Power Distribution in Electrical Grids

Safatul Islam, Deanna Johnson, Homa Shayan, Jonathan Utegaard
Mentors: Aalok Shah, Dr. Ildar Gabitov

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Abstract

Power in electrical grids is modeled using a set of ordinary differential equations. The system is reduced to one dimensional flow, lacking branches, and containing an infinite number of uniform loads. By converting a discrete model into homogeneous ODE's and rescaling ideal boundary conditions to initial conditions, stable solutions of voltage with respect to length are found. It is noticed that two stable solutions exist for a certain length; one 'normal' voltage, and another short voltage, identifying the presence of power outages and other irregular phenomena.

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1 Introduction and Background

1.1 Introduction

The ability to model power flow in electrical systems is crucial for the future of electrical networks. As technology progresses and energy consumption increases, better management of electrical systems is imperative to sustain the growing needs and curtail environmental damages. In 2010, energy related emissions accounted for 87% of greenhouse gas emissions in the US.[2] This level of usage cannot be sustained without innovation in the field. Specifically in Smart Grids, feed lines are structured to maximize production per length. Feedback from each individual consumer, or load, allows for better power distribution to particular regimes. Also understanding voltage fluctuations along a line will allow for innovations to deliver power in longer lines, thereby expanding the electrical grid and promoting transport of energy from distant sources.

Better distribution of power rests on the ability to comprehend its flow patterns. In this paper, a one-dimensional feed line with uniform loads is modeled to better understand the power distribution and voltage fluctuations along a line.

1.2 Background

In alternating current systems, current, voltage, and thus power - the product of two - are wave functions. In the case of power, the positive values will represent real power while the negative regions represent reactive power. In practice, real power represents energy that is consumed, while reactive power is 'generated' power due to energy storage in capacitors and inductors. Real and reactive power can be combined as a complex number representing apparent or total power. Similar reasoning exists for resistance and inductance combining to represent impedance.

$$S = P + jQ \tag{1}$$

$$z = r + jx \tag{2}$$

where
S is the apparent power
P, Q are the real and reactive power
z is the impedance
r, x are the resistance and inductance

The real and reactive power are important to this study as they represent natural parameters to the system describing an individual consumer's consumption or production. The above equations, combined with the fundamental Kirchoff's laws will lead to the initial descriptions of the system.

2 Developing The Model

2.1 Discrete Form

Starting with finite element model we write the equations for quantities that we are interested; real and reactive power and voltage, creating the system of equations (3), (4) and (5).

$$P_{k+1} - P_k = p_k - r_k \frac{P_k^2 + Q_k^2}{v_k^2} \tag{3}$$

$$Q_{k+1} - Q_k = q_k - x_k \frac{P_k^2 + Q_k^2}{v_k^2} \tag{4}$$

$$v_{k+1}^2 - v_k^2 = -2(r_k P_k + x_k Q_k) - (r_k^2 + x_k^2) \frac{P_k^2 + Q_k^2}{v_k^2} \tag{5}$$

where

$k = 0, ..., N$ enumerates buses of the feeder, buses represents the consumers

P_k,Q_k real and reactive power flowing from bus k to bus $k+1$

p_k, q_k overall consumption/production of real and reactive power at bus k

r_k, x_k line resistance and reactance connecting bus k to bus $k+1$

with Boundary Conditions

$$v_0 = 1, P_N = Q_N = 0$$

The boundary conditions modeled above represents an ideal system with known initial voltage; the voltage that the feeder is supplied with. The boundary conditions at the end with zero real and reactive power means that all the power supplied is being consumed.

