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The Stability of an Inverted Pendulum

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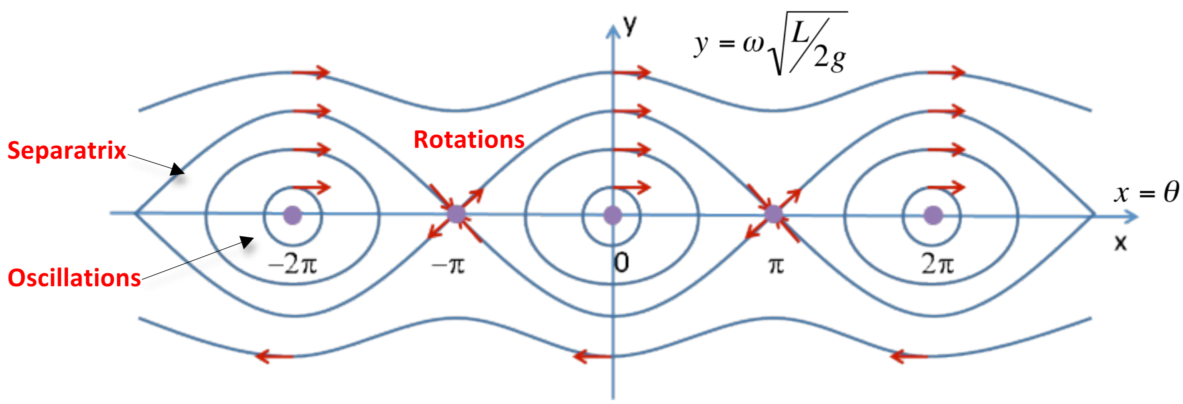
Abstract: The inverted pendulum is a simple system in which both stable and unstable state are easily observed. The upward inverted state is unstable, though it has long been known that a simple rigid pendulum can be stabilized in its inverted state by oscillating its base at an angle. We made the model to simulate the stabilization of the simple inverted pendulum. Also, the numerical analysis was used to find the stability angle.

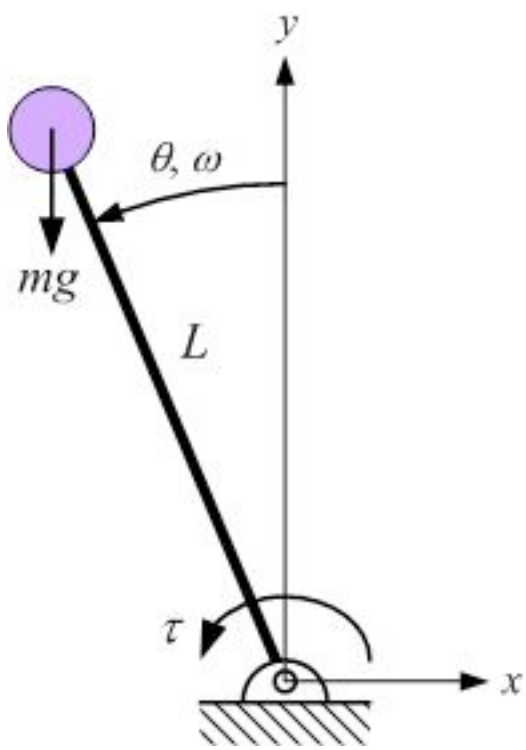
Introduction

The model of the simple pendulum problem is one the most well studied dynamical systems. Imagine a weight attached to the end of weightless rod that is freely swinging back and forth about some pivot without friction. The governing equation for this idealized mathematical model is given as $\frac{d^2\theta}{dt^2} + gl\sin\theta = 0$, where g is gravitational acceleration, l is the length of the pendulum, and θ is the angular displacement about downward vertical. If the pendulum starts at any given angle, we can expect to see one of the following happen: the pendulum will oscillate about the downward angle, $\theta = 0$; it will continue to rotate around the pivot; or it will stay still, at $\theta = 0, \pi$, but at $\theta = \pi$ any slight disturbance will cause the pendulum to swing downward.



