

Sinking Bubble in Vibrating Tanks
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Introduction:

It is experimentally observed that bubbles will sometimes sink to the bottom of their container. This strange phenomenon is only observed when the container is oscillating vertically (coaxial with gravity). A good example of this in real life was given to us anecdotally by Dr. Gabitov. In the early stages of 3-stage rocket testing, many rockets would explode shortly after take-off resulting in massive damages and wasted resources. It was later determined that the pressure sensors, that detonated small charges to separate the sections of the rocket mid-flight, were being triggered much too soon. This resulted in the fuel at the top of the tank being ignited which caused the uncontrolled explosion. The reason why the pressure sensors were being triggered too soon is that the vertical vibration of the rocket at take-off forced all the air bubbles in the fuel tank to sink to the bottom. This generated a region of very low pressure at the bottom of the fuel tank where the aforementioned sensor was placed. This result, and other experiments, spurred an interest into the behavior of bubbles in oscillating fluids.

Physical Model:

The physical action responsible for the sinking bubble is a property called added mass [1]. The basic idea is that the bubble must pull water along with it in order to move. This makes the bubble itself seem like it has more inertia than it actually does. How does this help the bubble move downward? To understand that first we must talk about the volume of the bubble. We modeled the bubble to be a sphere, so its radius is all we need to determine its volume and vice-versa. In addition, we modeled the bubble to be compressible. This means that its volume is changing as more or less pressure is exerted on it. From fluid mechanics, we learned the pressure in a fluid is determined solely by the depth in that fluid. The deeper you go, the more pressure you feel. Thus the size (volume) of the bubble will be smaller as it approaches the bottom of the container. This is the most important feature because the added mass increases with the size of the bubble. This makes sense because the larger an object is, the more water it has to push out of the way in order to move. The added mass will resist the motion upwards in the tank more than downwards because the size of the bubble gets bigger as it moves up and smaller as it moves down. Because the tank of water is oscillating, the bubble moves both up and down in the water allowing it to sink more than it rises as the tank is shaken. If the conditions are correct, the added mass will be the strongest force the bubble experiences and this will cause the bubble to sink all the way to the container's bottom.

What we mean by conditions are the following: the initial depth of the bubble, the amplitude of oscillations, and the frequency of oscillations. We take the size of the bubble to always be small (sufficiently small enough to exhibit isothermal compression) as well as the pressure outside the water to be large enough that everything in the water behaves nicely. Still there are other important forces on the bubble that we must take into account.

The first is the force from the rest of the water. Since the entire container is shaking, the water pushes on the bubble in the same direction the tank is shaking, regardless of what direction the bubble is trying to move.

There is also a drag force. This is basically like friction only stronger. Any time an object moves through a fluid (like a ball through air, or a bubble through water) it is in contact at all sides with the fluid. This creates a force opposing motion that depends on the surface area of the bubble and the speed at which the bubble is moving through the water.

There is also a so-called Archimedes force on the bubble. The Archimedes principle states that the volume displaced by an object in water will exert an equal and opposite force on that object. That means there is a constant upward force proportional to the volume of the bubble. This principle is why boats float and why bubbles normally rise to the top of their containers.

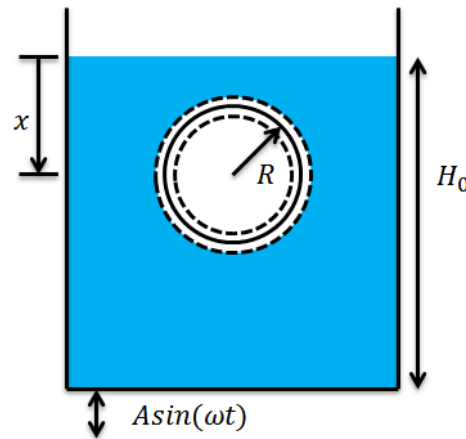
And lastly we considered the weight of the bubble. Gravity pulls constantly downward on the bubble since it has a small mass.

