SINKING BUBBLES IN AN OSCILLATING FLUID

Christian Gentry, James Greenberg, Nick Kearns,
Xi Ran Wang

STATEMENT OF PROBLEM, PHENOMENON

- It is observed that bubbles can sink in a quickly oscillating container

- Why? Added mass.

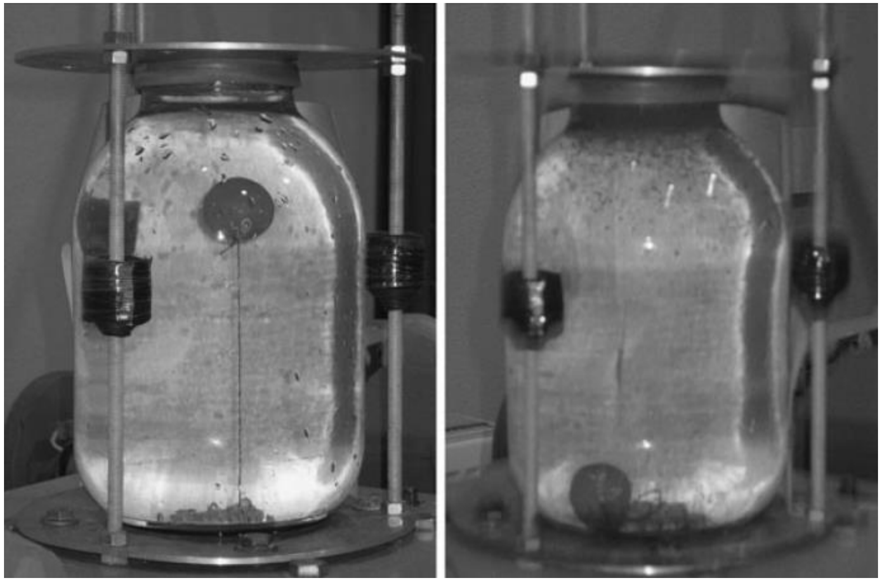


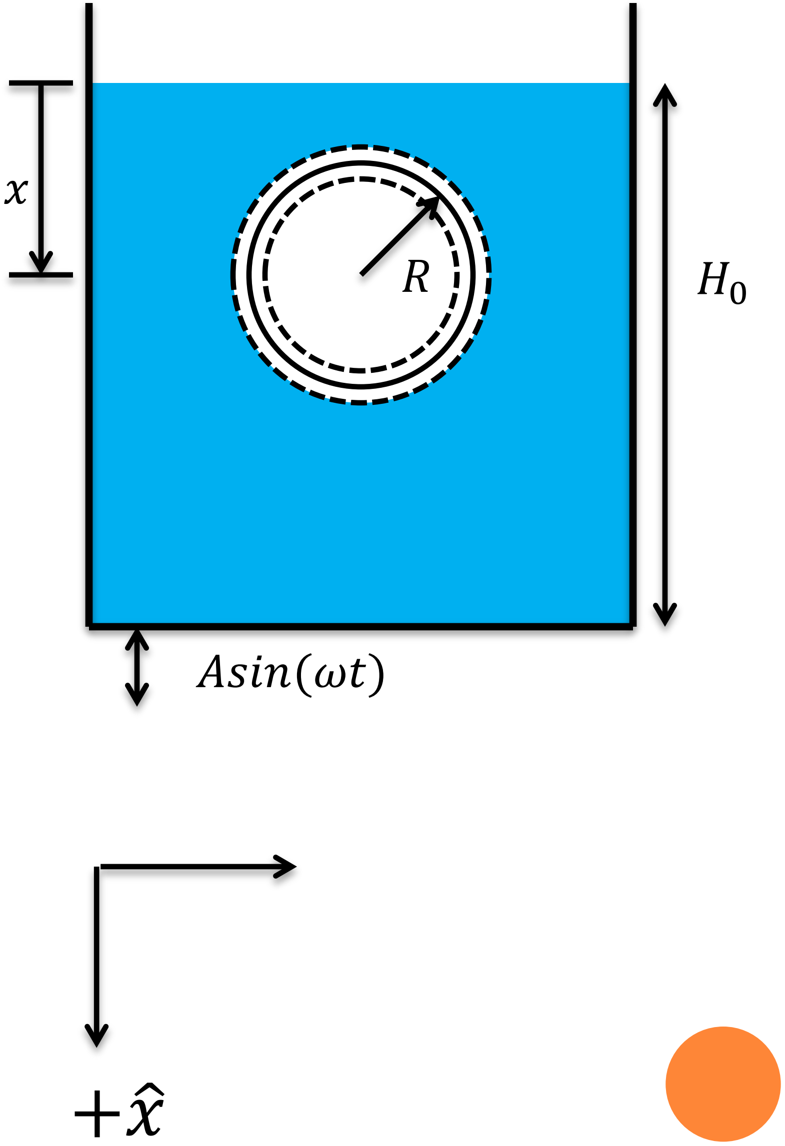
Figure credit [1]

APPLICATIONS

- Bubble sinking caused an unexpected accumulation of air at the bottom of rocket fuel tanks. Prevented early success of 3-stage rockets.
- Accurate model required for success of rockets as well as advanced liquid cooling technologies.[3]
- Practical application for separation of motions (slow and fast time scales).