The phase portrait above shows that the stability points for the simple pendulum are at $\theta = \pi n$, for $n = 0, \pm1, \pm2, \ldots$. For even n's, θ is a stable point and if given some angular velocity, ω, the pendulum will always oscillate around it, but for odd n's, θ is an unstable point, so even the smallest angular velocity will knock the pendulum off it and it will swing down toward its stable point.

The inverted pendulum system consists of a pendulum with its center of mass above its pivot point, which is mounted on the base with a weightless rod, or a pin. As shown above, this system should fall down to its stable point at $\theta = 0$, but if a high frequency oscillation with small amplitude is applied, the



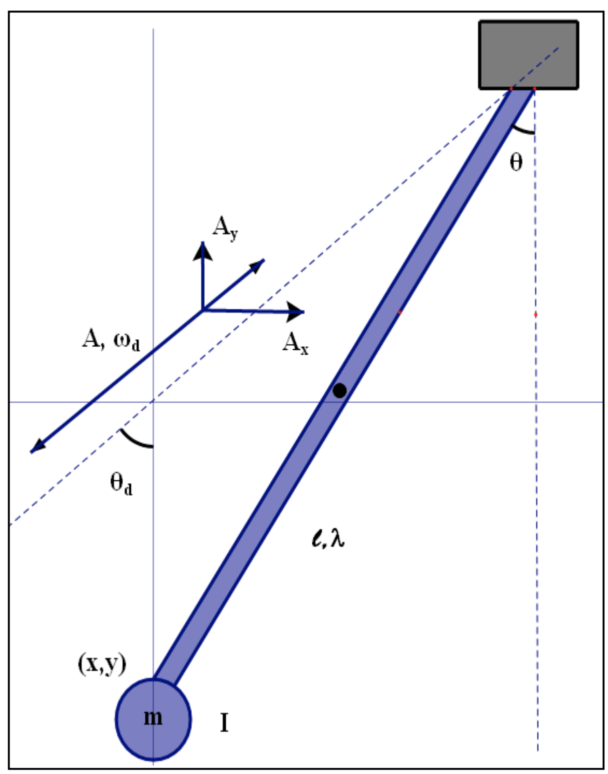
inverted pendulum will stabilize. If this high frequency oscillation is driven at an angle, it will create another oscillation about its new stable point, $\theta = \pi$. This other oscillation has large amplitude but low frequency. The goal of our study is to analyze and simulate the stabilization of the inverted pendulum.

The study of the inverted pendulum system is more than getting a pendulum to stay up right, its using the idea of combining two different oscillations, one with high frequency and small amplitude and the other with low frequency and large amplitude, to stabilize an unstable point. The most obvious application of the inverted pendulum is the segway because it is self-balancing, two-wheeled vehicle. Another application that isn't as straight forward would be a magnetic levitron. Permant magnets have continuously changing electromagnetic field for a small distance away from it. So as one magnet lays on a flat surface and another is placed on top, it will be repelled by the electromagnetic field, but if the top magnet is spinning, than it can stabalize its leviate position above the other magnet. This also works for any object that can be magnetized.

In our study, we will use a couple different methods to explain and prove this stability phenomenon. First, like the simple pendulum problem, we will formulate an idealized model equation that describes the pendulum's motion over time. Using the Euler- Lagrange equation to derive the idealized model equation of motion in dimensionless form, $\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha D(\theta,\tau) = 0$. Next we will study the effective potential energy of the driven pendulum. By averaging the potential energy over a period of the driving oscillation, we can derive effective potential equation. Now we can study the stability of the pendulum for different angles by applying the derivative test to the effective potential and create bifurcation diagram based on our analysis. Then, with numerical analysis, we can confirm the stable angle using MATlab's ODE 45 method. MATlab solves the derived governing equation with three driving angles of the pendulum and plots the result in phase portrait. In the future, we will look into graphically modeling the behavior of the driven pendulum with a dynamic manifold. And after studying the theoretical inverted pendulum, we will observe a real driven pendulum and compare the our predictions to the actual behavior of the pendulum.

