Power Distribution in Electrical Grids

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May 7, 2013

Abstract

Power in electrical grids is modeled using a set of ordinary differential equations. The system is reduced to static one dimensional flow, lacking branches, and containing an infinite number of uniform loads. By converting a discrete model into homogeneous ODE's and rescaling ideal boundary conditions to initial conditions, stable solutions of voltage with respect to length are found. It is noticed that two stable solutions exist for a certain length; one 'normal' voltage, and another short voltage, identifying the presence of power outages and other irregular phenomena. The effects of stochastic power consumption along the line is then studied using Monte Carlo simulations. It was found that these perturbations in power consumption of individual loads have relatively no overall effect on the stability of the system.

Contents

1 Introduction and Background

1.1 Introduction

The ability to model power flow in electrical systems is crucial for the future of electrical networks. As technology progresses and energy consumption increases, better management of electrical systems is imperative to sustain the growing needs and curtail environmental damages. In 2010, energy related emissions accounted for 87% of greenhouse gas emissions in the US.[2] This level of usage cannot be sustained without innovation in the field. Specifically in Smart Grids, feed lines are structured to maximize production per length. Feedback from sensors from each individual consumer, or load, allows for better power distribution to particular regimes and reduction of energy wastage. Also understanding voltage fluctuations along a line will allow for innovations to deliver power in longer lines, thereby expanding the electrical grid and promoting transport of energy from distant sources.

Better distribution of power rests on the ability to comprehend its flow patterns. In this paper, a one-dimensional feed line with uniform loads is modeled to better understand the power distribution and voltage fluctuations along a line.

1.2 Background

In alternating current systems, current, voltage, and thus power - the product of two - are wave functions. In the case of power, the positive values will represent real power while the negative regions represent reactive power. In practice, real power represents energy that is consumed, while reactive power is energy returned to the source due to energy storage in capacitors and inductors. Real and reactive power can be combined as a complex number representing apparent or total power. Similar reasoning exists for resistance and inductance combining to represent impedance.

$$S = P + jQ \tag{1}$$

$$z = r + jx \tag{2}$$

where
S is the apparent power
P, Q are the real and reactive power
z is the impedance
r, x are the resistance and inductance

The real and reactive power are important to this study as they represent natural parameters to the system describing an individual consumer's consumption or production. The above equations, combined with the fundamental Kirchoff's laws will lead to the initial descriptions of the system.

2 Developing The Model

2.1 Discrete Form

Starting with finite element model we write the equations for quantities that we are interested; real and reactive power and voltage, creating the system of equations (3), (4) and (5).

$$P_{k+1} - P_k = p_k - r_k \frac{P_k^2 + Q_k^2}{v_k^2} \tag{3}$$

$$Q_{k+1} - Q_k = q_k - x_k \frac{P_k^2 + Q_k^2}{v_k^2} \tag{4}$$

$$v_{k+1}^2 - v_k^2 = -2(r_k P_k + x_k Q_k) - (r_k^2 + x_k^2) \frac{P_k^2 + Q_k^2}{v_k^2} \tag{5}$$

where
$k = 0, ..., N$ enumerates buses of the feeder, buses represents the consumers
P_k,Q_k real and reactive power flowing from bus k to bus $k+1$
p_k, q_k overall consumption/production of real and reactive power at bus k
r_k, x_k line resistance and reactance connecting bus k to bus $k+1$

with Boundary Conditions

$$v_0 = 1, P_N = Q_N = 0$$

The boundary conditions modeled above represents an ideal system with known initial voltage; the voltage that the feeder is supplied with. The boundary conditions at the end with zero real and reactive power means that all the power supplied is being consumed, so that there is no leftover power in the line and thus no wastage.

2.2 Continuous and Homogeneous Form

To rewrite the modeled system in ODE's form one needs to transform the discrete finite element to a continuous form. Therefore a large number of consumers $N \gg 1$ is assumed. The system of equations can be represented in continuous form with limit $N \longrightarrow \infty$. Some more assumptions to simplify the system and reduce the number of parameters are considered. $\frac{r_k}{x_k}$ is set constant and the resistance and reactance can be represented as $r_k = r\frac{l_k}{L}$ and $x_k = x\frac{l_k}{L}$, where r and x are constant values. L total length of the feeder line and l_k length of line from bus k to bus $k+1$.
The other technique employed to obtain a homogenized system of equations is decomposing real and reactive power and voltage into a sum of two components. Therefore all the variables at node k can be decomposed as $F_k = F(z) + \tilde{F}(L_k)/N$ into $F(z)$ which is the change from node k to $k+1$ and $\tilde{F}(L_k)/N$ which is the averaging term. We define P_k, Q_k and v_k in term of a new variable $z = L_k$ where $L_k = \sum_{i=0}^{k-1} l_k$ and z represents the position along the feeder line. For real and reactive power, p_k and q_k are small varying fast whereas $p(z) = p_k\frac{L}{l_k}$ and $q(z) = q_k\frac{L}{l_k}$ are in $O(1)$ and varying smoothly.
Relating finite difference to derivatives $F_{k+1} - F_k \approx F'(z)l_k/L$ the system of equations (3), (4) and (5) can be represented in continuous homogenized form of ODE's equation (6).

$$\frac{d}{dz}\begin{pmatrix} P \\ Q \\ v \end{pmatrix} = \begin{pmatrix} p - r\frac{P^2+Q^2}{v^2} \\ q - x\frac{P^2+Q^2}{v^2} \\ -\frac{rP+xQ}{v} \end{pmatrix} \tag{6}$$

with Boundary Conditions

$$v_0 = 1, P(L) = Q(L) = 0$$

The ODE's represented in equation (6) is boundary value problem with mixed boundary conditions. Solving this boundary value ODE's for a known length of feeder line will result in evaluating real and reactive power and voltage along the given line.

