Discrete Spectrum of δ' Interaction Supported by Non-closed Manifolds

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Outline

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Introduction

- The operator can be formally written as

$$H = -\Delta + \beta^{-1}\delta'(\cdot - \Gamma)(\delta'(\cdot - \Gamma), \tilde{\cdot})$$

- Attractive δ' coupling $\beta(x) < 0$
- Γ is a manifold of codimension 1
- We are interested in the spectral properties of these operators

- Models with singular interactions are exactly solvable
- Suitable approximation for various systems
- Motion of a quantum particle on a graph-like structures

δ'-interaction on a line

- Operator acts as
$$-\Delta_{\beta,0} = -\Delta$$
with the domain
$$\mathcal{D}(-\Delta_{\beta,0}) = \{\psi \in \mathcal{H}^2(\mathbb{R} \setminus \{0\}) | \beta\psi'(0_+) = \beta\psi'(0_-) = \psi(0_+) - \psi(0_-)\}$$
- Corresponding quadratic form is
$$q(\psi) = (\nabla\psi, \nabla\psi) + \beta|\psi(0_+) - \psi(0_-)|^2$$
with the domain
$$\mathcal{D}(q) = \mathcal{H}^1(\mathbb{R} \setminus \{0\})$$
- Attractive δ'-interactions $\beta < 0$
- Ground state eigenvalue $-\frac{4}{\beta^2}$
- Essential spectrum $\sigma_{ess}(-\Delta_{\beta,0}) = [0, \infty)$

Definition of the operator-non-closed manifold

- the symmetric sesqulinear form

$$\mathfrak{a}^{\Sigma}_{\delta',\beta}[f,g] := (\nabla f_{\mathrm{i}}, \nabla g_{\mathrm{i}})_{\mathrm{i}} + (\nabla f_{\mathrm{e}}, \nabla g_{\mathrm{e}})_{\mathrm{e}} + \beta^{-1}(f_{\mathrm{e}}|_{\Sigma} - f_{\mathrm{i}}|_{\Sigma}, g_{\mathrm{e}}|_{\Sigma} - g_{\mathrm{i}}|_{\Sigma})_{\Sigma},$$
$$\mathrm{dom}\, \mathfrak{a}^{\Sigma}_{\delta',\beta} := \mathcal{H}^1(\Omega_{\mathrm{e}}) \oplus \mathcal{H}^1(\Omega_{\mathrm{i}}),$$

  is closed, densely defined and lower-semibounded in $L^2(\mathbb{R}^d)$
- linear mapping

$$\Lambda \colon \mathcal{H}^1(\Omega_{\mathrm{e}}) \oplus \mathcal{H}^1(\Omega_{\mathrm{i}}) \to L^2(\Sigma \setminus \Gamma), \qquad \Lambda f := f_{\mathrm{e}}|_{\Sigma\setminus\Gamma} - f_{\mathrm{i}}|_{\Sigma\setminus\Gamma},$$

- symmetric, densely defined and lower-semibounded form

$$\mathfrak{a}^{\Gamma}_{\delta',\beta}[f,g] := \mathfrak{a}^{\Sigma}_{\delta',\beta}[f,g], \qquad \mathrm{dom}\, \mathfrak{a}^{\Gamma}_{\delta',\beta} := \{f \in \mathrm{dom}\, \mathfrak{a}^{\Sigma}_{\delta',\beta} : \Lambda f = 0\}.$$

- the self-adjoint operator $-\Delta_{\Gamma,\beta}$ in $L^2(\mathbb{R}^d)$ is induced by the form $\mathfrak{a}^{\Gamma}_{\delta',\beta}$

δ'-interaction on a loop

- auxiliary self-adjoint Schrödinger operator $T_{d,\beta}$

$$\begin{aligned} T_{d,\beta}\psi &= -\psi'' \\ \mathcal{D}(T_{d,\beta}) &= \{\psi \in \mathcal{H}^2(0,d) \colon \psi'(0) = \psi'(d), \\ &\qquad \psi(0_+) - \psi(d_-) = \beta\psi'(d)\} \end{aligned}$$

- associated with the following form

$$\begin{aligned} &\mathfrak{t}_{d,\beta}[f,g] := (f',g')_{L^2(0,d)} + \beta^{-1}(f(d_-) - f(0_+))\overline{(g(d_-) - g(0_+))}, \\ &\mathcal{D}(\mathfrak{t}_{d,\beta}) := \mathcal{H}^1(0,d). \end{aligned}$$

Lemma

If $\beta < -d$, then the above self-adjoint operator $T_{d,\beta}$ is non-negative.