Model Equations

Consider a two-dimensional model for the driven pendulum where the pendulum rotates in the same plane as the driving oscillations. In the figure to the right, gravity is assumed to be acting downward. The horizontal x-axis is defined with positive to the right and the vertical y-axis is defined with positive pointing upward. Angles are measured from downward (0^o) with clockwise being positive. The table below describes the variables we used.



Symbol	Meaning
t	Time
x , y	Position of pendulum's center of mass
g	Acceleration due to gravity
θ	Angle of deviation of pendulum
m	Mass of pendulum
l	Distance from center of mass to base
Io	Rotational inertia about base
λ	Effective length of pendulum $Ioml2$
ωo	Natural frequency of pendulum gl
θd	Drive angle of the base
ωd	Drive frequency
Ax , Ay	x- and y- amplitudes of the driving

We applied principles from Lagrangian mechanics and the Euler-Lagrange equation to the driven pendulum in order to find the governing equation of motion. The Lagrangian $\mathcal{L}$ for Newtonian mechanics is the difference between the kinetic T and potential V energies:

(1) $$\mathcal{L} = \mathrm{T} - \mathrm{V}$$

The equation of motion is derived from the Euler-Lagrange equation:

(2) $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

[2]The kinetic energy of the pendulum was decomposed as the sum of rotational T_θ and translational kinetic energies are T_x and T_y. The potential energy is the product of the weight *mg* of the pendulum with the height *y* of its center of mass.

(3) $$T = T_x + T_y + T_\theta$$

(4) $$T_x = \frac{1}{2}m(l^2\dot{\theta}^2\cos^2\theta + A_x^2\omega_d^2\cos^2(\omega_d t) + 2A_x\omega_d\cos(\omega_d t)\cos\theta)$$

(5) $$T_y = \frac{1}{2}m(l^2\dot{\theta}^2\sin^2\theta + A_y^2\omega_d^2\sin^2(\omega_d t) + 2A_y\omega_d\sin(\omega_d t)\sin\theta)$$

(6) $$T_\theta = \frac{1}{2}I_0\dot{\theta}^2$$

(7) $$V = mgy = mg(l\cos(\theta) + A_y\cos(\omega_d t))$$

After solving for kinetic and potential energy using equations (3) – (7), we plugged the energies into equations (1) and (2) to find the equation of motion:

(8) $$\ddot{\theta} + \frac{g}{\lambda}\sin\theta + \frac{A\omega_d^2}{\lambda}(\cos(\theta_d)\cos(\omega_d t)\sin(\theta) - \sin(\theta_d)\sin(\omega_d t)\cos(\theta)) = 0$$

We introduced three dimensionless parameters and an expression for the driving effects: τ is a non-dimensional time, γ is a frequency parameter, α is a length parameter and $D(\theta,\tau)$ is the expression representing effects caused by the driving.

$$\tau = \omega_d t \qquad \gamma = \frac{\omega_0^2}{\omega_d^2} \qquad \alpha = \frac{A}{\lambda}$$

$$D(\theta,\tau) = \cos\theta_d \sin\theta\cos\tau - \sin\theta_d\cos\theta\sin\tau$$

τ is equal to the full periods of oscillation the base has gone through since some initial time. γ is inversely proportional to the drive frequency; smaller γ represents faster drive frequencies. α is proportional to the amplitude of the driving oscillations. Writing our equation of motion in terms of these dimensionless parameters gives us the non-dimensional form of the equation of motion:

(9)
$$\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha D(\theta,\tau) = 0$$

Averaging Methods & Effective Potential

We applied averaging techniques and found the average or effective potential energy of the driven pendulum. [1]The effective potential $Ueff$ will be a measure of the average potential energy the pendulum has at certain pendulum angles of deviation θ. We did this by averaging the potential energy over a period of the driving oscillation. The first step is to separate the slow and fast components of the pendulum. ϕ is the slower angle of the pendulum while ξ is the rapid oscillations of the base. $F(\theta)$ is the slower force of gravity while $f(\theta,\tau)$ is the rapidly oscillating force from the driving.