2.3 Re-scaled Form

The system of ODE's described in equation (6) can then be simplified into a initial value problem. Assuming p is constant, we define a new variable $s = \frac{\sqrt{|p|r}}{v(L)}(L - z)$. This re-scaling changes the positioning of the line and the end of line in terms of s would be the beginning. We define these new dimensionless variables to represent P, Q and v:

$$\varrho(s) = \sqrt{\frac{r}{|p|}}\frac{P(z)}{v(L)} \tag{7}$$

$$\tau(s) = \sqrt{\frac{r}{|p|}}\frac{Q(z)}{v(L)} \tag{8}$$

$$\upsilon(s) = \frac{v(z)}{v(L)} \tag{9}$$

These re-scalings will result in equation (10):

$$\frac{d}{ds}\begin{pmatrix} \varrho \\ \tau \\ \upsilon \end{pmatrix} = \begin{pmatrix} sign(p) - \frac{\varrho^2+\tau^2}{\upsilon^2} \\ A - B\frac{\varrho^2+\tau^2}{\upsilon^2} \\ -\frac{\varrho+B\tau}{\upsilon} \end{pmatrix} \tag{10}$$

with Initial Conditions

$$\upsilon(0) = 1, \varrho(0) = \tau(0) = 0$$

As the boundary conditions suggest this from is initial value problem. Solving equation (5) for some value of s_* where $s : 0 \to s_*$ we obtain $\varrho(s_*)$, $\tau(s_*)$ and $\upsilon(s_*)$. Then we can compute the value of L and the end values.

$$L = \frac{s_*}{\upsilon(s_*)\sqrt{|p|r}} \tag{11}$$

$$v(L) = \frac{1}{\upsilon(s_*)} \tag{12}$$

$$P(0) = \frac{\varrho(s_*)\sqrt{|p|/r}}{\upsilon(s_*)} \tag{13}$$

$$Q(0) = \frac{\tau(s_*)\sqrt{|p|/r}}{\upsilon(s_*)} \tag{14}$$

By solving initial value problem described in equation (10) we solve for the length of feeder line and end values for that length which is more efficient. The result will be described in next section.

3 Reproduction of Results

The initial value problem and the boundary value problem were solved in order to produce the graphs for this section. The Matlab function ode45 was used to solve the initial value problem, which consisted of the system of rescaled ODEs, shown above by Equation (10). The solutions were then used to calculate the end voltage and power utilization, which were graphed against the length of the line.

The Matlab function bvp4c was used to solve the boundary value problem, which was made up of the original system of ODEs, shown above by Equation (6). These solutions were used to generate the graphs of the voltage and power along the line. This problem was slightly more difficult to solve than the initial value problem, because this function requires a guess of the solution. This becomes important, as is shown later, because two stable solutions are possible for the same length in some cases. Therefore, it was important to choose a guess that resulted in the desired solution.

The first graph produced was that showing the end voltage versus the length of the line, shown by Figure 1. Once the solutions to initial value problem were found, the equations used to rescale the original ODEs, shown by Equations (11) and (12), were used to solve for the end voltage and the length, which generated this graph. The parameters p and q were set to -1 and -0.5, respectively. This implies constant power consumption, because these parameters are both constant and negative. An interesting feature of this graph is that there is a maximum length, slightly larger than L = 0.6. The length of the line is limited because there is also a limit to the amount of power being consumed by the system.

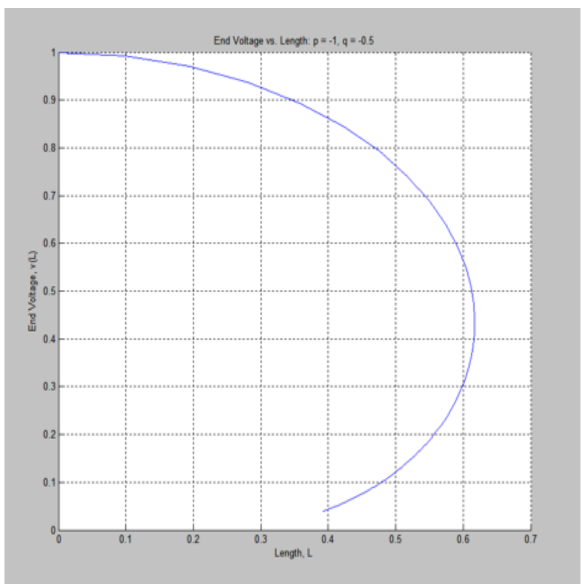

Figure 1: Voltage at the end of the line versus the length of the line, for $p = -1$ and $q = -0.5$

The next graph, shown by Figure 2, displays a similar graph of the end voltage versus the length of the line. The parameter q, however, has now been changed from -0.5 to 0. This means that there is no longer any reactive power consumption. So, the maximum length can be longer because there is not as much power being consumed along the line. The graph confirms this idea by showing that the maximum length is now $L = 0.7$.

