Miyant'e Newton, Ken Peng, Eric Sonera
May 5, 2013
Math 485

Electromechanical Properties of Bones

Abstract

Bones are an integral part of the human body that exhibit piezoelectric properties, meaning that when stress is applied to the bone, the bone produces a current within itself. This is known as the piezoelectric effect. Material that are piezoelectric can similarly exhibit the reverse piezoelectric effect, meaning that when a current is applied, the material compresses on itself. This property has many applications such as applying a current to a fractured bone to promote compression of the bone to allow it to heal faster. However, too much of a current will destroy the tissue of the bone,thus killing the cells. The goal of our research is to determine the electrical conductivity of the human bone to establish the maximum amount of current that can be applied before the bone tissue begins to become damaged. Our model will expand upon that which is described by R. Casas and I. Sevostianov by considering the interaction between the Haversian and Volkmann canals of bones and their effect on the overall conductivity of the bone.

Introduction

It is well known by everyone that bones are a crucial component to the human body. Bones provide stability of structure so that humans may stand upright and bones also provide protection for vital organs such as the brain, lungs and heart. Red and white blood cells are also made within bone marrow so bones can be seen to house important physiological processes as well. Though the importance of bones is well known, intensive study on bone structure, also known as osteology, only began within the last century. These studies led to the discovery of many intriguing properties which commands further exploration.

One of the properties discovered, was that bones are dielectric. This implies that they themselves are semi-conductive. In addition, they are anisotropic, so not only are bones conductive, they conduct differently depending on the direction of the flow of the current through the bone. The major find was that bones are piezoelectric materials. This piezoelectric property implies that a voltage could be applied to the bone and the bone itself would compress. It is also well known that if a bone is fractured, compressing the fracture shortens the healing time. So in theory, if there were many small fractures in the bone, a voltage could be applied to the bones which would then cause the bone to compress and allow the fractures to heal quicker. However, since bones are semi-conductive this means that if a voltage is applied to a bone, a current will be produced which could cause damage to tissue inside the bone which would effectively kill the bone and prevent it from healing. Therefore, the motivation behind the model is to accurately describe the flow of any current in the bone so that the exploitation of the piezoelectric property can be safely applied as a treatment.

Before modeling the flow of current, certain physical properties of the bone needs to be addressed. The structure of bone is quite complicated and can be seen as a network of different parts all intertwined in a specific manner. Since the goal is to model the total conductivity of a bone, the structure of the bone can be simplified as a sum of three different parts: Haversian canals, Volkmann canals, and the rest of the bone which will be referred to as the bone matrix. The Haversian canals run parallel to the length of the bone, are an average 5 mm in length, and are approximately 50 microns in diameter. Volkmann canals connect Haversian canals together and they are orientated in the perpendicular direction of the Haversian canals. Volkmann canals are about 0.5 mm in length with an average diameter of 5 microns. The aspect ratio, the ratio of length to diameter, of the Haversian and Volkmann canals are both equal to 100 and, they have the same filling factor of 2%. Since the canals contain capillaries and nerve tissues which both are much more conductive than the bone matrix, the canals can be approximated to be metallic. The provided information allows the start of the formation of an accurate model.

Before current can be modeled the bone must first be simplified even further so that electrodynamic techniques can be properly used. Since Casas and Sevostianovs' paper, **Electrical resistivity of cortical bone: Micromechanical modeling and experimental verification,** was used as a reference when constructing the model, their bone model as well as an alternative bone model will be described along with the benefits and pitfalls of each model. In Casas and Sevostianovs' paper they made three assumptions; one assumption was that the total conductivity of the bone matrix was σ, independent of direction, another assumption was that no canals had any interactions with each other and therefore no canals intersected and the last assumption, was that each canal was viewed as an elongated ellipsoid. Casas and Sevostianov then, using well know electrodynamic equations, calculated the contribution of conductivity of one Haversian canal, using the fill factor and averaging techniques, calculated the average contributions to conductivity of all the Haversian canals. They used a similar technique and found the contribution to conductivity of all the Volkmann canals and then summed the three contributions (bone matrix, Haversian canals, and Volkmann canals) together to equate the total conductivity.

The model becomes simple and straightforward so finding parameters becomes almost trivial which makes the model as one of the most attractive choices however there is a flaw in the assumptions. One of the assumptions was the the canals have no interactions with one another however if there are no intersections of canals, specifically Volkmann and Haversian canals, then biologically the bone in study would be dead and no longer heal. This implies that any answer derived from the model may have substantial error in it. Therefore, in order for the model to be accurate it must account for at least some interaction between the Volkmann and Haversian canals.