(10)
$$I_0\ddot{\theta} = I_0(\ddot{\phi} + \ddot{\xi}) = (F(\theta) + f(\theta,t))$$

Considering ξ as the difference between ϕ and θ, we can do a first-order Taylor approximation to (10). Making the assumption that ξ is insignificantly small and ωd is significantly large, we can ignore negligible terms. After averaging the equation (10) over the period of oscillations of the base, we derived equation (11).

(11)
$$I_0\ddot{\phi} \cong F(\phi) + \frac{d}{d\theta} f(\phi,t)\xi$$

Now, let's define the $Ueff$:

(12)
$$I_0\ddot{\phi} = -\frac{dU_{eff}}{d\phi}$$

Writing the forces using the terms from the equation of motion, we have:

$$F(\theta) = -I_0\gamma\sin\theta \qquad f(\theta,t) = -I_0\alpha D(\theta,t)$$

Using these expressions for force and plugging them into (11) and (12), we solved for $Ueff$:

(13) $$U_{eff} = I_0\left(-\gamma\cos\theta + \frac{\alpha^2}{4}(\cos^2\theta_d \sin^2\theta + \sin^2\theta_d \cos^2\theta)\right)$$

With $Ueff$, we analyzed the stability of the pendulum for different θ. Equilibrium angles θeq are predicted to occur where the derivative of (13) evaluates to zero:

(14) $$\frac{dU_{eff}}{d\theta} = 0 = I_0 \sin\theta\left(\frac{\alpha^2}{2}(\cos^2\theta_d \cos\theta_{eq}) + \gamma\right)$$

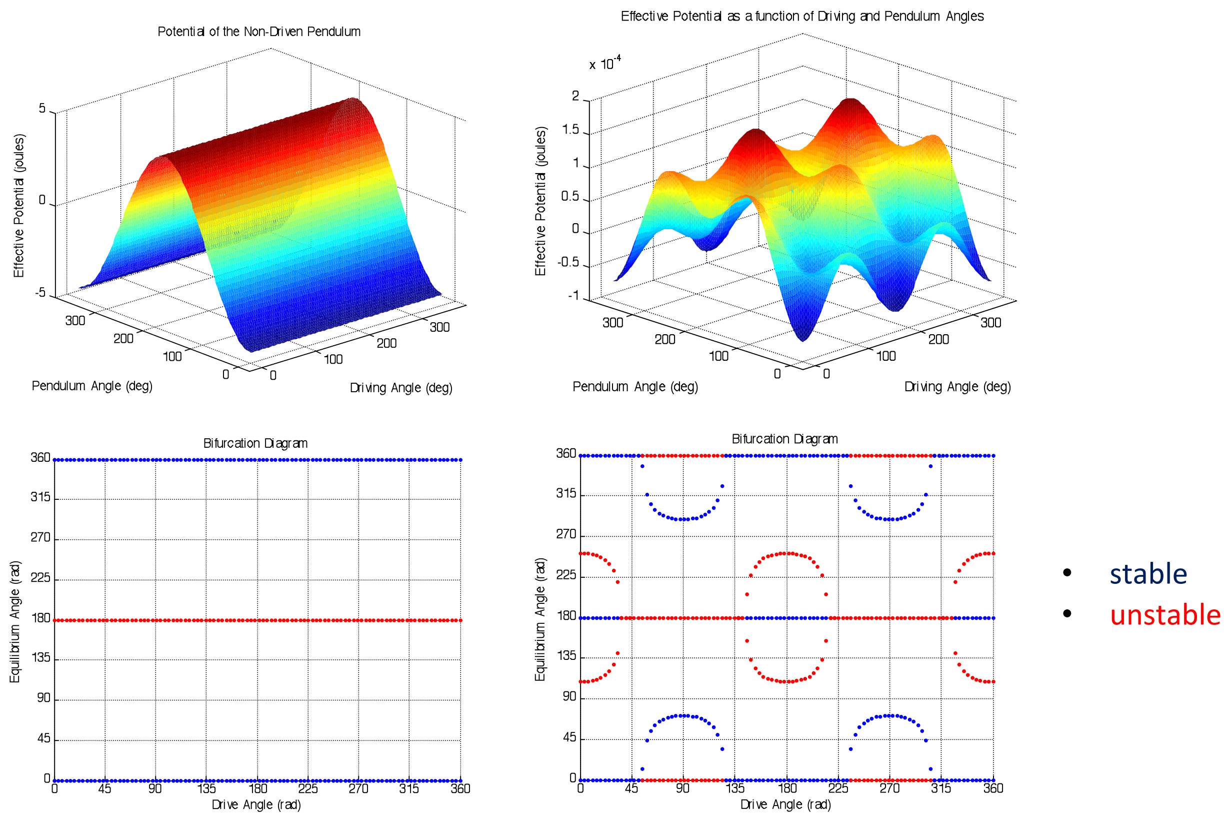
There are three different types of equilibrium angles: hanging down $\theta eq=0$, standing up $\theta eq=\pi$, or leaning to the side $\theta_{eq} = \pm\arccos^2(\gamma\alpha^2)\cos^2\theta_d$. To see whether these equilibrium angles are stable or unstable, we looked to see if the second derivative of (13) at these angles is positive – meaning stable – or negative – meaning unstable.