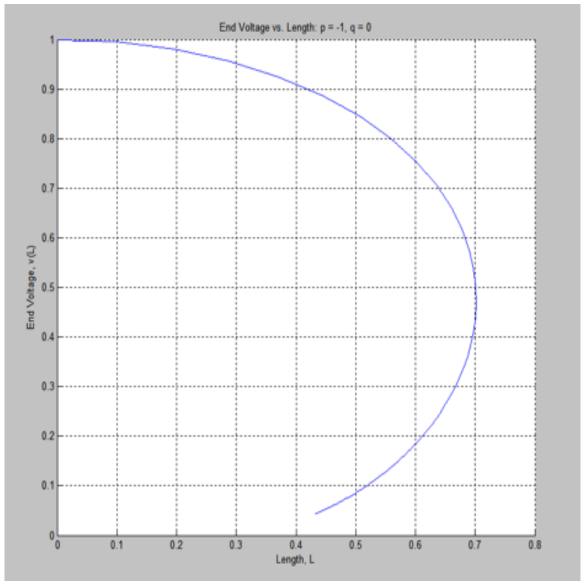

Figure 2: Voltage at the end of the line versus the length of the line, for p = -1 and q = 0

An important concept shown by these graphs is that there are two stable solutions to this system. This means that for the same length and system parameters, there can be two possible voltages at the end of the line. It was seen in Matlab that the solution for the voltage along the line depended on the guess of the solution used in the boundary value problem solver. Figure 3 shows how the voltages along a line of length $L = 0.5$ can be different depending on the guess. The green curve shows the voltage along the line when a guess of 0.1 is used for the solver, and this results in an end voltage around 0.13. The red curve shows the voltage along the line when a guess of 1 is used, which results in a higher voltage of around 0.77.

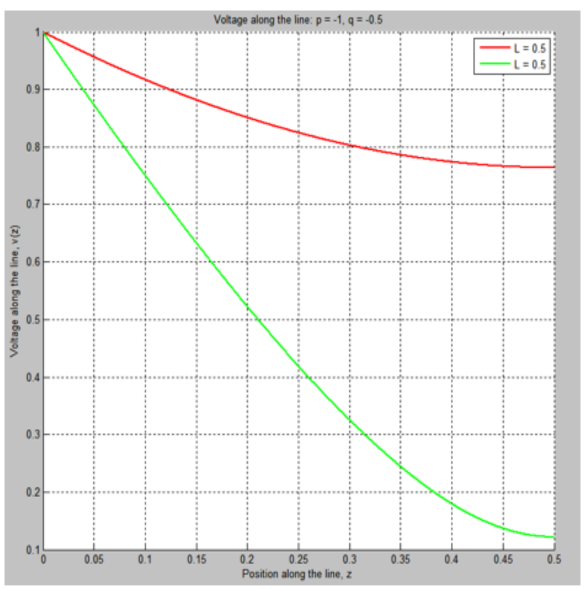

Figure 3: Voltages along a line, with $L = 0.5$; the red curve shows the solution for a guess of 1, while the green curve shows the solution for a guess of 0.1

Figures 4 and 5 describe, mathematically, how the system can produce two end voltages for the same length. Figure 4 shows that $\nu(s)$ is increasing over the entire range of s. Then, because the end voltage is simply $1/\nu(s)$, there will also be a unique end voltage for every value of s. Figure 5, on the other hand, shows that because L is proportional to the ratio of s to $\nu(s)$, the length can have the same value for different values of s. So, because the graphs are plotted against the same range of s, it is seen that there may exist two end voltages for a single length.

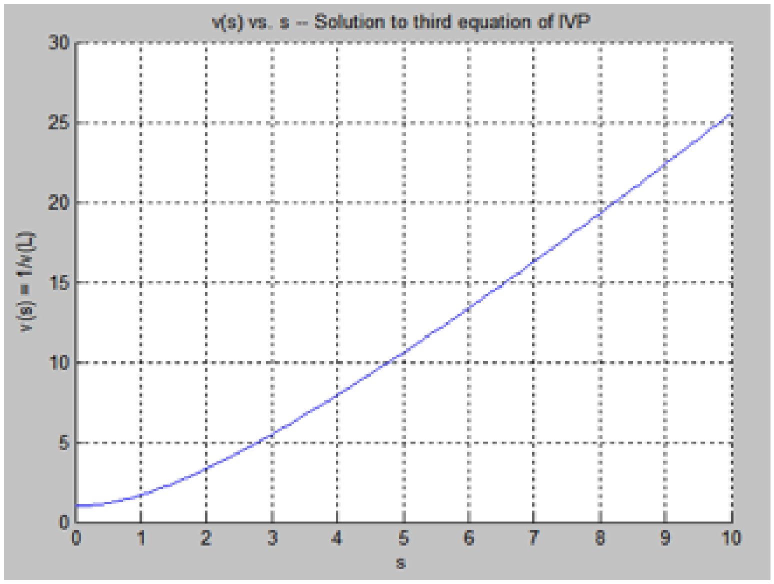

Figure 4: Graph of $\nu(s)$ vs. s; shows there is a unique end voltage for every s.