2.2 Continuous and Homogeneous Form

To rewrite the modeled system in ODE's form one needs to transform the discrete finite element to a continuous form. Therefore we assume large number of consumers $N \gg 1$. The system of equations can be represented in continuous form with limit $N \longrightarrow \infty$. Some more assumptions to simplify the system and reduce the number of parameters are considered. $\dfrac{r_k}{x_k}$ is set constant and the resistance and reactance can be represented as $r_k = r\dfrac{l_k}{L}$ and $x_k = x\dfrac{l_k}{L}$, where r and x are constant values. L total length of the feeder line and l_k length of line from bus k to bus $k+1$.
The other technique employed to obtain a homogenized system of equations is decomposing real and reactive power and voltage into a sum of two components. Therefore all the variables at node k can be decompose as $F_k = F(z) + \tilde{F}(L_k)/N$ into $F(z)$ which is the change from node k to $k+1$ and $\tilde{F}(L_k)/N$ which is the averaging term. We define P_k, Q_k and v_k in term of a new variable $z = L_k$ where $L_k = \sum_{i=0}^{k-1} l_k$ and z represents the position along the feeder line. For real and reactive power, p_k and q_k are small varying fast whereas $p(z) = p_k\dfrac{L}{l_k}$ and $q(z) = q_k\dfrac{L}{l_k}$ are in $O(1)$ and varying smoothly
Relating Finite difference to derivatives $F_{k+1} - F_k \approx F'(z)l_k/L$ the system of equations (3), (4) and (5) can be represented in continuous homogenized form of ODE's equation (6).

$$\frac{d}{dz}\begin{pmatrix} P \\ Q \\ v \end{pmatrix} = \begin{pmatrix} p - r\frac{P^2+Q^2}{v^2} \\ q - x\frac{P^2+Q^2}{v^2} \\ -\frac{rP+xQ}{v} \end{pmatrix} \tag{6}$$

with Boundary Conditions

$$v_0 = 1, P(L) = Q(L) = 0$$

The ODE's represented in equation (6) is boundary value problem with mixed boundary conditions. Solving this boundary value ODE's for a known length of feeder line will result in evaluating real and reactive power and voltage along the given line.

2.3 Re-scaled Form

We can take the system of ODE's described in equation (6) one step father and simplify it into a initial value problem. Assuming p =constant, we define a new variable $s = \frac{\sqrt{|p|r}}{v(L)}(L - z)$. This re-scaling changes the positioning of the line and the end of line in terms of s would be the beginning. We define these new dimensionless variables to represent P, Q and v:

$$\varrho(s) = \sqrt{\frac{r}{|p|}}\frac{P(z)}{v(L)} \tag{7}$$

$$\tau(s) = \sqrt{\frac{r}{|p|}}\frac{Q(z)}{v(L)} \tag{8}$$

$$\upsilon(s) = \frac{v(z)}{v(L)} \tag{9}$$

These re-scalings will result in equation (10):

$$\frac{d}{ds}\begin{pmatrix} \varrho \\ \tau \\ \upsilon \end{pmatrix} = \begin{pmatrix} sign(p) - \frac{\varrho^2+\tau^2}{\upsilon^2} \\ A - B\frac{\varrho^2+\tau^2}{\upsilon^2} \\ -\frac{\varrho+B\tau}{\upsilon} \end{pmatrix} \tag{10}$$

with Initial Conditions

$$\upsilon(0) = 1, \varrho(0) = \tau(0) = 0$$

As the boundary conditions suggest this from is initial value problem. Solving equation (5) for some value of s_* where $s : 0 \rightarrow s_*$ we obtain $\varrho(s_*)$, $\tau(s_*)$ and $\upsilon(s_*)$. Then we can compute the value of L and the end values.

$$L = \frac{s_*}{\upsilon(s_*)\sqrt{|p|r}} \tag{11}$$

$$v(L) = \frac{1}{\upsilon(s_*)} \tag{12}$$

$$P(0) = \frac{\varrho(s_*)\sqrt{|p|/r}}{\upsilon(s_*)} \tag{13}$$

$$Q(0) = \frac{\tau(s_*)\sqrt{|p|/r}}{\upsilon(s_*)} \tag{14}$$

By solving initial value problem described in equation (10) we solve for the length of feeder line and end values for that length which is more efficient. The result will be described in next section.

3 Results

The initial value problem and the boundary value problem were solved in order to produce the graphs for this section. The Matlab function ode45 was used to solve the initial value problem, which consisted of the system of rescaled ODEs, shown above by Equation (10). The solutions were then used to calculate the end voltage and power utilization, which were graphed against the length of the line.

The Matlab function bvp4c was used to solve the boundary value problem, which was made up of the original system of ODEs, shown above by Equation (6). These solutions were used to generate the graphs of the voltage and power along the line. This problem was slightly more difficult to solve than the initial value problem, because this function requires a guess of the solution. This becomes important, as is shown later, because two stable solutions are possible for the same length in some cases. Therefore, it was important to choose a guess that resulted in the desired solution.