PHYSICAL MODEL

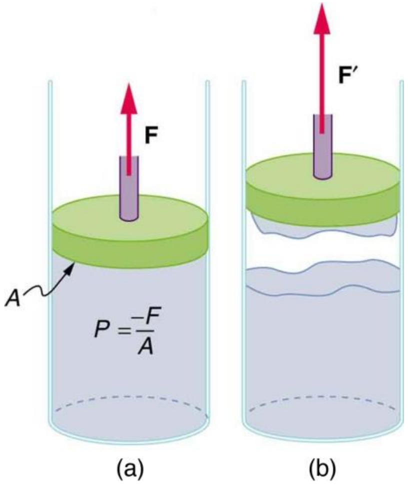
- Variables and parameters
  - x – Bubble depth
  - R – Bubble radius
  - H_0 – Total height of water
  - A – Amplitude of oscillations
  - ω – Frequency of oscillations
  - t – Time



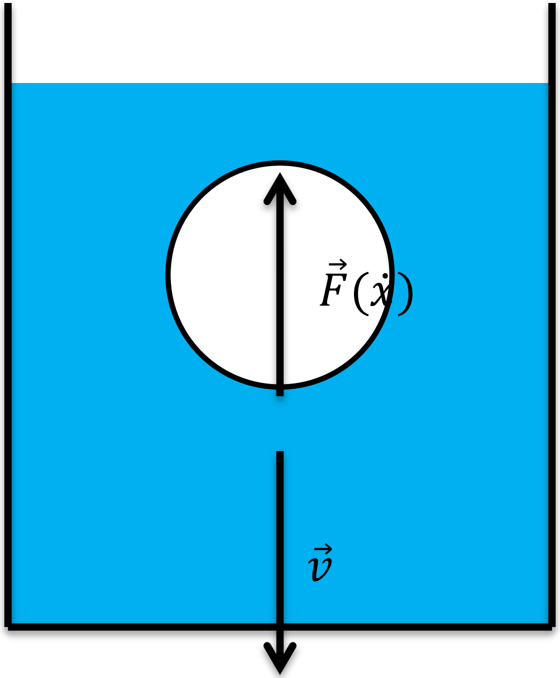
MODEL ASSUMPTIONS

- Spherical bubble
  - Small bubble
  - Compressible
- Quasistatic
- External pressure >> Pressure in water
  - Prevents cavitation

- $V_b = \frac{4}{3}\pi R^3$
  - $R < 2cm$
- $P_x V_b = P_e V_{b0}$
- $P_e \gg \rho H_0(g + A\omega^2)$

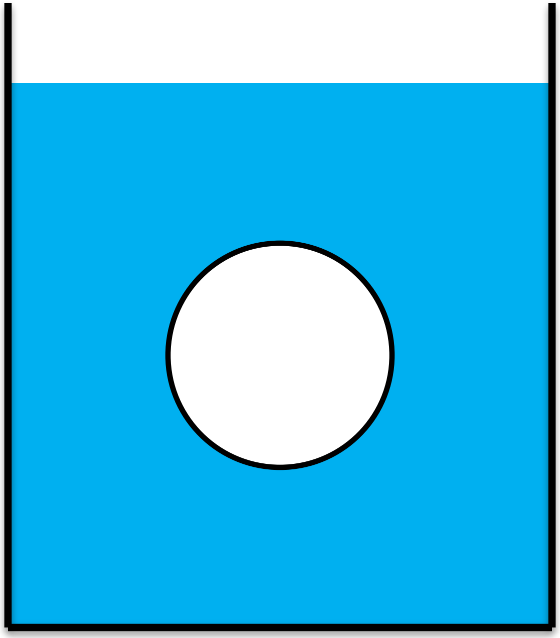


"FRICTIONAL" FORCES – DRAG



- Bubble has an instantaneous velocity
- The fluid resists the motion of the bubble
- $F(\dot{x}) = \frac{1}{2}\rho\dot{x}^2 C_d A$
  - C_d – drag coefficient
  - A – surface area