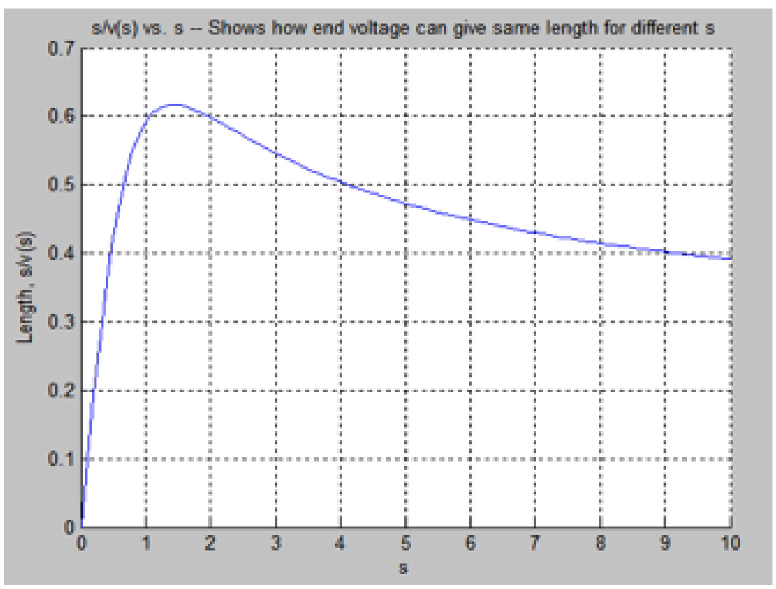

Figure 5: Graph of L vs. s; shows there are different values of s with the same length

Most of the time, the guesses resulted in the higher end voltage, which is good when analyzing how this relates to a real scenario. In real situations, a lower end voltage would be very bad and potentially dangerous, because it could lead to power outages. If the lower voltage is found to be the solution, and the line is used to power a load that requires a voltage higher than this, then it will not be able to accommodate the load. The same idea applies to the power used by the system, as seen later on.

The next main graph that was produced shows the power utilization versus the length of the line, as seen in Figure 6. The power utilization is defined to be the power injected at the beginning of the line divided by the power consumed along the line, given by $p * L$. This graph uses the same constant power consumption as the first graph of end voltage versus length, which means the parameters p and q are set to -1 and -0.5. Because the parameters and conditions of the system remain the same, there will be the same limit to the amount of power that can be consumed by the system. This, in turn, will limit the length to the same maximum value of around $L = 0.6$.

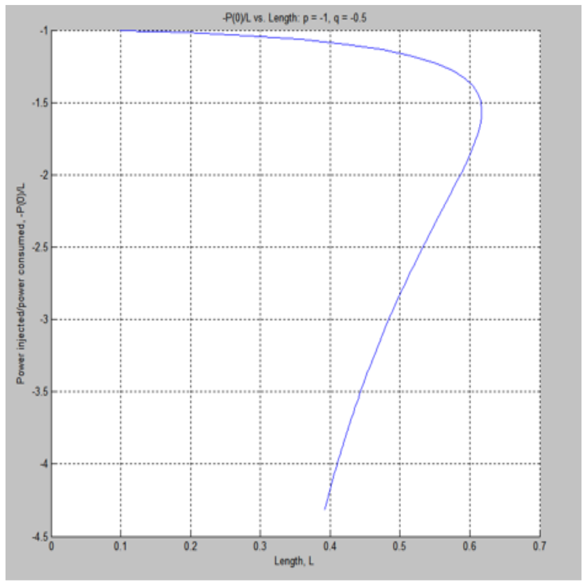

Figure 6: Power utilization vs. length of the line

This graph, again, shows that there will be two stable solutions for the power utilization at the same length. This is related to the bistability of the end voltage for a given length. Because there are two different possibilities for the power utilization at a length, L, there will be two possible initial powers that may be injected at the head of the line to satisfy the system. This can be seen by looking at $L = 0.4$ as an example. The higher power utilization, around -1, implies that the system requires an initial injected power of around 0.4. However, the power utilization may also be around -4, which would imply that the system needs an injected power of around 1.6. This can be seen by looking at the graph of the power along the line, shown in Figure 7. The solid lines show the real power, which is the type being considered here.

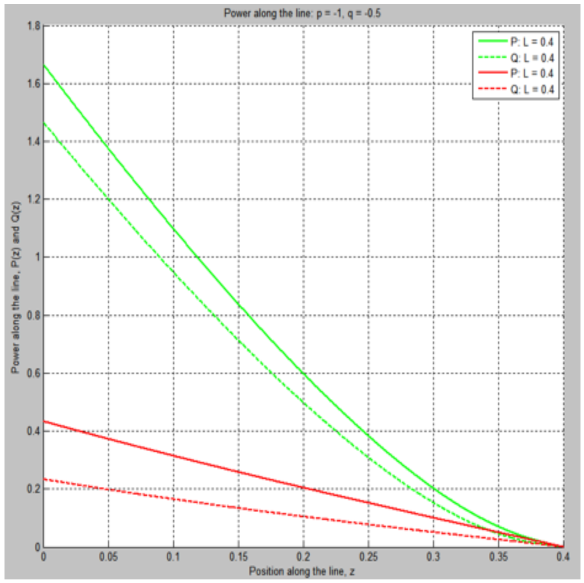

Figure 7: Power along a line, with $L = 0.4$; the green curve shows the higher power utilization (around -1), while the red curve shows the lower power utilization (around -4)

So, when the system tends towards the lower power utilization, it requires that four times the power be injected at the head of the line in order to satisfy the conditions of the system. This means that the system would need to be supplied with more power in order to meet all of the conditions. As previously mentioned, it would be bad to get the lower solution in a real situation, because there may not be enough power along the line to satisfy the power requirements of the loads. This could lead to power outages, because the loads would not be supplied with enough power.

