alpha\beta\eta z - 2\alpha\beta\tau z - 2\alpha\eta z - 2\beta\tau z\Big)$$

$$\frac{h(x(z,t),t)}{t} = \frac{1}{(\alpha z-1)^2(\beta z+1)^2}\Big(\alpha^2\left(-z^2\right)\left(\eta(\beta z+1)^2 - \beta\tau\left((\beta+2)z^2 - 2z + 1\right)\right) + \alpha\left(\eta\left(2z^2-2z+1\right)(\beta z+1)^2 - 2\beta\tau z\left((\beta+2)z^2-2z+1\right)\right) + \beta\tau\left((\beta+2)z^2-2z+1\right)\Big),$$

where $M = \eta t$ and $T = \tau t$.

(a) $\beta = 1/10$, $N = 1000$

(b) $\beta = 25$, $N = 1000$

(c) $\alpha = 3/4$, $N = 600$

(d) $\alpha = 3/4$, $\beta = 1$, $T = 500$, $M = 500$, $N = 800$

Thank you for your attention!

References I

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