Using fluid mechanics and the basic assumption that the events in the container are quasistatic, we were able to model the motion of the bubble in an oscillating container.

PHYSICAL MODEL

○ Variables and parameters

- x – Bubble depth
- R – Bubble radius
- H_0 – Total height of water
- A – Amplitude of oscillations
- ω – Frequency of oscillations
- t – Time



To model the sinking bubble problem we start with a few basic constants. We have an open container with a water level a distance H_0 from the bottom. A bubble then pre-exists in the water at a depth x_0 below the surface of the water. This bubble is spherical and compressible which means that its radius will change and its volume depends only on the radius. In addition, we apply small amplitude oscillation of magnitude A to the container at an angular frequency ω . Thus, the vertical motion of the tank is characterized by

$$A * \sin(\omega t)$$

Then we considered the mathematical meaning of each force. First of all, the Archimedes force which is buoyant force is always upward, so it is negative with

$$F_{buoyant} = -\rho g V_b,$$

where V_b is the instantaneous volume of the bubble solved from the quasistatic relation $P_e V_{b0} = P_x V_b$, where P_x is the pressure at depth x , V_{b0} is the volume of the bubble at the surface, and P_e is the external pressure.

$$F_{buoyant} = -\rho g V_{b0} \left[1 - \frac{\rho x g}{P_e} \left(1 + \frac{A \omega^2}{g} \sin \omega t \right) \right]$$

Since the density of water and gravitational acceleration are both constant, the buoyancy force is proportional to the volume of bubble. When bubble is at the bottom of tank, the volume is smaller, so the Archimedes force is smaller as well.

The equation for the drag force in turbulent flow is

$$F(\dot{x}) = \frac{1}{2} \rho \dot{x}^2 C_d A.$$

C_d is the drag coefficient and it is related to the viscosity of the fluid. From this equation, we see that the drag force is proportional to square of velocity and the surface area. As the depth of the bubble decreases, the surface area increases, so friction is proportionately larger.

We now get to the reason why the bubble can sink: added mass. In the case of bubble floating, the water above the bubble's surface is barrier for bubble to move up. In order to move up, it has to push this water out of its way and pull an equal amount of water along behind it. In this process this water can be treated as extra mass which bubble needs to carry around to be able to move. That result of this phenomenon is a larger observed inertia

$$M = m + m_0$$

where m is the intrinsic mass of the bubble.

$$m_0 = \frac{1}{2} \rho_{fluid} V_b = \frac{1}{2} \rho_{fluid} V_{b0} \left[1 - \frac{\rho x g}{P_e} \left(1 + \frac{A \omega^2}{g} \sin \omega t \right) \right]$$

is the added mass, which changes with time because we assume the bubble is moving due to the oscillating fluid. Also when bubble's position stays constant, the added mass is not observable because the inertial mass of the bubble is not observable.

In the process of developing the governing equation, we consider Newton's 2nd Law

$$F = \frac{dp}{dt}$$

with time varying mass and velocity. After taking the derivatives

$$F = (m + m_0)\ddot{x} + \dot{m}_0\dot{x}$$

Plugging in all of the forces that are derived above, we get

$$(m + m_0)\ddot{x} + \dot{m}_0\dot{x} = -F(\dot{x}) + (m - \rho_{fluid}V_b)(A\omega^2 \sin(\omega t) + g)$$

Recall that

$$V_b = V_{b0} \left[1 - \frac{\rho x g}{P_e} \left(1 + \frac{A\omega^2}{g} \sin \omega t \right) \right]$$

$$F(\dot{x}) = \frac{1}{2} \rho \dot{x}^2 C_d A * \text{sign}(\dot{x})$$

$$m_0 = \frac{1}{2} \rho V_b$$

Analytical Results

From the governing equation, we see that the forcing functions have dependence on two time scales: one slow time scale (t) and one fast, periodic time scale (ωt). We also see the bubbles overall behavior has these two distinct timescales in the computational results. Since the interesting behavior (whether or not the bubble sinks) occurs on the slow timescale, we use the method of direct separation of motion detailed in Vibrational Mechanics [3]. In this method, we seek a solution that is the sum of a slow and fast function, where the fast function depends on some periodic timescale, (ωt):