Figure 8 shows the next graph, which displays the voltage along the line for lengths $L = 0.2$ and $L = 0.5$. The parameters p and q are both negative and constant, which implies constant power consumption. This is confirmed by the shape of the voltage curve, which is decreasing across the entire interval, because as the power decreases along the line, the voltage will also go down.

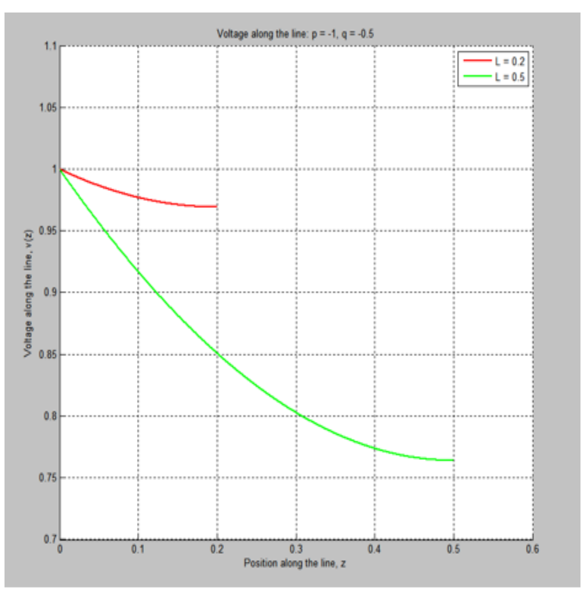

Figure 8: Voltage along the line, for $L = 0.2$ and $L = 0.5$

An interesting observation arose when experimenting with the program generating this graph. When a length of $L = 0.7$ was used, Matlab reported an error saying that a singular Jacobian matrix was encountered. This is consistent with the results of the graph of the end voltage versus length, which showed that the maximum length of the system was only around $L = 0.6$. Because the parameters p and q are the same for both graphs, this is the same system. When a length greater than the maximum was used, the boundary value problem could not be solved. In real situations, this means that for the given parameters, it is physically impossible for a line of the given length to meet all of the boundary conditions.

Figure 9 shows a similar graph of the voltage along the line, except that the parameter q has been changed from -0.5 to 0. It can be seen that a length of $L = 0.7$ is possible, as it is graphed along with the voltage along a line of length $L = 0.4$. This shows that because less power is being consumed along the line, a greater length is possible.

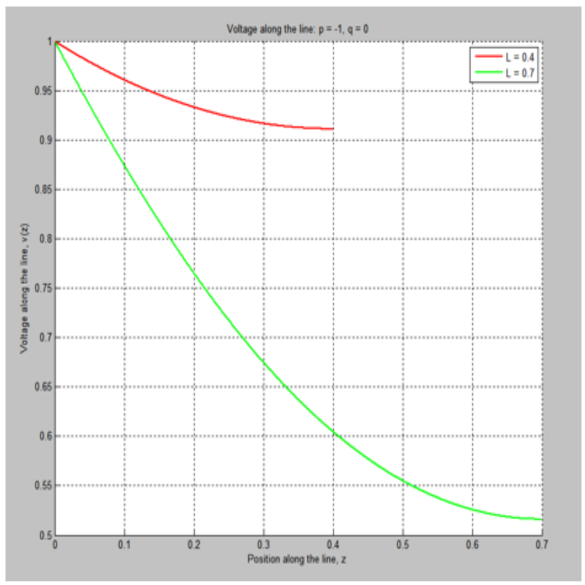

Figure 9: Voltage along the line, for $L = 0.4$ and $L = 0.7$

The next graph, Figure 10, shows the real and reactive power along the line, plotted for lines of length $L = 0.2$ and $L = 0.5$. The solid lines are used to show the real power and the dashed lines are used to show the reactive power along each line. The parameters $p = -1$ and $q = -0.5$ give a constant power consumption, which can be seen in the graph because each type of power is decreasing over the entire interval. The real and reactive power at the end of the lines both equal zero, satisfying the boundary conditions. This graph also shows that more power needs to be injected at the beginning of a line with a greater length, in order accommodate the amount of power being consumed along the line under the same parameters.