(15) $$\frac{d^2U_{eff}}{d\theta^2} = I_0\left(\frac{\alpha^2}{2}(\cos^2\theta_d \cos^2\theta_{eq}) + \gamma\cos\theta_{eq}\right)$$

The conditions for stability are summarize in the table below for each equilibrium:

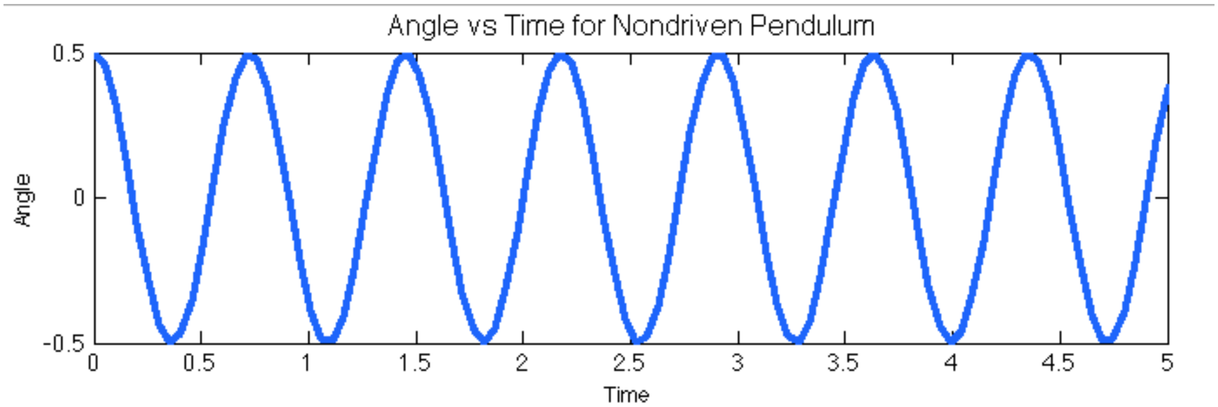
θeq	**Stable**	**Unstable**
0	$\gamma > \frac{\alpha^2}{2}\cos^2\theta_d$	$\gamma < -\frac{\alpha^2}{2}\cos^2\theta$
$\pm\arccos^2(\gamma\alpha^2)\cos^2\theta_d$	$4\gamma^2\alpha^2 - \frac{\alpha^2}{2}\cos^2\theta_d > 0$	$4\gamma^2\alpha^2 - \frac{\alpha^2}{2}\cos^2\theta_d < 0$
π	$\gamma < \frac{\alpha^2}{2}\cos^2\theta_d$	$\gamma > \frac{\alpha^2}{2}\cos^2\theta_d$