The first graph produced was that showing the end voltage versus the length of the line, shown by Figure 1. Once the solutions to initial value problem were found, the equations used to rescale the original ODEs, shown by Equations (11) and (12), were used to solve for the end voltage and the length, which generated this graph. The parameters p and q were set to -1 and -0.5, respectively. This implies constant power consumption, because these parameters are both constant and negative. An interesting feature of this graph is that there is a maximum length, slightly larger than L = 0.6. The length of the line is limited because there is also a limit to the amount of power being consumed by the system.

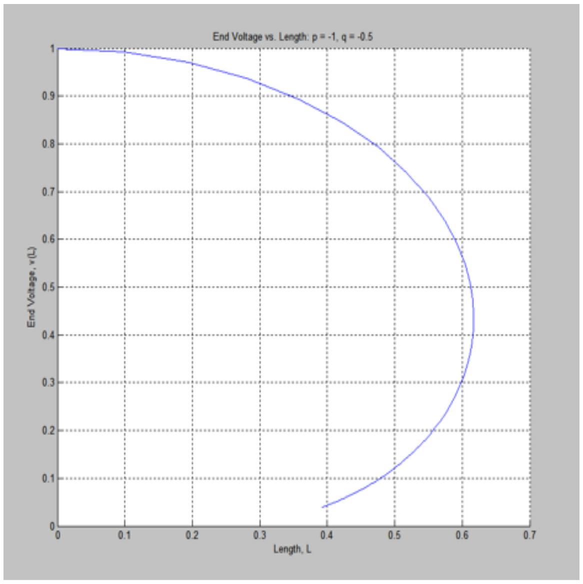

Figure 1: Voltage at the end of the line versus the length of the line, for p = -1 and q = -0.5

The next graph, shown by Figure 2, displays a similar graph of the end voltage versus the length of the line. The parameter q, however, has now been changed from -0.5 to 0. This means that there is no longer any reactive power consumption. So, the maximum length can be longer because there is not as much power being consumed along the line. The graph confirms this idea by showing that the maximum length is now L = 0.7.

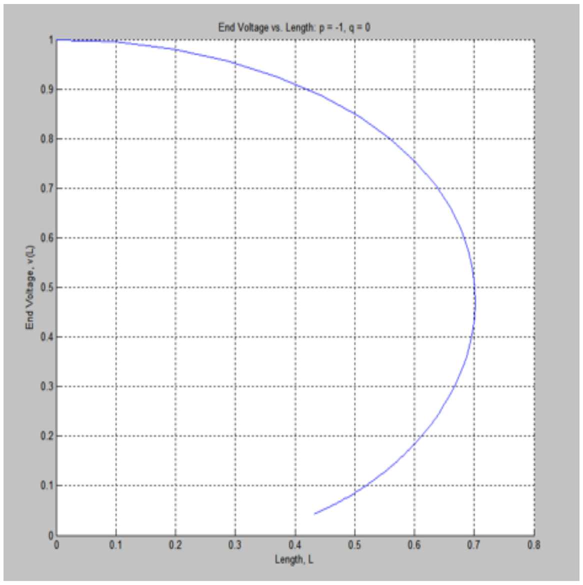

Figure 2: Voltage at the end of the line versus the length of the line, for p = -1 and q = 0

An important concept shown by these graphs is that there are two stable solutions to this system. This means that for the same length and system parameters, there can be two possible voltages at the end of the line. It was seen in Matlab that the solution for the voltage along the line depended on the guess of the solution used in the boundary value problem solver. Figure 3 shows how the voltages along a line of length L = 0.5 can be different depending on the guess. The green curve shows the voltage along the line when a guess of 0.1 is used for the solver, and this results in an end voltage around 0.13. The red curve shows the voltage along the line when a guess of 1 is used, which results in a higher voltage of around 0.77.