$$\langle h(\tau) \rangle = \frac{1}{2\pi} \int_0^{2\pi} h(\tau) d\tau$$

$$x(t, \tau) = X(t) + \Psi(t, \tau)$$

$$\langle \Psi(t, \tau) \rangle = 0$$

$$\langle X(t) \rangle = X(t)$$

$$\langle x(t, \tau) \rangle = X(t)$$

Because the fast function Ψ is periodic, its average value over the period is zero. Since the slow function X is slow relative to Ψ , its average value over the period is the function itself. Thus, the average value of the bubbles trajectory is the slow function. Therefore, the equation of motion in regards to the slow timescale is simply the average of the overall equation of motion while the equation of motion for the fast timescale is comprised of any leftover terms. After plugging in the solution $X+\Psi$ into the equation of motion, taking the average, and making simplifications based on our original assumptions and neglecting higher order terms, we obtain the slow equation of motion:

$$(m + m_{01})\ddot{X} - \gamma \cdot m_{01} W \frac{X}{H_0} (\langle \ddot{\Psi} \sin \omega t \rangle + \omega \langle \dot{\Psi} \cos \omega t \rangle) = - \langle F(\dot{X} + \dot{\Psi}) \rangle + (m - \rho V_{b0})g + \gamma \cdot \rho V_{b0} W \frac{X}{H_0} \frac{A\omega^2}{2}$$

and the fast equation of motion:

$$(m + m_{01})\ddot{\Psi} = -4\rho R_0^2 \psi_\infty (\dot{\Psi}^2 \operatorname{sgn} \dot{\Psi} - \langle \dot{\Psi}^2 \operatorname{sgn} \dot{\Psi} \rangle) + (m - \rho V_{b0}) A \omega^2 \sin \omega t$$

Because the slow equation contains terms that depend on the fast solution, we first solve the fast equation. Solving the fast equation allows the slow equation to be transformed into the following:

$$m_{01}\ddot{X} + \frac{16}{\pi} \rho \psi_\infty R_0^2 \dot{X} B \omega = \gamma \cdot W^2 \frac{X}{H_0} \frac{\rho V_{b0} g}{2} \cdot \left(1 - \frac{2}{3} \frac{\theta \frac{A^2}{R_0^2}}{2 \left(1 + \sqrt{1 + \theta \frac{A^2}{R_0^2}} \right) + \theta \frac{A^2}{R_0^2}} \right) - \rho V_{b0} g$$

$$\theta = \frac{16^2 \psi_\infty^2}{\pi^4 \chi^4}$$

Because the bubble moves slowly, we can make assume that

$$\ddot{X} \ll 1$$

to reduce this equation into a first order equation:

$$\dot{X} \approx v \left(\frac{X}{X_0} - 1 \right)$$

where

$$X_0 = \frac{2H_0}{\gamma \cdot W^2} \frac{2 \left(1 + \sqrt{1 + \theta \frac{A^2}{R_0^2}} \right) + \theta \frac{A^2}{R_0^2}}{2 \left(1 + \sqrt{1 + \theta \frac{A^2}{R_0^2}} \right) + \frac{\theta A^2}{3 R_0^2}} \quad v = \frac{\pi^2}{12\psi_\infty} \frac{R_0}{B} \frac{g}{\omega}$$

We see that this equation has one fixed point at $X = X_0$, which is unstable. This means that if the bubble does not start at X_0 , it will sink or float to $\pm\infty$. However, since the tank containing the fluid is finite, we impose boundary conditions at 0 and H_0 which both become stable fixed points. This averaged equation is compared to the computational results in the computational results section.