An alternative view of the bone is conductive ellipsoids inside a semi-conductive matrix. In this model the canals are also viewed as elongated ellipsoids however, instead of viewing the canals as separate, the Volkmann canals will be part of the matrix, giving it a uniform conductivity perpendicular to the Haversian canals. The rationale involves using the fill factor of the canals. Since the canals have the same fill factor but differ in size by a large magnitude, this implies that there is a significantly higher quantity of Volkmann canals then there are Haversian canals. Due to this significant difference in amounts of the canals and the large difference of conductivity between canals and the bone matrix, it can be approximated that the Volkmann canal and the bone matrix can be viewed as a homogenized substance with a new conductivity, φ, that points along the axis perpendicular to the Haversian canals. For this approximation it must be assumed that the number of of Volkmann canals are roughly uniform within their plane. This assumption isn't drastically far fetched since there is no biological reasoning against it. After the homogenization, the bone can be viewed as a set of conductive ellipsoids inside a lesser conductive matrix whose direction of conductivity is perpendicular to the ellipsoids. This model better accounts for the interaction between the two canals and is still relatively simple since the electrodynamic equations for the system are well known.

However no matter which bone model is used there is still the issue of dealing with charge flowing in different directions due to the anisotropic nature of the bone. Unlike a metallic rod where the current will flow in a unidirectional fashion, when current travels along a bone it will branch out and go in different directions making it more complicated to track all of the current and every flow. Another technique must be introduced so that the problem can become manageable.

Explanation of Tensors

As previously discussed, bones are anisotropic, meaning that the conductivity of the bone may differ dependent on the direction considered. This creates a problem in modeling the electrical conductivity of the bones. To deal with this problem, the mathematical concept of tensors is introduced. Tensors, in a basic definition, are multidimensional arrays that describe some physical property. In our case, tensors will help us to describe the electrical conductivity of the bone in any direction. The rank or order of the tensor describes the dimensions of the array itself and is dependent upon the number of indices needed to describe the array. In general, a rank zero tensor is a scalar, a rank one tensor is a vector, and a rank two tensor is a matrix. One important feature of a tensor is that when using a tensor in a mathematical expression, the result should remain the same no matter the space that the tensor is used in.

There are three types of tensors that describe this, contravariant, covariant, and mixed tensors. Contravariant tensors describe the change in a displacement vector from one coordinate space to another. For example;

$$A'^{\mu} = \frac{\partial x_{\mu}}{\partial x_{\sigma}} \cdot A^{\sigma}$$

$$\begin{pmatrix} \mu = 1,2 \\ \sigma = 1,2 \end{pmatrix}$$

Describes a transformation from the coordinate system σ to the μ coordinate system by describing the change of the two coordinate basis vectors in μ with respect to the two coordinate basis vectors in σ. In this example, this describes a rank one contravariant tensor, which is denoted by the index labeled as a superscript. This definition uses the Einstein summation convention, which states that anytime a product between two objects contains the same index as a subscript on one term and a superscript on another, (or in this case in the numerator of one term and the denominator of another) it is implied that the product is summed over all possible values of the index. For example;

$$y = \sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$$

Similarly, a covariant tensor describes the transformation of a gradient vector from one coordinate space to another and is described mathematically as;

$$A'_i = \frac{\partial x_j}{\partial x'_i} A_j$$

This describes a rank one covariant tensor, or covector, that describes the change of a gradient vector in coordinate space j to the coordinate space i. A covariant tensor is described by the index placed as a subscript. A mixed tensor is a tensor that has indices as both a subscript and superscript. Therefore, the smallest rank that a mixed tensor can have is a rank of two. The usefulness of tensors can be seen in this one property in that we can describe some physical property in a coordinate space that is more intuitive and then transform it to a space that has more physical meaning. For example, when describing the flux of water through a cylinder, a good approach may be to describe the flux out of the cylinder in cylindrical coordinates where computation will be simpler and then transform the result into cartesian coordinates. The primary tensors used in the model in the paper by R. Casas and I. Sevostianov are the resistivity contribution tensor, **R** and the conductivity contribution tensor, **K**, which is further described with respect to Eshelby's tensor $\mathbf{s}^c$.