Absence of the discrete spectrum-results

Monotone piecewise C^1 curve

The curve $\Gamma \subset \mathbb{R}^2$ is monotone piecewise C^1 if it is given by the parametrization

$$\gamma(r) = x_0 + (r\cos(\phi(r)), r\sin(\phi(r))) ,$$

where $x_0 \in \mathbb{R}^2$, $r \in (0, R)$ with $R \in \mathbb{R}^+$ and mapping $\phi(r)$ is piecewise C^1.

Theorem

Let a curve $\Gamma \subset \mathbb{R}^2$ be monotone piecewise-C^1. Then

$$\sigma(-\Delta_{\Gamma,\beta}) \subseteq \mathbb{R}^+ \quad \text{for} \quad \beta(r) \leq -2\pi r\sqrt{1+(r\phi'(r))^2}\,, \qquad r \in (0, R)\,. \tag{1}$$

Furthermore if $\mathbb{R}^2 \setminus \Gamma$ is quasi-conical, then $\sigma(-\Delta_{\Gamma,\beta}) = \mathbb{R}^+$.

Absence of negative eigenvalues for non-closed curves-revisited

- (cm1)Γ is a compact piecewise C^1 smooth curve,
- (cm2) Γ is obtainable by a conformal mapping M from a curve $\tilde{\Gamma}$. The conformal map M transforms the unit disc D to a different subset of $\mathbb{C}$. Furthermore the preimage $\tilde{\Gamma}$ of the curve Γ is a monotone curve parametrized as

$$\tilde{\Gamma}(r) = (r\cos\phi(r), r\sin\phi(r)) \quad \text{for} \quad r \in (0,1)$$

with the endpoints at the boundary of the unit disc D and the center of the disc D.

Theorem

Let the curve $\Gamma \subset \mathbb{R}^2$ satisfy (cm1) and (cm2). Then $\sigma(-\Delta_{\Gamma,\beta}) = [0,\infty)$ for all $\beta(z) \leq -2\pi r\sqrt{1+(r\phi'(r))^2}\sqrt{J_M(z)}$ where r is related to $\tilde{\Gamma}$.

Circle arc

Theorem

Let a curve Γ *be parametrized as*

$$\Gamma_\epsilon = \{(R\cos\phi, R(1-\sin\phi))|\phi \in (\epsilon, 2\pi-\epsilon)\}, \quad R \in \mathbb{R}^+.$$

Then the operator $-\Delta_{\Gamma_\epsilon,\beta}$ *is positive if* $\beta \le -\frac{8\pi R}{\tan\frac{\epsilon}{2}}$.

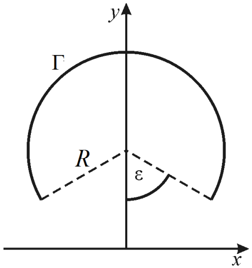

Figure: Sketch of a non-closed segment of the circle in the coordinate system

Line segment

Theorem

Let a curve Γ *be a line segment of length* L. *Then the operator* $-\Delta_{\Gamma,\beta}$ *has no negative eigenvalues if* $\beta \leq -\pi L$. *Furthermore if* $-\frac{2L}{\pi} \leq \beta \leq 0$ *then* $\sigma_d(-\Delta_{\Gamma,\beta}) \cap \mathbb{R}^- \neq \emptyset$.

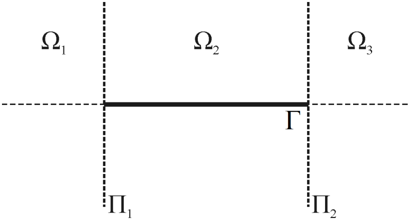

Figure: Separation of $\mathbb{R}^2$ into three regions Ω_k

Absence of negative eigenvalues for non-closed hypersurface

Key inequality

Let Γ be a non-closed compact Lipschitz manifold of the codimension 1. Then there exists a constant $C = C(\Gamma) > 0$ such that

$$\|(\psi_{\Gamma_+} - \psi_{\Gamma_-})\|^2_{L^2(\Gamma)} \leq C\|\nabla\psi\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^n)}$$

holds for any $\psi \in H^1(\mathbb{R}^n \setminus \Gamma)$.

Theorem

Let Γ be a non-closed compact Lipschitz manifold of the codimension 1. Then there exists β_0 such that the operator $-\Delta_{\Gamma,\beta}$ is positive as long as $\beta < \beta_0$. Furthermore the spectrum is $\sigma(-\Delta_{\Gamma,\beta}) = \sigma_{ess}(-\Delta_{\Gamma,\beta}) = \mathbb{R}^+$.

Summary

- Definition of δ'-interaction supported by a compact non-closed manifold
- Behavior of the point spectrum governed by a behavior of a δ'-interaction on a circle
- Absence of the negative eigenvalues state for sufficiently small coupling for non-closed hypersurface
- Explicit sufficient condition for the case of δ'-interaction supported by non-closed curves

References

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Thank you for your attention