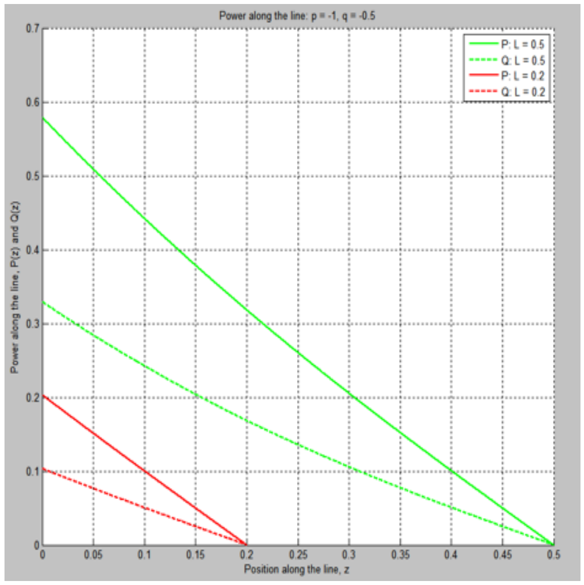

Figure 10: Power along the line, for $L = 0.2$ and $L = 0.5$

Figure 11 shows the graphs analyzed above, which were produced in Matlab by solving the initial and boundary value problems for parameters $p = -1$ and $q = -0.5$. Figure 12 shows the graphs displayed in Wang, Turitsyn, and Chertkov's article for the same parameters. This shows that the graphs and results of the article were successfully reproduced.

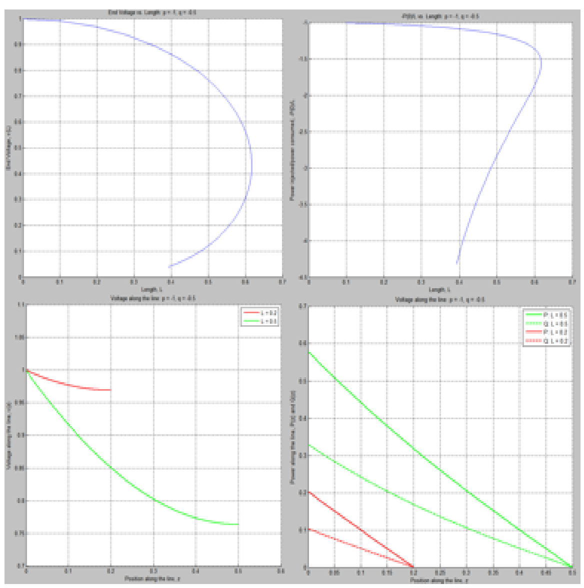

Figure 11: Graphs produced from Matlab, guided by the work in the article

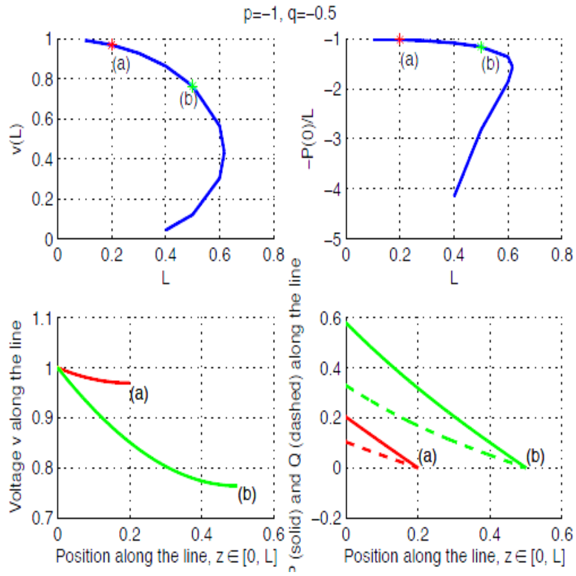

Figure 12: Graphs presented in the article

4 Adding Stochasticity

Throughout these previous results, the power consumption p was assumed to be constant along the line. However, it is clear that this is not the case for real world power distribution systems, seeing as every customer does not use the same amount of energy as other consumers. Hence, a stochastic component to this power consumption was added to observe how this varies our solutions to the system.

There are many possible ways in which stochasticity could be added to the system, but in this paper only one practical method of adding stochasticity was focused on. A Wiener process was implemented to the power consumption parameter in the boundary value conditioned ODE's. The Wiener process was chosen because its defining characteristics translate well to practical situations that a power distribution system might experience. First, the Wiener process has an expected value of 0, which means that the power consumption can be of the form $p = p_0 + W_z$, where p_0 is a constant and W_z is a random variable at position z along the line, and this will have an expected power consumption of p_0 (due to the linearity of the expectation operator). Second, a Wiener process is a collection of independent random variables such that $W_b - W_a$ is approximately normal with mean 0 and variance $b - a$ for all $0 \leq a < b$. Thus, this choice for power consumption allows the assumption that the changes in power usage from one customer to another are completely independent, and that there is not a drastic variation in power usage along the line.

In order to substitute $p_0 + W_z$ into the equations for the power consumption, the amount of random variables in the process must be specified. In the calculations, 100 steps were chosen in the process, meaning that the power line can be partitioned into 100 steps of equal length, with a power consumption of $p_0 + W_n$ at the n^{th} step along the line. Generation of these 100 random variables was performed using the Matlab program bpath1, which was developed by Jeffrey M. Moehlis.[4] In aligning the work with the previous results, the real power consumption was set to $p_0 = -1$, and the power consumption parameter $p(z)$ was defined to be a step function that assigned to each position z (between 0 and L, the length of the line) along the line the consumption value $p_0 + W_k$, where $k = \lfloor 100\frac{z}{L} \rfloor$. With the substitution $p(z) = -1 + W_{\lfloor 100\frac{z}{L} \rfloor}$, the power consumption is defined to be a randomly generated, yet everywhere computable function that represents one profile of power consumption. Therefore, this definition of power consumption can be used in the ODE's to solve the boundary value problem for distinct, randomly generated power consumption profiles.