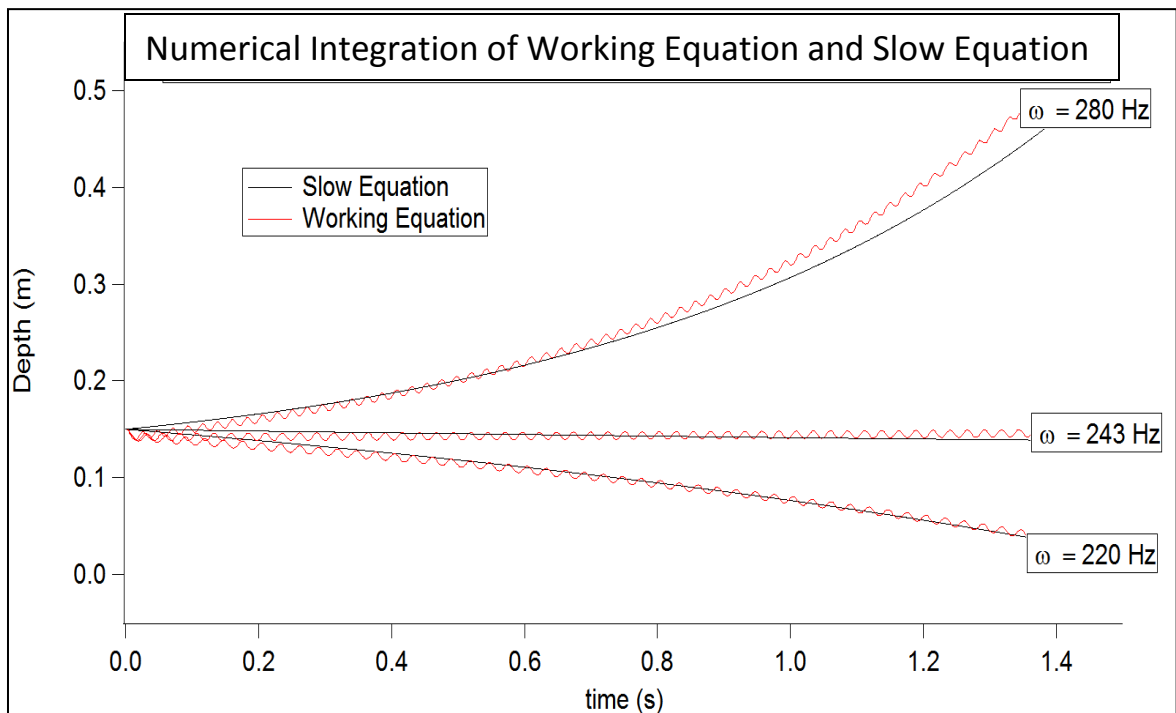
Computational Results:

After achieving the working equation, the acceleration of the depth was solved for to allow for a numerical integration method to solve it. A velocity-verlet integration method was implemented to advance to depth as a function of time. This was compared to the integration of the first-order slow equation derived above. This involves advancing the velocity of the bubble as a function of time and then advancing the position based upon the time-stepped velocity.

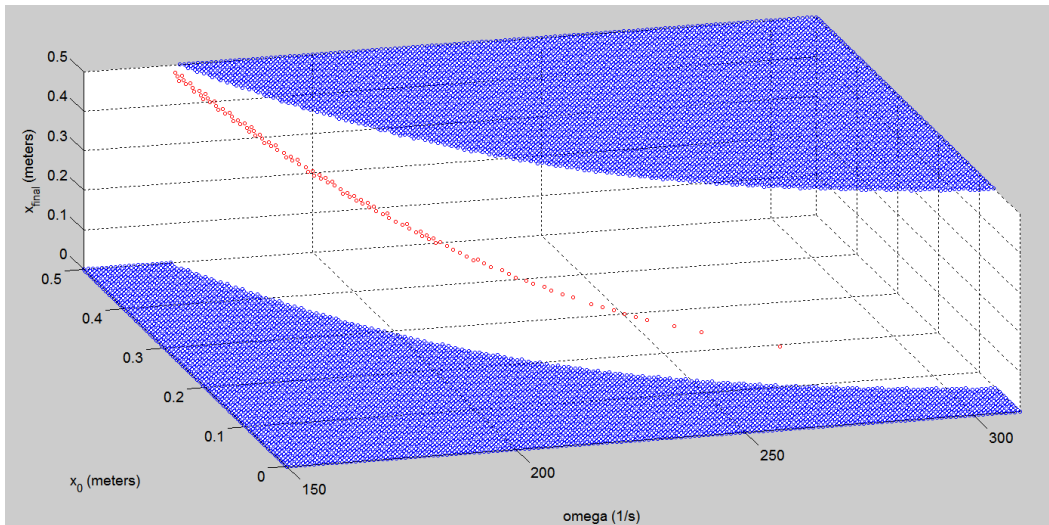
Using this integration method, the depth of the bubble is advanced over many time steps. To do so we chose the following parameter values and varied the initial depth and oscillation frequency to view different behavioral regions.

