Droplet states in the XXZ model on general graphs

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Symmetric graph products

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected (possibly countably infinite) graph. For any $N \in \mathbb{N}_0, N \leq |\mathcal{V}|$, define the N-th symmetric graph product $\mathcal{G}_N$ of $\mathcal{G}$, to be the graph $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$ with vertex set

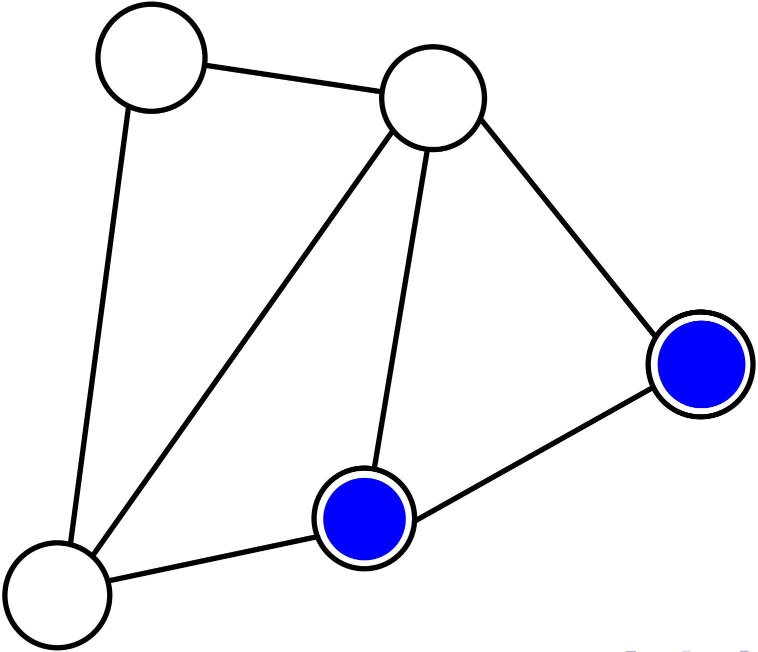
$$\mathcal{V}_N = \{X \subset \mathcal{V} : |X| = N\}$$