Model on the Conductivity of Bone

The model created by R. Casas and I. Sevostianov first uses Maxwell's equations to derive the dual equations used to represent the divergence of the electric field, and the electric current density. Because the bone is dielectric, the volume of the bone is inversely proportional to the electric field, and is also inversely proportional to the electric field gradient. However, one must account for the highly conductive Haversian and Volkmann canals whose volumes are proportional to the electric field;

$$\Delta \boldsymbol{E} = \frac{V^*}{V_0} \boldsymbol{R} \cdot \boldsymbol{J}$$

Naturally, the above considerations also apply to the equation for the electric field gradient when expressed in its alternate form;

$$\Delta \boldsymbol{J} = \frac{V^*}{V_0} \boldsymbol{K} \cdot \boldsymbol{E}$$

Due to the anisotropic nature of the bone, one cannot model the resistivity and conductivity of the bone in a unidirectional fashion. As such, one must use a resistivity contribution tensor, **R**, for the inhomogeneity, and a conductivity tensor, **K**, for the inhomogeneity (both of rank-2). Resistivity and conductivity are inverses of one another, and the conductivity tensor **K** is simply the inverse tensor of the resistivity tensor R. For ease in modelling, one can approximate the Haversian and Volkmann canals as ellipsoidal in shape. This allows one to use Eshelby's results for an inhomogeneity of ellipsoidal shape to model the conductivity of the bone, where the conductivity of the matrix material (k_0), the conductivity of the inhomogeneity (k_1), and Eshelby's tensor are all considered;

$$\boldsymbol{K} = k_0\left(\boldsymbol{s}^C - \frac{k_0}{k_1 - k_0}\boldsymbol{I}\right)^{-1}$$

In the case of the bone, the conductivity of the Haversian and Volkmann canals is much greater than that of the bone material so k_1 in the model is much greater than k_0. For relatively large values of k_1, we can approximate by taking the limit as k_1 goes to infinity, which is as follows;

$$\boldsymbol{K} = k_0\left(\boldsymbol{s}^C\right)^{-1}$$

Since the K-tensor is proportional to the inverse of the Eshelby tensor for an inhomogeneity, this tensor must be expressed in useful terms. In the case of a spheroidal inhomogeneity with a certain aspect ratio,γ, relevant results were provided by Carlslaw & Jaeger. **n** is the unit vector with respect to the spherical axis of symmetry (which in this case is the minor axis). f_0 represents the aforementioned spherical axis of symmetry.

$$\boldsymbol{s}^C = f_0(\boldsymbol{I} - \boldsymbol{nn}) + (1 - 2f_0)\boldsymbol{nn}$$

f_0 is itself an expression wherein shape factor g of the spheroid is considered and taken into account;

$$f_0 = \frac{\gamma^2(1-g)}{2(\gamma^2 - 1)}$$

When the aspect ratio is > 1, a prolate spheroid is being described, and when it is < 1 it is an oblate spheroid. These two cases are considered below;

$$g = \begin{cases} \frac{1}{\gamma\sqrt{1-\gamma^2}}\arctan\frac{\sqrt{1-\gamma^2}}{\gamma}, & \text{oblate shape } (\gamma < 1) \\ \frac{1}{\gamma\sqrt{\gamma^2-1}}\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right), & \text{prolate shape } (\gamma > 1) \end{cases}$$

As the paper approximates the canals to be prolate spheroids with a very large aspect ratio >>> 1, one can approximate the value of g by taking the limit as γgoes to infinity. Since the growth in the denominator expression greatly outstrips the growth in the numerator, this term goes to zero. Using this value of g, one can then approximate f_0 by plugging in the g that is acquired, and then taking the limit as γ goes to infinity. This value, in turn, can then be plugged into the Eshelby tensor, resulting in the following expression;

$$\boldsymbol{s}^C = f_0(\boldsymbol{I} - \boldsymbol{nn})$$

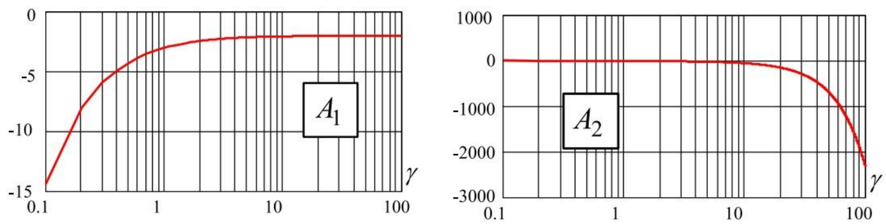
Applying dimensionless analysis, the paper arrives at the following expression of the K-tensor with dimensionless factors A_1 and A_2 where A_1 is proportional to the conductivity of the matrix material and A_2 is proportional to the conductivity of the inhomogeneity:

$$\boldsymbol{K} = -k_0(A_1\boldsymbol{I} + A_2\boldsymbol{nn})$$

$$A_1 = \frac{k_0 - k_1}{k_0 + (k_1 - k_0)f_0}, \quad A_2 = \frac{(k_0 - k_1)^2(1 - 3f_0)}{[k_1 - 2(k_1 - k_0)f_0][k_0 + (k_1 - k_0)f_0]}$$