$$\begin{aligned}
 x_0 &= 0.15 \text{ m} \\
 H_0 &= 0.5 \text{ m} \\
 A &= 0.002 \text{ m} \\
 R_0 &= 0.0029 \text{ m} \\
 \rho_{fluid} &= 1000 \frac{\text{kg}}{\text{m}^3} \\
 \rho_{bubble} &= 1.2 \frac{\text{kg}}{\text{m}^3} \\
 P_e &= 10^5 \text{ Pa}
 \end{aligned}$$

By doing so, we produced the following graph which only varies in terms of the oscillation frequency.

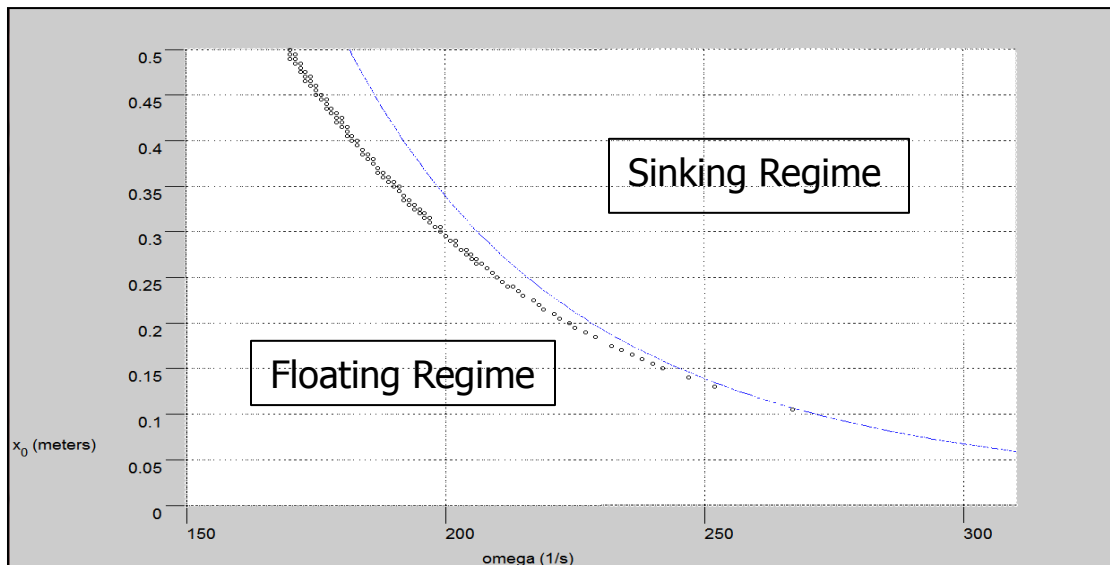


There are three types of end behavior: the bubble sinks, the bubble floats, or the bubble is trapped near its initial depth. We notice that the slow equation behaves exactly as the working equation until you get to longer timescales. This is due to dropping the acceleration term from the slow equation. To further show this behavior, a bifurcation diagram is produced by using the above parameters and by performing the integration on the working over x_0 from 0 to H_0 with a step size of $0.01 * H_0$ meters and ω from 150 to 310 with a step size of $1 \frac{1}{5}$. This produces the three dimensional bifurcation diagram that can be seen below.