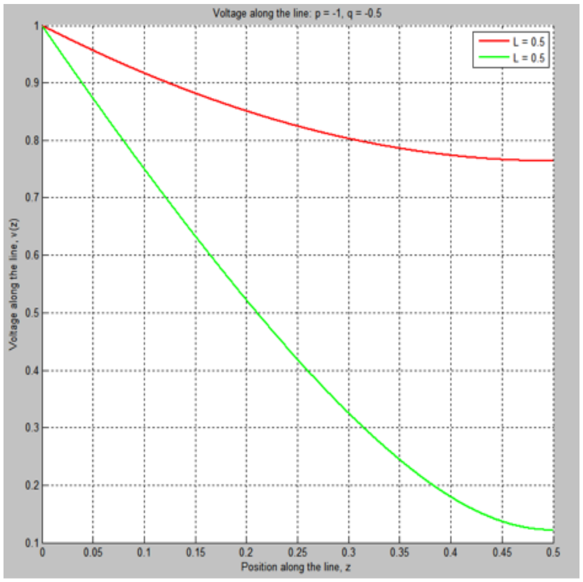

Figure 3: Voltages along a line, with L = 0.5; the red curve shows the solution for a guess of 1, while the green curve shows the solution for a guess of 0.1

Figures 4 and 5 describe, mathematically, how the system can produce two end voltages for the same length. Figure 4 shows that $\nu(s)$ is increasing over the entire range of s. Then, because the end voltage is simply $1/\nu(s)$, there will also be a unique end voltage for every value of s. Figure 5, on the other hand, shows that because L is proportional to the ratio of s to $\nu(s)$, the length can have the same value for different values of s. So, because the graphs are plotted against the same range of s, it is seen that there may exist two end voltages for a single length.

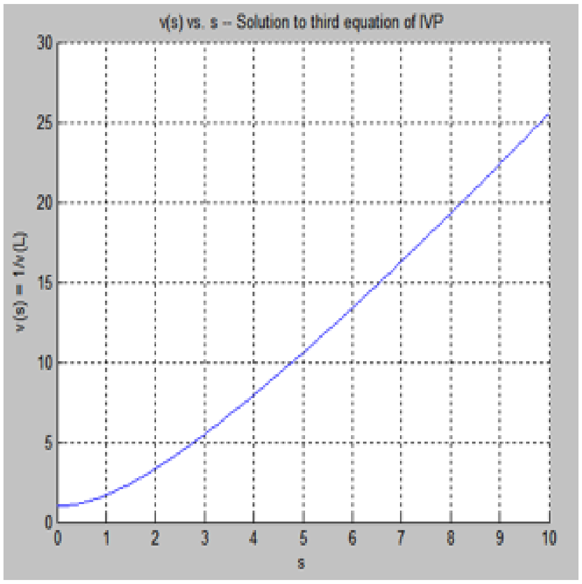

Figure 4: Graph of (s) vs. s; shows there is a unique end voltage for every s

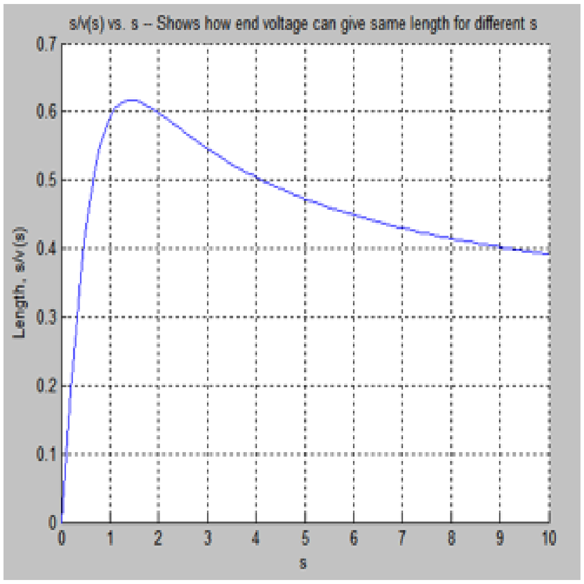

Figure 5: Graph of L vs. s; shows there are different values of s with the same length

Most of the time, the guesses resulted in the higher end voltage, which is good when analyzing how this relates to a real scenario. In real situations, a lower end voltage would be very bad and potentially dangerous, because it could lead to power outages. If the lower voltage is found to be the solution, and the line is used to power a load that requires a voltage higher than this, then it will not be able to accommodate the load. The same idea applies to the power used by the system, as seen later on.

The next main graph that was produced shows the power utilization versus the length of the line, as seen in Figure 6. The power utilization is defined to be the power injected at the beginning of the line divided by the power consumed along the line, given by $p * L$. This graph uses the same constant power consumption as the first graph of end voltage versus length, which means the parameters p and q are set to -1 and -0.5. Because the parameters and conditions of the system remain the same, there will be the same limit to the amount of power that can be consumed by the system. This, in turn, will limit the length to the same maximum value of around $L = 0.6$.