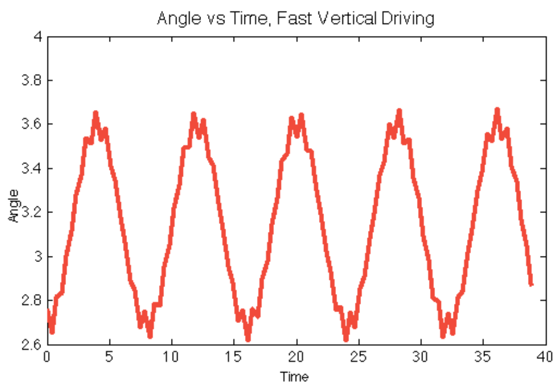
Using (13), we used MATlab to make surface plots of $Ueff$ as a function of drive angle θ_d and pendulum angle θ. Then, we also used the stability conditions summarized in the last table to create a bifurcation diagram for the pendulum. Surface plots and bifurcation diagrams are shown below for the undriven – to the left – and driven – on the right – pendulums.

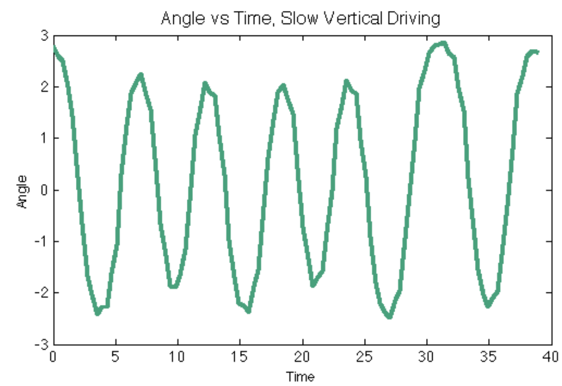


Numerical Study

Using the derived governing equation of motion (9) from the Lagrangian analysis, we can take a numerical look at how any particular pendulum will behave when driven at any particular angle. For the purposes of our analysis, we are using the parameters associated with our Black and Decker jigsaw, including its effective length, driving frequency, and driving amplitude, and will change our angular displacement to be about the upward vertical. By implementing MATlab's ODE45 method, we can set the initial angle and initial velocity for the pendulum and plot the solution for the pendulum's angle vs. the time passed since the pendulum's release. ODE45 uses a Runge-Kutta variable step method to solve our differential equation, which MATlab then plots. First we plot the non-driven pendulum so that we can compare the graphs of a pendulum driven at high and low frequencies with a simple case.







Still using this method, we will compare the behavior of a driven pendulum at three different driving angles: $\theta=0$, drives about the vertical; $\theta=\pi/4$, drives about the diagonal; and $\theta=\pi/2$, drives about the horizontal.

For the sake of avoiding redundancy, symmetrical angles have been disregarded. By setting appropriate initial conditions, we have observed oscillatory behavior for each driving angle, indicating the stability points therein. As expected, the stability point for the horizontal driving angle is located below the horizontal, due to gravity. Notably, the diagonal driving angle has produced differing results from the other two driving angles. It oscillates about $\theta=0$, as expected, but with lengthy and wide oscillations. This is most likely due to the volatility of the equilibrium angle as the angle slightly increases from $\theta=\pi/4$. Further analysis and experimentation will test the accuracy of this model with the actual behavior of the jigsaw.

Work Cited

[1]Shew, Woody. “Inverted Equilibrium of a Vertically Driven Pendulum”. *wooster.edu*. College of Wooster. 24 Apr, 1997. Web. 23 Mar, 2013.

[2]VanDalen J., Gordon. “The driven pendulum at arbitrary drive angles”. *arvix.org.* Department of Physics at University of California and Embry-Riddle Aeronautical University. 2 Feb, 2008. Web. 24 Mar, 2013.