We limit the range of initial position to starting within the water tank. We limit the frequency of oscillation by the requirements of assumptions made during the derivation of the working equation. For greater than 310, the cavitation requirement no longer holds, and for ω less than 150 the vibrations are not strong.

From this diagram, we can see that there are two stable solutions, the sinking and floating bubble, and an unstable solution, the trapped bubble, which agrees with our original simulations. We then compare this to the fixed point analysis of the slow equation, to determine how closely the numerical solution agrees with the analytical solution in its end behavior.



From this comparison, we notice that the integration of the working equation agrees quite well with the fixed-point analysis of the slow equation quite well, especially in the regime of high frequency oscillation and small initial depth. Contrasting this, there is a slight divergence at

low frequency oscillation and large initial depth that is most likely due to dropping the acceleration term in the slow equation.

Analyzing the bifurcation diagram can tell us a few things. Examining behavior of the system for small H_0 shows that there is no initial height at which the bubble can be placed that will cause the bubble to sink, the bubble will float even if initially placed at its maximum depth. Now we can consider the behavior of the solutions for small initial depth. The bifurcation diagram indicates that there does not exist an oscillation frequency which will cause the bubble to sink if it is placed at a small enough initial depth. This is only true for our ranges of ω , as it appears that the bifurcation diagram has not turned around entirely yet at $\omega = 310$. A restructuring of the original derivation of the working equation that would allow for $\omega > 310$ is needed to confirm whether or not the bifurcation diagram turns around at large ω and small x_0 .

Computational analysis shows that our model of a bubble in an oscillating tank displays three sets of behavior. It also shows that there are two clear timescales of the bubble's motion: the rapid position changes and the drift velocity to the final state. This confirms the assumption of two well defined timescales that we plan to use to separate the working equation and analytically solve it.

Conclusion:

In the absence of oscillations, bubbles do not sink. This is due to the buoyancy force being much greater than the gravitational force since the mass of the bubble is very small. When oscillations are added to the system, the bubble gains an added mass which is an apparent inertial increase that results from water being displaced as the bubble moves in the fluid. When the oscillations are fast enough, the added mass will cause the downward acceleration due to gravity to be greater than the upwards buoyancy acceleration. Thus, the bubble sinks. This is shown in both the computational and analytical results.

Our computational results showed that the two most important factors in the behavior of the bubble in the oscillating fluid are the frequency of oscillations and the initial depth of the bubble. If the bubble initially begins at a depth that is too close to the top of the fluid containing vessel, it will not sink regardless of the oscillating frequency. This is due to the overall downward force from the water above the bubble is small enough compared to the buoyancy force that the added mass from the oscillations cannot overcome the upward force. If the bubble starts at a large depth, lower frequencies of oscillations can be used to cause bubble to sink. This is because at the greater depths, the downward force from the water above the bubble is large, so only a small added mass is needed to overcome the buoyancy force. For each frequency there is a special initial depth at which the bubble will remain stationary in the fluid. This is because the force from the above water combined with the added mass is perfectly balanced with the buoyancy force.

The frequency of oscillations is also a very important parameter in the behavior of the bubble. If the frequency is too low, the added mass of the bubble will never be enough to overcome the buoyancy. At large frequencies, the added mass is also large so the bubble can

sink at smaller initial depths. The relationship between the initial depth, frequency of oscillation, and behavior of the bubble is well shown in the bifurcation diagram.

In the future, we may also want to consider the behavior of the system under different conditions. For example, since the sinking bubble became a problem during the flight of rockets, it would be reasonable to explore the system's behavior in an environment in which the external pressure is very low, such as on the moon. Additionally, this model assumes the container is completely rigid, so adding elasticity to the container as it oscillates may produce interesting results. Finally, we may want to consider the behavior of clusters of bubbles. In the paper by Kana and Dodge, sinking bubbles were observed to create large bubble clusters at the bottom of the tank which led to violent behavior in the system [2].

Works Cited

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