From here, the effects of these randomized power consumption profiles on the solution to the boundary value problem were examined by preforming Monte Carlo simulations. In each simulation, a distinct power consumption profile was created, the boundary value problem was solved for the two lengths $L = .2$ and $L = .5$, and the solutions were plotted. These solutions were then averaged and the average variance was computed so that the typical effect of a randomized power consumption could be revealed.

5 Results

Figure 13 below depicts the aggregated results of 1000 simulations, as described above, along with the averaged solution and the solution in the case of a constant power consumption of -1.

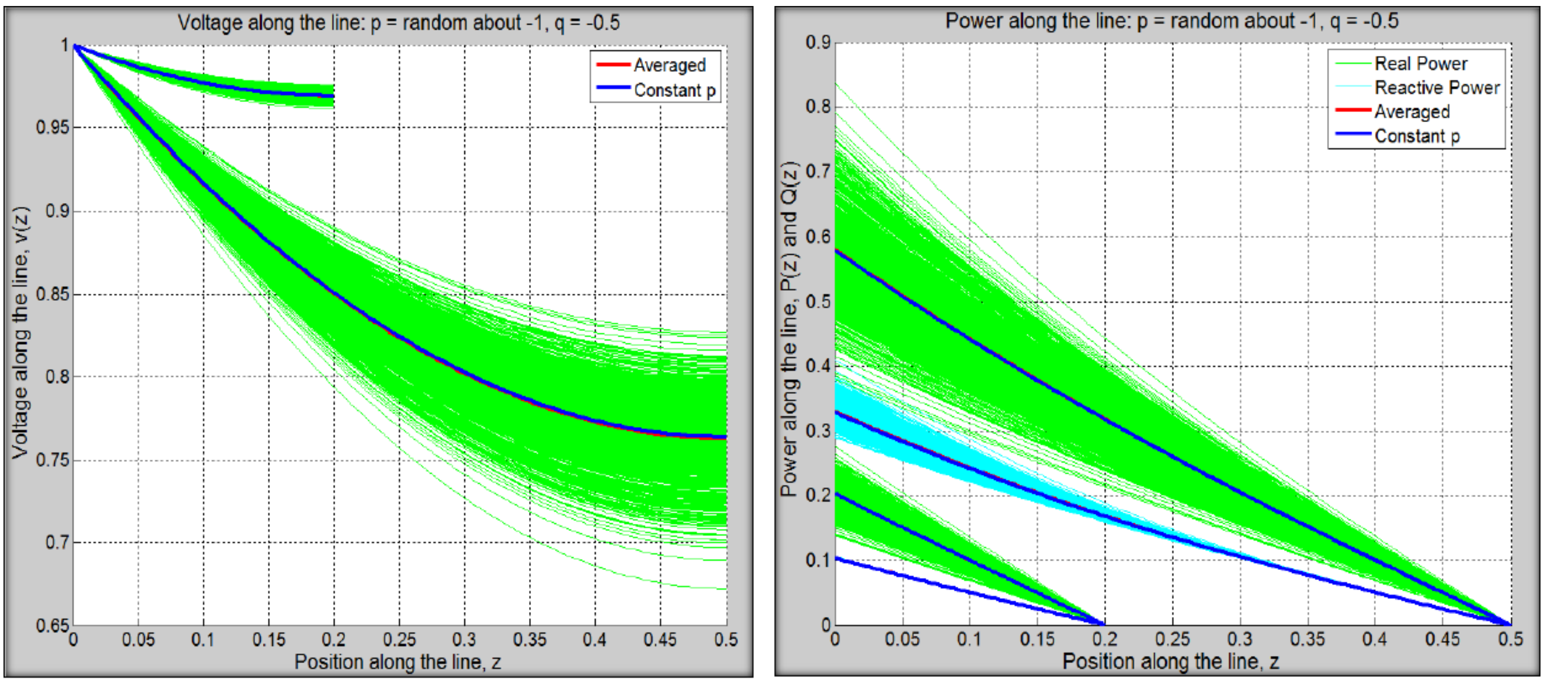

Figure 13: Each green line represents an individual solution to the boundary value problem for a distinct, randomly generated power consumption profile. The averaged solution (red) appears to lay exactly on the solution with constant power consumption.

As can be seen, the individual solutions average almost exactly to that of the solution in which power consumption is constant, and the variance of these paths from this average for $L = .2$ is less than 2.5×10^{-5}. Moreover, each one of the solutions has an end voltage of greater than .65, which shows that all of these stochastically generated power consumption cases lie on the upper branch of the initial value problem. These results show that it is statistically valid to assume a constant power consumption, since these stochastic perturbations tend to have little overall effect on the system. In addition, these simulations suggest that slight variations in power consumption alone does not serve as an explanation as to why voltage drops occur, since none of the solutions appeared to lie on the lower branch of the initial value problem.