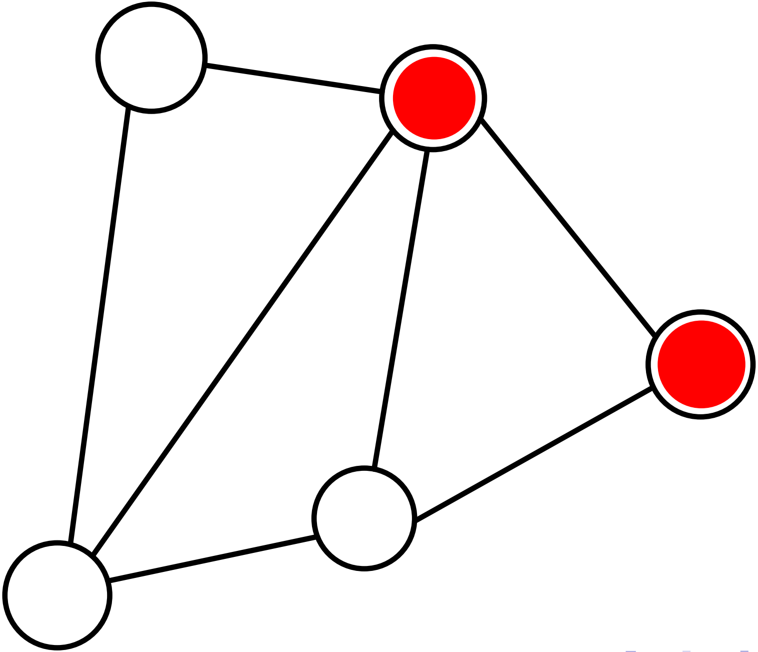
and edge set

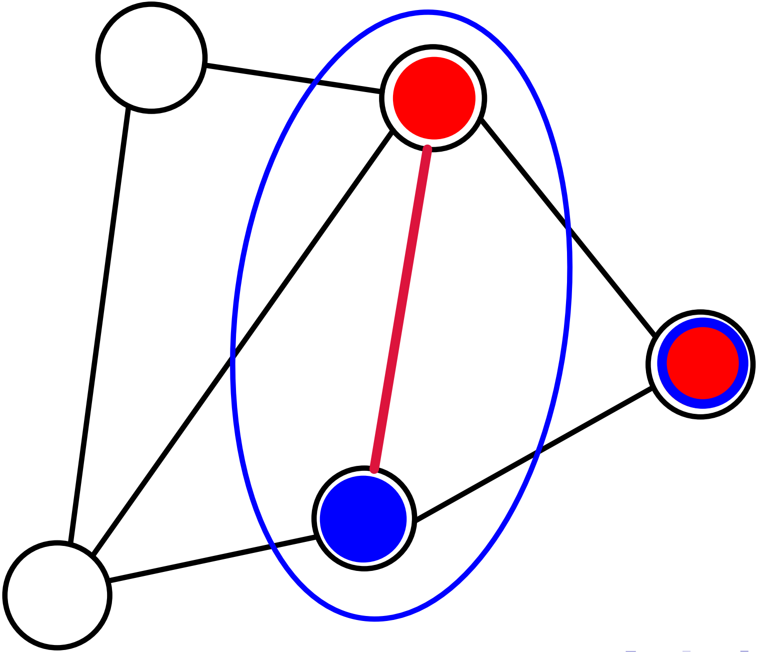
$$\mathcal{E}_N = \{\{X, Y\} : X, Y \in \mathcal{V}_N : X \triangle Y \in \mathcal{E}\}.$$

Studied under different names:

- "N-graph of $\mathcal{G}$" (Johns)
- "N-tuple vertex graphs" (Alavi, Behzad, Erdős, Lick, Liu, Zhu)
- "N-token graphs" (Rivera et al.)
- "N-th symmetric product" (Audenaert et al. , Ouyang)