Since A_1 and A_2 are expressed in terms of f_0, k_0, and k_1, they can be reduced further as $k_0 <<< k_1$. The behavior of A_1 and A_2 given an aspect ratio is shown in the graph below:

$$A_1 = \frac{-1}{f_0}, \quad A_2 = \frac{1-3f_0}{f_0(1-2f_0)}$$



Alternative Approach

Instead of considering the conductivity of the entirety of the bone, we will consider the conductivity of a particular cross-sectional volume to determine the conductivity in the vertical and horizontal directions. First, consider the conductivity of the bone in the vertical direction. In therefore, the parts of the bone of interest are the haversian canals and the bone matrix itself. First a cross sectional volume is taken consisting with a particular length L_B. Next, a voltage is applied to this cross section from the top to the bottom. Therefore, we can assume that the Haversian canals and the bone matrix have the same voltage applied to both. Since both the matrix and Haversian canals have resistances, we can then say that the two form a parallel circuit. In addition, since the Volkmann canals are perpendicular to the long axis of the bone, when the voltage is applied to the top of the cross section, the voltage across the entirety of the canals is the same and thus does not conduct electricity. Using the idea of parallel circuits we say that the total resistance in the vertical direction is equal to the product of the resistances of the Haversian canals and the bone matrix divided by the sum of the two:

$$R_{total} = \frac{R_B R_C}{R_B + R_C}$$

We then relate resistance to resistivity to determine the overall resistivity of the system which turns the previous equation into the form:

$$\rho_o = \frac{S_C}{\kappa(L_{total})} \frac{\rho_C \rho_B L_B L_C}{L_C \rho_C S_B + L_B \rho_B S_C}$$

Where ρis the resistivity of the total cross sectional volume, the bone, or the Haversian canal, L total is the total length of the cross section and κ is the fill factor of the canals to the bone. However, since we can choose our cross section to be of any length that we choose, we will choose the length to be equal to the length of the canals which will simplify our equation to:

$$\rho_o = \frac{S_C}{\kappa} \frac{\rho_C \rho_B}{\rho_C S_B + \rho_B S_C}$$

We then solve for the effective conductivity by taking the inverse of the effective resistivity. In order to determine the conductivity in the Horizontal direction, we use the same approach as previously described. However, unlike the Haversian canals which are all oriented in the vertical direction, the Volkmann are perpendicular to the long axis of the bone, but can be oriented in random angles. In order to take this into account, we use a probability distribution function in order to determine the average effective length of a Volkmann canal and use that for our calculations. If we were to apply an electric field in the x direction, if a Volkmann canal is angled off of the electric field, we can determine the effect of the electric field on the canal based on its projection onto the field:

$$x(\alpha) = l cos(\alpha)$$

Next, we consider the probability of the angles that the Volkmann canals can take. In this case we consider angles from 0 to π because anything after will be repeated. Since we assume that the canals are randomly oriented we say that the probability of any given angle is 1/π. With this information we use a probability distribution function to solve for the average effective length:

$$R(x)dx = p(\alpha)d\alpha \longrightarrow R(x) = p(\alpha)\frac{d\alpha}{dx} = \frac{-1}{\pi l\sqrt{1-\left(\frac{x}{l}\right)^2}}$$

$$\int_0^l R(x)x dx = \frac{-1}{\pi l}\int_0^l \frac{x}{\sqrt{1-\left(\frac{x}{l}\right)^2}}dx = \frac{2l}{\pi}$$

With this information, we can use our original equation to solve for the total resistivity of the cross sectional volume to determine the conductivity in the horizontal directions.