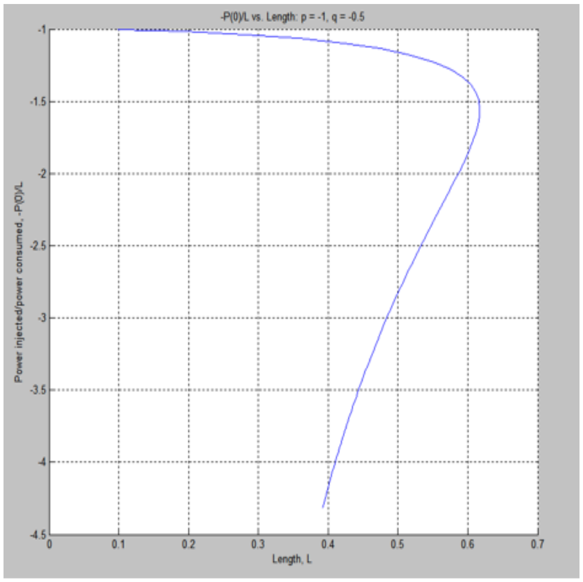

Figure 6: Power utilization vs. length of the line

This graph, again, shows that there will be two stable solutions for the power utilization at the same length. This is related to the bistability of the end voltage for a given length. Because there are two different possibilities for the power utilization at a length, L, there will be two possible initial powers that may be injected at the head of the line to satisfy the system. This can be seen by looking at $L = 0.4$ as an example. The higher power utilization, around -1, implies that the system requires an initial injected power of around 0.4. However, the power utilization may also be around -4, which would imply that the system needs an injected power of around 1.6. This can be seen by looking at the graph of the power along the line, shown in Figure 7. The solid lines show the real power, which is the type being considered here.

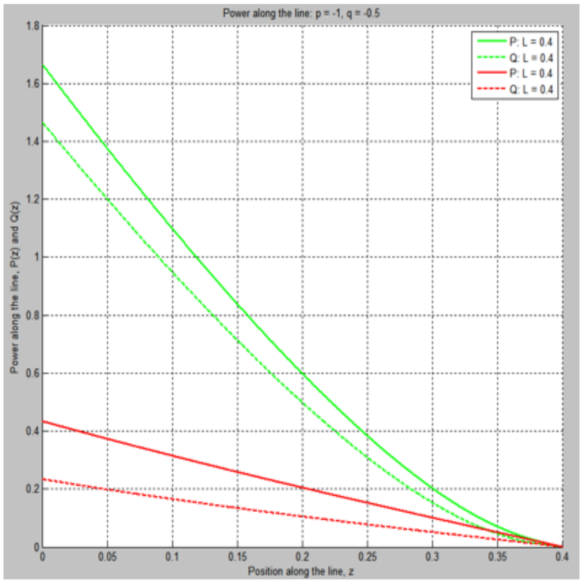

Figure 7: Power along a line, with L = 0.4; the green curve shows the higher power utilization (around -1), while the red curve shows the lower power utilization (around -4)

So, when the system tends towards the lower power utilization, it requires that four times the power be injected at the head of the line in order to satisfy the conditions of the system. This means that the system would need to be supplied with more power in order to meet all of the conditions. As previously mentioned, it would be bad to get the lower solution in a real situation, because there may not be enough power along the line to satisfy the power requirements of the loads. This could lead to power outages, because the loads would not be supplied with enough power.

Figure 8 shows the next graph, which displays the voltage along the line for lengths $L = 0.2$ and $L = 0.5$. The parameters p and q are both negative and constant, which implies constant power consumption. This is confirmed by the shape of the voltage curve, which is decreasing across the entire interval, because as the power decreases along the line, the voltage will also go down.