Definitions: Adjacency operator on $\mathcal{G}_N$ and potential S_N

- On $\ell^2(\mathcal{V}_N)$, the adjacency operator A_N is defined via

$$(A_N f)(X) = \sum_{Y:\{X,Y\}\in\mathcal{E}_N} f(Y)\,.$$

- The potential S_N is the operator of multiplication on $\ell^2(\mathcal{V}_N)$

$$(S_N f)(X) = S_N(X) f(X) = |\partial X| f(X)\,,$$

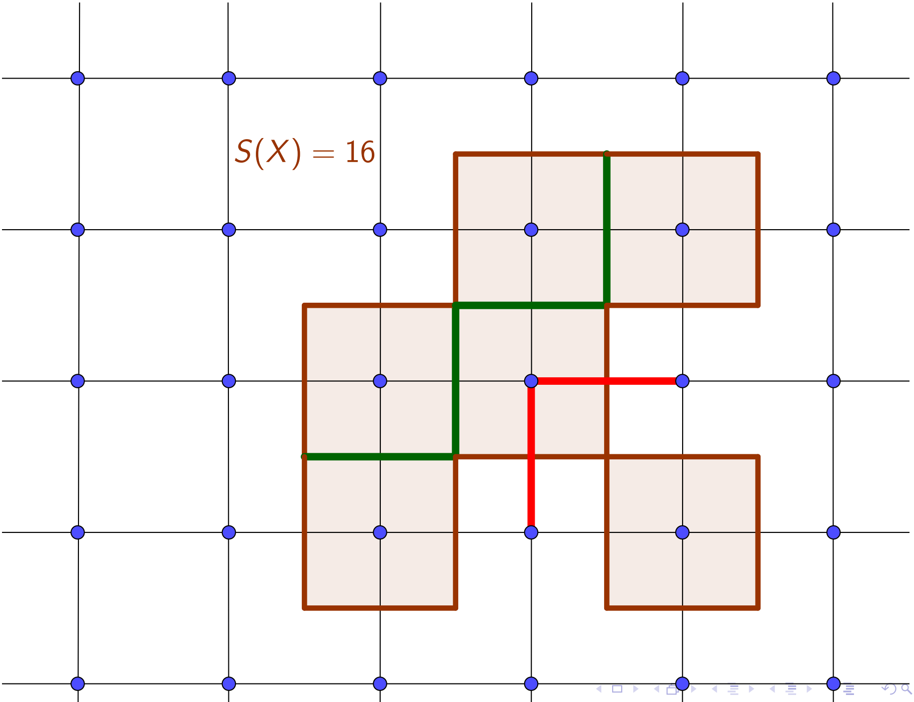
where the edge boundary ∂X is given by

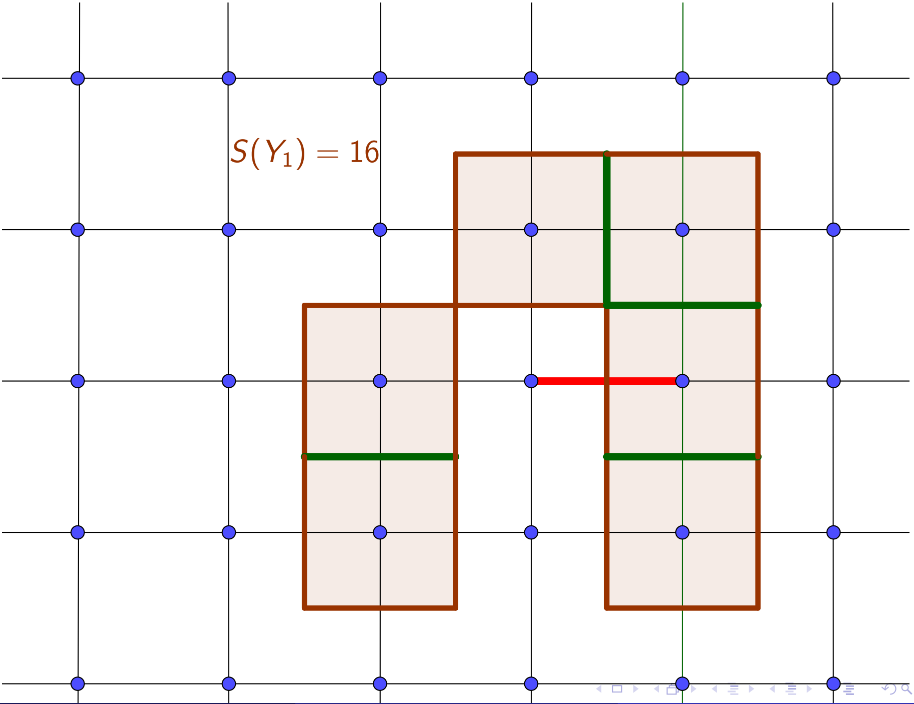
$$\partial X = \{\{x,y\} \in \mathcal{E} : x \in X, y \notin X\}\,.$$

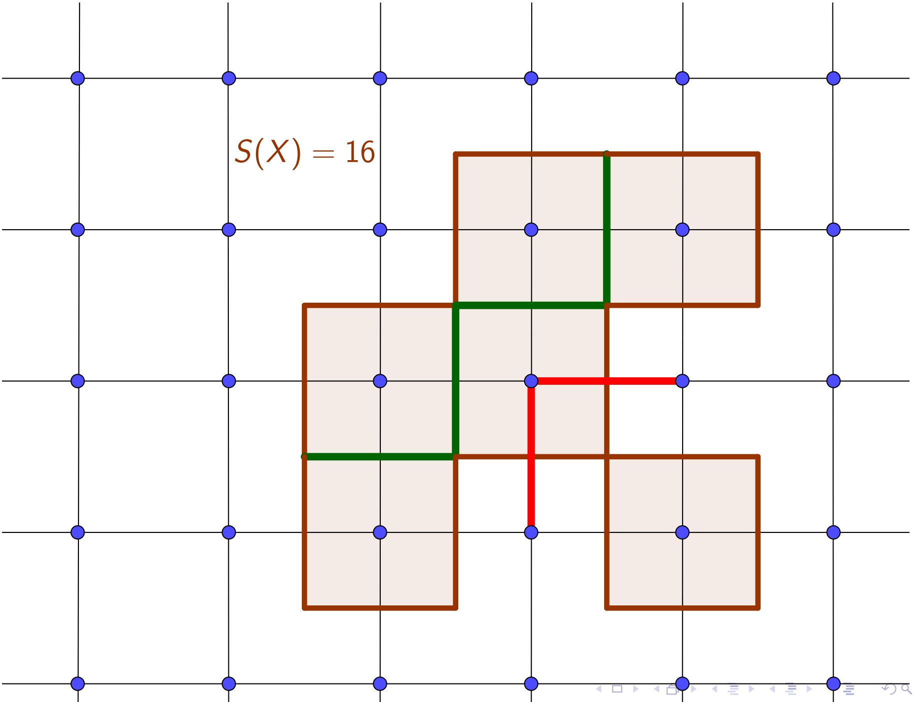
Remark: We interpret $S_N(X) = |\partial X|$ as the "surface" of the configuration X.