Results

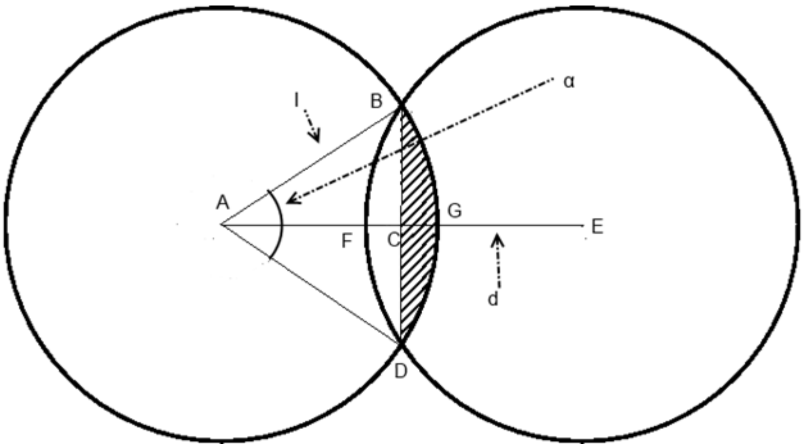
From our calculations, we determined that the overall conductivity of the bone in the horizontal and vertical directions to be .034 and .013 $(\Omega m)^{-1}$, respectively. With this information, we can construct our conductivity tensor **k** as:

$$\begin{bmatrix} 0.034 & 0 & 0 \\ 0 & 0.034 & 0 \\ 0 & 0 & 0.013 \end{bmatrix}$$

The x and y coordinate directions have the same conductivity because the Volkmann canals span all radial directions of the bone. Since we assume that they are randomly oriented, in addition to the vast number of these canals, that the conductivity will be the same in either direction. In addition, all off diagonals are zero because as stated previously, we determine the projection of the Volkmann canals onto our electric field in the x or y direction. Therefore there must also be a projection in the coordinate direction not of interest. However, since we say that the canals are randomly oriented, we assume that there exist another canal with the opposite orientation, thus cancelling the conductivity in the direction not of interest. When comparing values to those in the paper, we see that our values are much smaller than those that Casas and Sevostianov acquired. However, we note that the paper considers the conductivity of the total bone and not a particular cross section. Regardless, introducing the total numbers of Haversian and Volkmann canals show our values to be off by a factor of 2. Thus, we came to the conclusion that the conductivity values that we have calculated are off because we do not consider the overlapping of the Haversian canals, which will connect the Haversian and Volkmann canals together, thus increasing conductivity.

Overlap of Haversian Canals

To find the conductivity of the intersections, we must first find the area of the average intersection between two canals. First, we look at the intersection between the two canals looking from above. This looks much like the intersection between two circles of equal radius (image below):



The area of the sector enclosed by the lines AB and AG is equal to the angle formed times the radius, or:

$$S = \alpha * l^2$$

However, we are interested in finding the shaded area separate from the rest of the sector, as this gives us half of our intersection. To do this, we must subtract the area A of triangle formed by ABD from the aforementioned sector. Multiplying the result by 2 gives us the entire intersection, represented by the following:

$$S_0 = 2 * (S - A)$$

The area of our triangle is 2*((d/2)*l*sin(α)/2), or simply d*l* sin(α)/2. Substituting this and our S into S_0, we have the following:

$$S_0 = 2 * (\alpha * l^2 - \frac{d * l}{2} * \sin(\alpha))$$

Since the sine function is the measure of the opposite side of the acute angle divided by the hypotenuse, l*sin(α) is itself equal to the length of the line formed by BD. Using the Pythagorean Theorem, this length can be represented by the other sides of the triangle:

$$1 * \sin(\alpha) = \sqrt{l^2 - \frac{d}{2}^2}$$

The angle α can be found using this derivation. Solving for α results in the following representation:

$$\alpha = \arcsin\left(\frac{\sqrt{l^2 - \frac{d}{2}^2}}{l}\right)$$

Substituting both of these alternative representations into our formula for the area of the intersection allows us to represent this area without using α.

$$S_0 = 2 * (\arcsin\left(\frac{\sqrt{l^2 - \frac{d}{2}^2}}{l}\right) * l^2 - \frac{d}{2} * \sqrt{l^2 - \frac{d}{2}^2})$$

Now we must also take into account the area of the canals in a lengthwise direction. This is as simple as finding the absolute value of the difference between the total length of a canal and the length of the intersection.