It is also clear that the variation of the solutions in which $L = .2$ is much less than that of the solutions for $L = .5$. Although this is not a completely trivial result, it does align with intuition in that a longer line can be influenced more by the factors of line resistance, reactance, and fluctuations in power consumption. In noticing these differences, we found it of interest to study how the variance of our consumption parameter influences the variance of the solutions. To do this, we fixed $L = .2$ and let the variance of $p(z)$ range over one tenth of integral powers of 2 from -4 to 4, then compared these variances to the variances of the solutions. Figure 14 shows a scatterplot of these results.

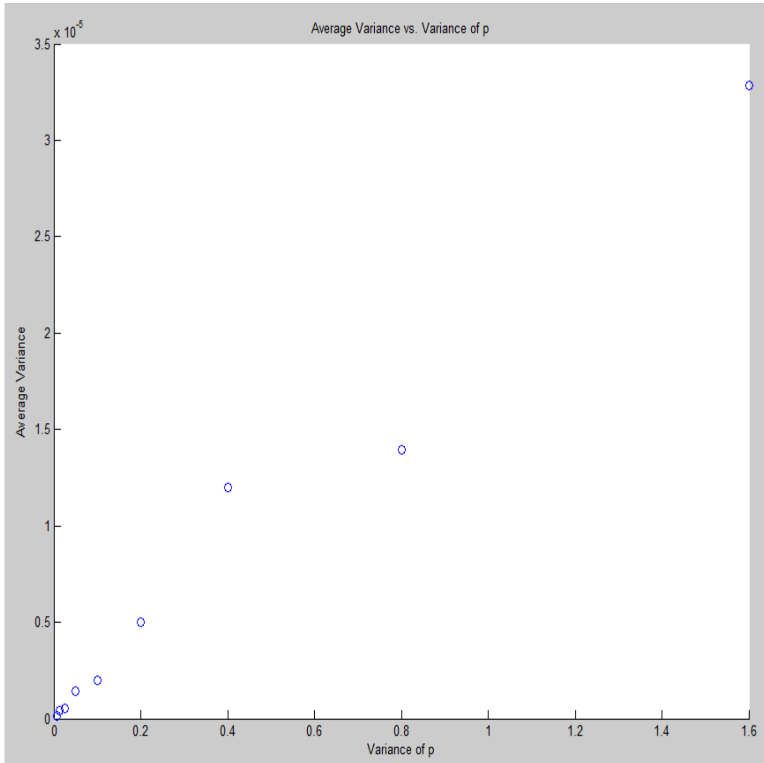

Figure 14: Here, variance of $p(z)$ is adjusted within the bpath1.m code.[4] The variance of the solutions was computed using the difference of moments formula: $\mathrm{Var(X)=E(X^2)-(E(X))^2}$.

This plot appears to suggest a proportional relation between the variance of $p(z)$ and the variance of the solutions. However, the proportion appears to be of the order 10^{-5}, which further confirms our results that variations in power consumption across the line has little effect on the overall tendencies of the solutions to the boundary value problem.

6 Conclusions and Path Forward

The major points of this paper can be summarized as follows:

- The work of Wang, Turitsyn and Chertkov[4] shows that there can be two solutions to the system of ODE's for large enough L when there is constant and uniform power consumption along the line. One of these solutions corresponds to a desirable system, in which the voltage along the line is satisfactory, and the other corresponds to an undesirable scenario in which the system experiences a power failure.
- Adding stochasticity to the power consumption in this model does very little in terms of varying the solutions of the boundary value problem. Thus,it is statistically valid to assume a constant power consumption along the line. Further, it was shown that the variations of the power consumption profiles contributes linearly to the variance of the solutions to the boundary value problem, as a factor of approximately 1.6×10^{-5}.
- It appears as though the solution with the higher end voltage is a more stable and probable solution than the solution with a lower end voltage. This result is suggested (but not verified) by seeing that in none of the simulations did the solution have a drastically low end voltage.

There are still many paths of further research of these models of energy flow that will contribute to a better understanding electrical distribution systems. Specifically, the other two assumptions - static and one dimensional flow - made in the development of this model need to be addressed. By examining time dependence and thereby a dynamic model, the actual stability of the two stable voltages can be better understood. This may lay insight into what can cause a 'jump' or voltage drop from one solution to the other. However, another way to study this phenomenon could be to take the boundary value problem and perturb the initial guess to the solution and examine the effects of these perturbations on the end voltage. From this, one might be able to give threshold conditions under which the system could end up on the lower branch of the initial value problem. Probabilities of being on the upper and lower branches could then be calculated to give a sense of the chance of having a lower or higher voltage at the end.

It is also important to address the challenge of transforming these models to incorporate branching nodes in the feeder line. This would provide a much more sophisticated and realistic model, since most electrical distributions systems are designed as a grid, and not a straight line. This problem can be approached by building from the discrete form of the differential equations, and considering specific cases for varying number of buses and/or branching nodes. Discrete analysis on a system of equations, with the same ideal boundary conditions at each endpoint as presented here, can then allow for a multi-dimensional model.

7 Acknowledgments

Many thanks are extended to both Professor Ildar Gabitov of the mathematics department at the University of Arizona and Aalok Shah for respectively motivating and mentoring this research.

8 References

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