Example:

- $\mathcal{G} = (\mathbb{Z}^2, \{\{x, y\} : \|x - y\|_1 = 1\})$.
- $N = 6$.


$S(X) = 16$


$S(Y_1) = 16$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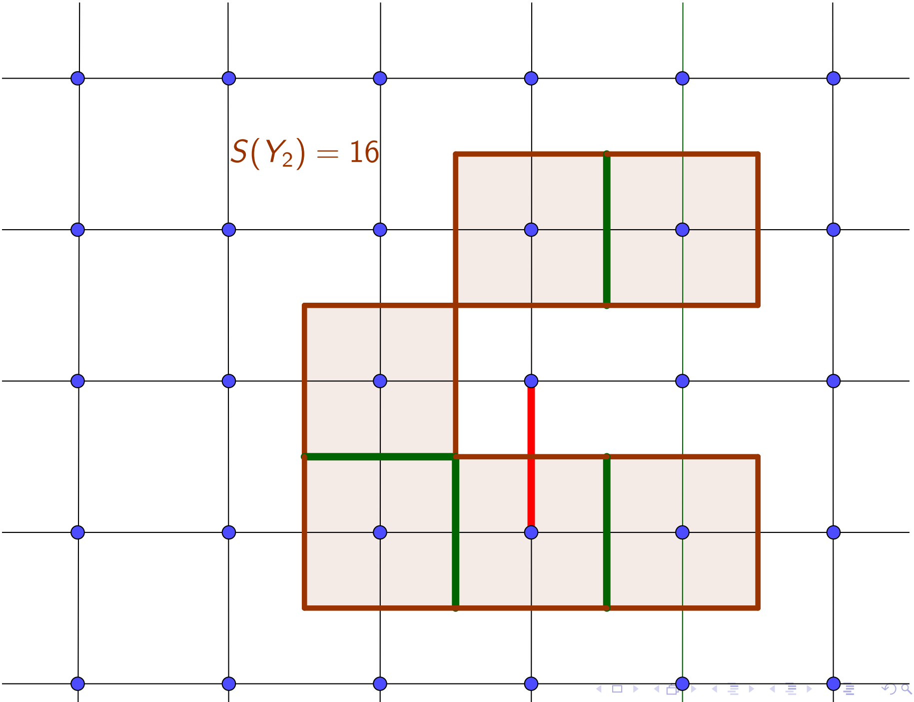
$S(Y_2) = 16$

Definition of the XXZ Hamiltonian on infinite graphs — the Schrödinger way

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with finite/countably infinite vertex set, which is of bounded of bounded maximal degree. We then define:

$$H_{\mathcal{G}} := \bigoplus_{N=0}^{|\mathcal{V}|} \underbrace{\left(-\frac{1}{2\Delta} A_N + \frac{1}{2} S_N\right)}_{=:H_{\mathcal{G}}^N} .$$

Remarks:

- For any $N \in \mathbb{N}$, we have that $\|H_{\mathcal{G}}^N\| \leq \frac{Nd_{max}}{2}\left(1 + \frac{1}{\Delta}\right)$, however, $H_{\mathcal{G}}$ itself is unbounded.
- In the infinite case, if $N \geq 1$, we have $S_N(X) \geq 1$ for any $X \in \mathcal{V}_N$. This implies that

$$H_{\mathcal{G}}^N = -\frac{1}{2\Delta}\mathcal{L}_N + \frac{1}{2}\left(1 - \frac{1}{\Delta}\right) S_N \geq \frac{1}{2}\left(1 - \frac{1}{\Delta}\right)$$

for any $N \geq 1$. $\Rightarrow E_0 = 0$ is the ground-state energy of $H_{\mathcal{G}}$ (coming from $N = 0$) with a gap of width $\geq \frac{1}{2}\left(1 - \frac{1}{\Delta}\right)$ above it.