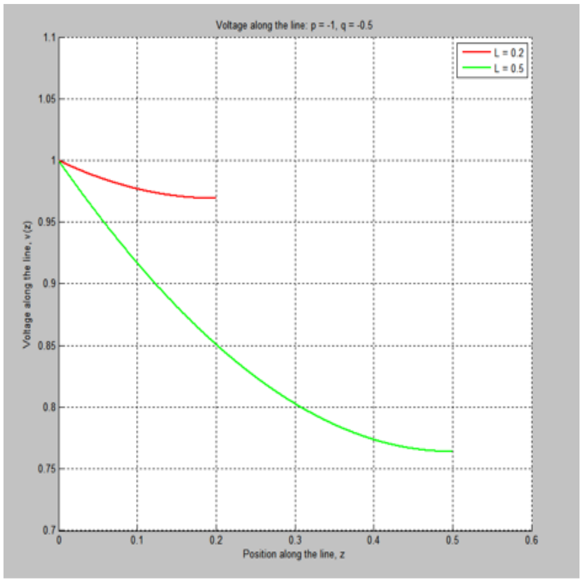

Figure 8: Voltage along the line, for L = 0.2 and L = 0.5

An interesting observation arose when experimenting with the program generating this graph. When a length of $L = 0.7$ was used, Matlab reported an error saying that a singular Jacobian matrix was encountered. This is consistent with the results of the graph of the end voltage versus length, which showed that the maximum length of the system was only around $L = 0.6$. Because the parameters p and q are the same for both graphs, this is the same system. When a length greater than the maximum was used, the boundary value problem could not be solved. In real situations, this means that for the given parameters, it is physically impossible for a line of the given length to meet all of the boundary conditions.

Figure 9 shows a similar graph of the voltage along the line, except that the parameter q has been changed from -0.5 to 0. It can be seen that a length of $L = 0.7$ is possible, as it is graphed along with the voltage along a line of length $L = 0.4$. This shows that because less power is being consumed along the line, a greater length is possible.

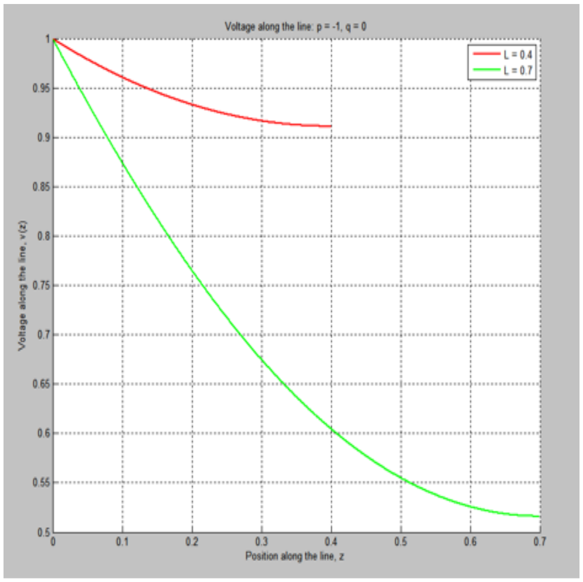

Figure 9: Voltage along the line, for L = 0.4 and L = 0.7

The next graph, Figure 10, shows the real and reactive power along the line, plotted for lines of length $L = 0.2$ and $L = 0.5$. The solid lines are used to show the real power and the dashed lines are used to show the reactive power along each line. The parameters $p = -1$ and $q = -0.5$ give a constant power consumption, which can be seen in the graph because each type of power is decreasing over the entire interval. The real and reactive power at the end of the lines both equal zero, satisfying the boundary conditions. This graph also shows that more power needs to be injected at the beginning of a line with a greater length, in order accommodate the amount of power being consumed along the line under the same parameters.

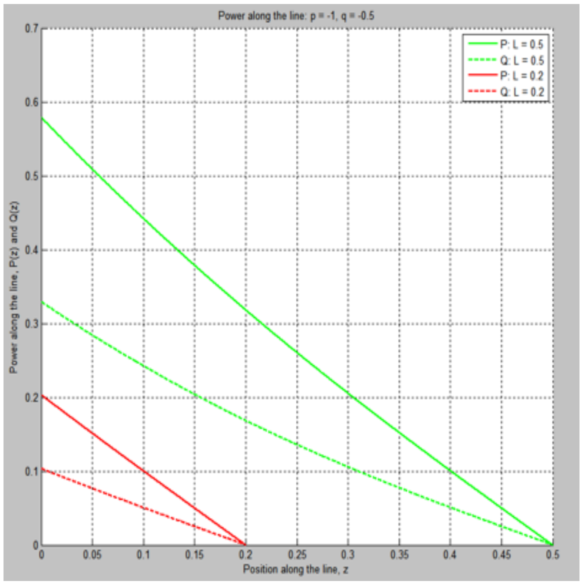

Figure 10: Power along the line, for L = 0.2 and L = 0.5

Figure 11 shows the graphs analyzed above, which were produced in Matlab by solving the initial and boundary value problems for parameters $p = -1$ and $q = -0.5$. Figure 12 shows the graphs displayed in the article for the same parameters. This shows that the graphs and results of the article were successfully reproduced.