Placing two canals in a 3-dimensional Cartesian coordinate system, setting the origin as the center of one of these canals, one can now find the lengthwise intersection as a function of z, where L is the length of the canal and z is the center of the other canal:

$$ב = mz - b$$

$$0 = m(-L) + b; \qquad 0 = m(L) + b$$

$$L = b; m = 1; ב = L - |z|$$

Furthermore, the distance (d) can be expressed in terms of x and y:

$$d = \sqrt{x^2 + y^2}$$

We can now consider the area of the entire intersection as a function of x, y, and z:

$$f(x, y, z) = ב * S_0 \quad \rightarrow$$

$$f(x, y, z) = (L + |z|) * [2 * \left(\arcsin\left(\frac{\sqrt{l^2 - \frac{\sqrt{x^2 + y^2}}{2}^2}}{l}\right) * l^2 - \frac{\sqrt{x^2 + y^2}}{2} * \sqrt{l^2 - \frac{\sqrt{x^2 + y^2}}{2}^2} \right)]$$

Since the function f(x,y,z) is symmetric, it is useful to convert to polar coordinates to integrate the function.

$$f(\varphi, \mathrm{r}, z) = (L + |z|) * [2 * \left(\arcsin\left(\frac{\sqrt{l^2 - \frac{r^2}{2}}}{l} \right) * l^2 - \frac{r}{2} * \sqrt{l^2 - \frac{r^2}{2}} \right)]$$

Now we can set up the integral, integrating with respect to φ, r, and z:

$$\int_0^L (L - |z|) dz \iint_0^{2\pi} 2 * \left(\arcsin\left(\frac{\sqrt{l^2 - \frac{r^2}{2}}}{l} \right) * l^2 - \frac{r}{2} * \sqrt{l^2 - \frac{r^2}{2}} \right) d\varphi dr$$

Integrating with respect to φ and z, we are left with the following single integral for r:

$$2\pi L^2 \int_{d_0}^{2l} 2 * \left(\arcsin\left(\frac{\sqrt{l^2 - \frac{r^2}{2}}}{l} \right) * l^2 - \frac{r}{2} * \sqrt{l^2 - \frac{r^2}{2}} \right) dr$$

This integral can subdivided into two integrals and solved to obtain the following:

$$\frac{3}{2}\pi^2 L^2 l^4$$

Using the total number of canals (N) and dividing by volume allows us to express the average of the overlap in the following way:

$$\omega = \frac{N(N-1)}{2V} * (\frac{3}{2}\pi^2 L^2 l^4)$$

N can be rewritten as the fill factor K times the volume.

$$\omega = \frac{K(KV-1)}{2} * \left(\frac{3}{2}\pi^2 L^2 l^4\right)$$

The average volume of the Volkmann canals (V_v) can be approximated as the volume of a cylinder described below (n is described below, as well):

$$V_V = \pi * z_V {l_V}^2$$

$$n = z_V * \frac{N}{L}$$

Knowing the average value of the overlap tells us the space in which there are possible intersections. Since the average intersection is proportional to the average of the overlap as well as n, and inversely proportional to the average volume of the Volkmann canals, it can be expressed in the following way:

$$A_I = \frac{\omega}{V_V} * n$$

Works Cited

Carslaw, H. S., and J. C. Jaeger. Conduction of Heat in Solids. Oxford: Clarendon, 1959. Print.

Kosterich, J. D., Foster, K. R., & Pollack, S. R. (1983). Dielectric permittivity and electrical conductivity of fluid saturated bone.IEEE Transactions on Biomedical Engineering, 30, 81–86.

Kosterich, J. D., Foster, K. R., & Pollack, S. R. (1984). Dielectric properties of fluid-saturated bone – The effect of variation in conductivity of immersion fluid. IEEE Transactions on Biomedical Engineering, 31, 369–374.

Mangan, Thomas C. A Gentle Introduction to Tensors

Peeters, Kasper and Kees Dullemond. Introduction to Tensor Calculus. 2008

Rockwood, , and Green. *Fractures in Adults*. 6th. Lippincott Williams & Wilkins, 2006. eBook. http://www.msdlatinamerica.com/ebooks/RockwoodGreensFracturesinAdults/sid1337350.html.

R. Casas, I. Sevostianov. Electrical resistivity of cortical bone: Micromechanical modeling and experimental verification. International Journal of Engineering Science 62 (2013) 106–112.

SEER Training Modules, *Module Name*. U. S. National Institutes of Health, National Cancer Institute.
http://training.seer.cancer.gov.

Weinberger, C., Cai, W., & Barnett, D. (2005). Lecture Notes -- Elasticity of Microscopic Structures. Stanford University. 29 Mar 2013.
http://micro.stanford.edu/~caiwei/me340b/content/me340b-notes_v01.pdf.