Classical droplets

- The spectral minimum of S_N is realized by states $\delta_{\widehat{X}}$, where $\widehat{X} \in \mathcal{V}_N$ solves the discrete isoperimetric problem

$$|\partial \widehat{X}| = \min\{|\partial X| : X \subset \mathcal{V}, |X| = N\} =: S_{N,1} \,.$$

We call such states "classical droplets" as they minimize their surface.

- Let P_N denote the orthogonal projection onto the space of classical droplets (and $\bar{P}_N := I - P_N$.) Denote by $S_{N,2}$ the second smallest eigenvalue of S_N. It is clear that $S_{N,2} \geq S_{N,1} + 1$ and moreover that

$$\bar{P}_N H_{\mathcal{G}}^N \bar{P}_N \geq \frac{1}{2}\left(1 - \frac{1}{\Delta}\right) S_{N,2} \bar{P}_N \,.$$

- We thus call

$$\delta_N := \left(0, \frac{1}{2}\left(1 - \frac{1}{\Delta}\right) S_{N,2}\right) \cap \sigma(H_{\mathcal{G}}^N)$$

the "droplet spectrum" of $H_{\mathcal{G}}^N$.

The existence of gaps above the droplet spectrum – uniformly in the particle number

Assume that the graph $\mathcal{G}$ is such that $\sup_N S_{N,1} < \infty$ and $\sup_N S_{N,2} < \infty$ ("quasi one-dimensional graph", e.g. strip). Treating the kinetic term in

$$H_{\mathcal{G}}^N = -\frac{1}{2\Delta} A_N + \frac{1}{2} S_N \,,$$

as a perturbation shows that – if Δ is large enough – there is no spectrum between $\frac{S_{N,1}}{2} + \frac{\|A_N\|}{2\Delta}$ and $\frac{S_{N,2}}{2} - \frac{\|A_N\|}{2\Delta}$.
Problem: $\|A_N\|$ grows with the particle number, so this won't yield a condition on Δ that shows the existence of a spectral gap uniformly in the particle number.

Decompose the vertex set $\mathcal{V} = \mathcal{V}^{(1)} \cup \mathcal{V}^{(2)}$, where $\mathcal{V}^{(2)}$ are the droplet configurations and $\mathcal{V}^{(1)} = \mathcal{V} \setminus \mathcal{V}^{(2)}$. A more refined analysis of the adjacency term between $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ yields the following

Proposition

All energies $E \in \left(\frac{1}{2} \left(S_{N,1} + \frac{\|A_N^{(2)}\|}{\Delta} \right), \left(\frac{1}{2} \left(1 - \frac{1}{\Delta}\right) S_{N,2} \right) \right)$ such that

$$\frac{d_1 d_2}{4\Delta^2} < \left(\frac{1}{2} \left(1 - \frac{1}{\Delta}\right) S_{N,2} - E \right) \cdot \left(E - \frac{1}{2} \left(S_{N,1} + \frac{\|A_N^{(2)}\|}{\Delta} \right) \right) \qquad (1)$$

are not in the spectrum of $H = -\frac{1}{2\Delta} A_N + \frac{1}{2} S_N$.

Proposition

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are not in the spectrum of $H = -\frac{1}{2\Delta} A_N + \frac{1}{2} S_N$.