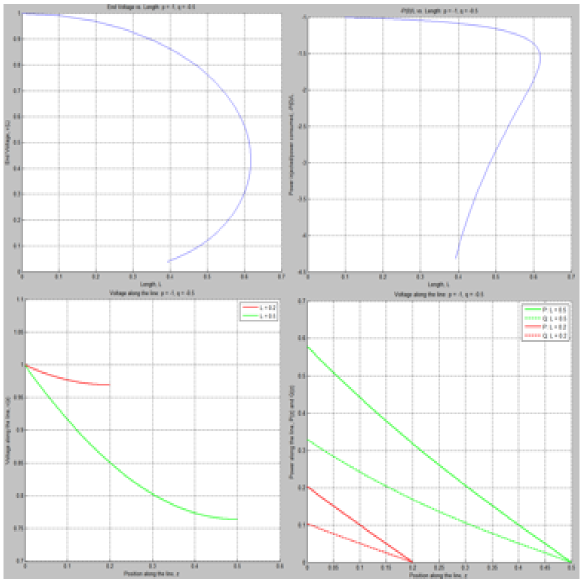

Figure 11: Graphs produced from Matlab, guided by the work in the article

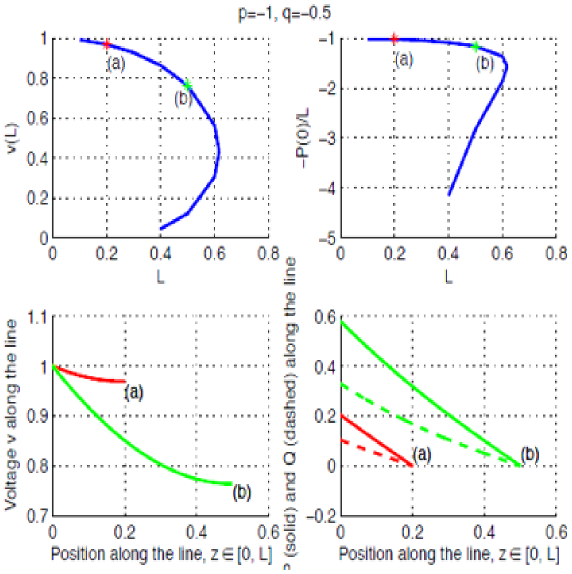

Figure 12: Graphs presented in the article

4 Future Work

In all of the simulations that were run by Wang, Turitsyn, and Chertkov, each one set the parameter p equal to a constant, thereby assuming that the feeder line was experiencing constant power consumption throughout. However, we know that this cannot be the case in an actual power distribution system because there are surely going to be minute differences among the amount of power that different customers use. Therefore, our future goal is to add a stochastic process to the parameter p in order to mirror the actual randomness in consumption that a power distribution system experiences.

Our plan is to perform multiple, independent simulations of the power line, which will entail creating a random Wiener process that will be added to an average consumption rate $p_0 = -1$, solving the boundary value problem, and recording features of the solution. We have chosen to add a Wiener process, W_t, to our power consumption because these types of processes have convenient properties for our purposes, such as having a mean of 0, and having normally distributed differences between independent time steps. Once the boundary value problem is solved, we will be interested in finding the difference between our simulation and the one that was found by Wang, Turitsyn, and Chertkov, and look to see if there is an average tendency towards or away from their result. This analysis will involve standard statistical methods such as averaging, summing residuals, and calculating variances.

From our work, we hope to find that our simulations involving stochasticity effectively confirm the plausibility of Wang, Turitsyn, and Chertkovs decision to allow the power consumption to be constant. In order to do this, we will look to see if there is a central tendency towards the solution obtained by Wang, Turitsyn, and Chertkov. Moreover, we expect that the stochastic component may drastically affect the different solutions we obtain, and we will be interested in analyzing any anomalies that arise from our computations.

5 References

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