Remarks:

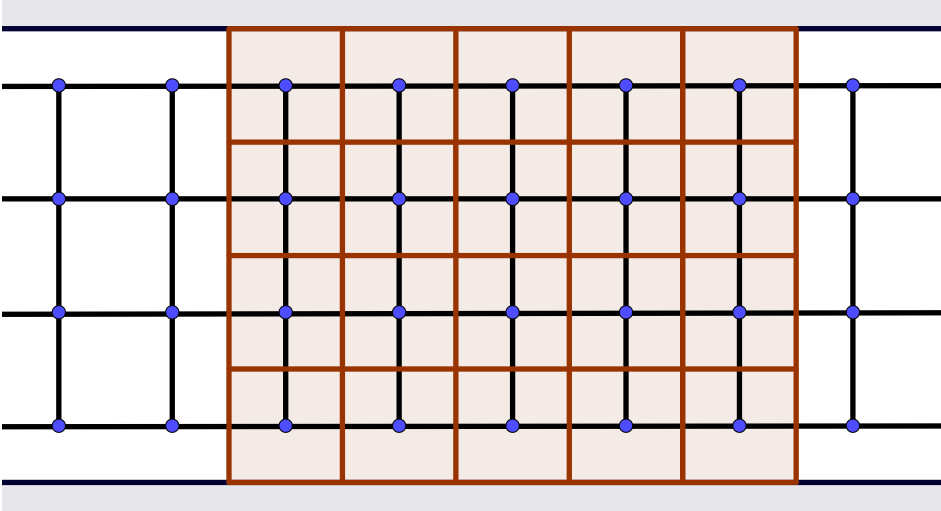
- Here, $A_N^{(2)}$ is the adjacency operator on $\mathcal{G}_N^{(2)} = (\mathcal{V}_N^{(2)}, \mathcal{E}_N^{(2)})$, where $\mathcal{E}_N^{(2)} = \{\{X, Y\} \in \mathcal{E}_N : X, Y \in \mathcal{V}_N^{(2)}\}$.
- The numbers d_1, d_2 estimate the boundary degree

$$d_1 := \sup_{X \in \mathcal{V}_N^{(1)}} |\{Y \in \mathcal{V}_N^{(2)} : X \sim Y\}|$$

$$\text{and} \quad d_2 := \sup_{Y \in \mathcal{V}_N^{(2)}} |\{X \in \mathcal{V}_N^{(1)} : X \sim Y\}|.$$

The strip of width M

Let $\mathcal{V} = \mathbb{Z} \times \{1, 2, \ldots, M\}$, where $M \geq 3$. The edge set $\mathcal{E}$ is given by ℓ^1 next neighbors. For simplicity, we only consider large enough particle numbers that are multiples of M. In this case, the classical droplets are given by rectangular configurations.



Let us now determine the necessary quantities:

- $S_{N,1} = 2M$.
- $S_{N,2} = 2M + 2$.
- $d_1 d_2 = 2M$.
- $\|A_N^{(2)}\| = 0$.

Applying the previous proposition we thus get that any energy $E \in (M, (1 - \frac{1}{\Delta})(M+1))$ is not in the spectrum of $H_{\mathcal{G}}^N$ if it satisfies the additional condition

$$\frac{M}{2\Delta^2} < \left(\left(1 - \frac{1}{\Delta}\right)(M+1) - E\right)(E - M).$$

- For large enough Δ such energies certainly do exist.
- This is independent of the particle number.

An example of growing droplet surface: The two-dimensional lattice

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathbb{Z}^2$ and $\mathcal{E} = \{\{x, y\} : \|x - y\|_1 = 1\}$.
For simplicity, let us only consider particle numbers N that are squares, i.e. for which $\sqrt{N} \in \mathbb{N}$. The classical droplets are given by square configurations. In particular, we have $S_{N,1} = 4\sqrt{N} \to \infty$ as $N \to \infty$.
Nevertheless, we can apply the previous proposition with

- $S_{N,1} = 4\sqrt{N}$, $S_{N.2} = 4\sqrt{N} + 2$
- $d_1 d_2 = 4\sqrt{N}$
- $\|A_N^{(2)}\| = 0$

in order to get

$$\frac{\sqrt{N}}{\Delta^2} < \left(\left(1 - \frac{1}{\Delta}\right)(2\sqrt{N} + 1) - E\right)(E - 2\sqrt{N}),$$

which is not uniform in N anymore, but still better than the rough norm estimate of A_N.

Future work

- Properties of eigenfunctions in the droplet spectrum (e.g. entanglement – Beaud-Warzel)
- Translation invariant graphs – a.c. spectrum?
- Randomness – many-body localization? (Elgart-Klein-Stolz, Beaud-Warzel)
- Scattering theory – fission and fusion of droplets?

Thanks for your attention!