BUBBLE VOLUME



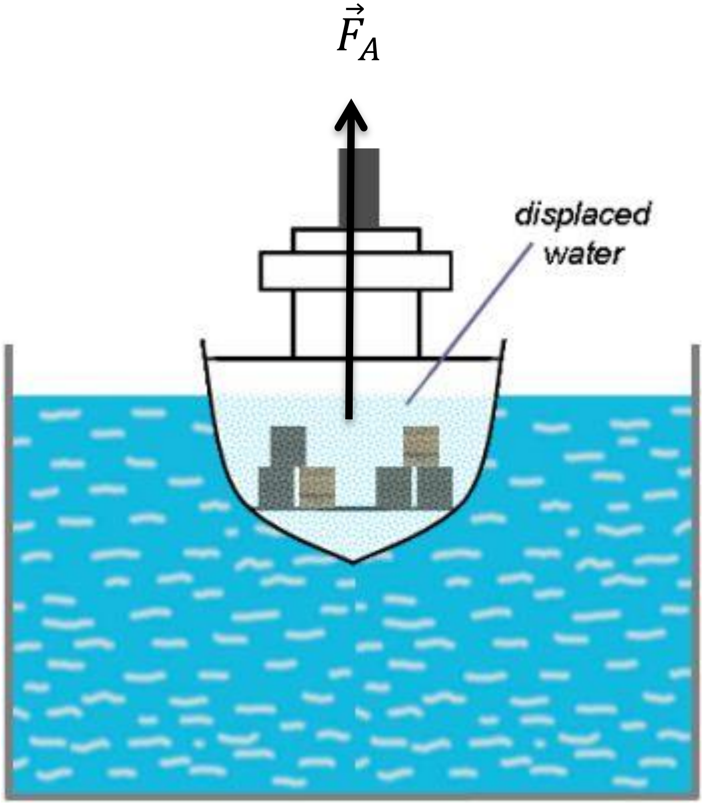
- Bubble doesn't move same distance as the tank
- Causes bubble depth to change
  - Bubble expands/shrinks

$$V_b=V_{b0}\left[1-\frac{\rho x g}{P_e}\left(1+\frac{A\omega^2}{g}\sin\omega t\right)\right]$$

ARCHIMEDES' FORCE

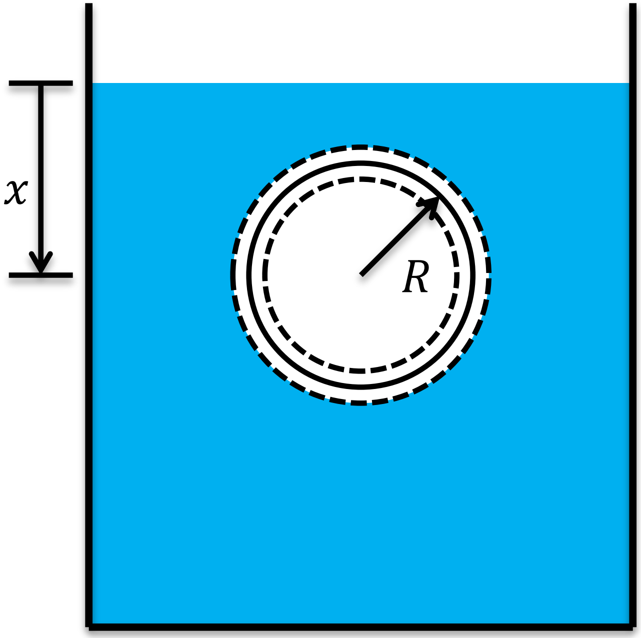
- Buoyant force
- Always upwards
- Why bubbles usually float

$$F_A = -\rho V_b g$$



PHENOMENON OF ADDED MASS

- Bubble has to displace the water around it to move
- Results in a larger observed inertia for bubble (bubble gained mass)



$$m_0 = \frac{1}{2}\rho V_b$$

m_0 changes with time!

$$M = m + m_0$$

added mass

Intrinsic mass of bubble

GOVERNING EQUATION – PART 1

- Newton's Second Law

$$\cancel{\vec{F} = M\vec{a}}$$

- Only valid for constant mass (inertia)
- Added mass changes with bubble size
- More general:

$$\vec{F} = \frac{d\vec{P}}{dt} \quad ; \quad \vec{P} \equiv M\vec{v}$$

$$F = \frac{dP}{dt}$$

$$F = \frac{d((m + m_0)\dot{x})}{dt}$$

$$F = (m + m_0)\frac{d\dot{x}}{dt} + \dot{x}\frac{d(m + m_0)}{dt}$$

$$\boxed{F = (m + m_0)\ddot{x} + \dot{m}_0\dot{x}}$$

Added Mass

GOVERNING EQUATION – PART 2

- Plugging in all the forces we get:

$$(m + m_0)\ddot{x} + \dot{m}_0\dot{x} = -F(\dot{x}) + (\mathrm{m} - \rho V_b)(\mathrm{A}\omega^2 \sin \omega t + \mathrm{g})$$

- To get equation of motion, plug in:

$$m_0 = \frac{1}{2}\rho V_b$$

$$F(\dot{x}) = \frac{1}{2}\rho\dot{x}^2 C_d A * \mathrm{sign}(\dot{x})$$

$$V_b = V_{b0}\left[1 - \frac{\rho x g}{P_e}\left(1 + \frac{A\omega^2}{g}\sin \omega t\right)\right]$$

SEPARATION OF TIMESCALES

- Interesting behavior is on slow timescale relative to fast oscillations
- Separate timescales by using method of direct separation of motions [2]

$$< h(\tau) >= \frac{1}{2\pi} \int_{0}^{2\pi} h(\tau) d\tau$$

$$x(t,\tau) = X(t) + \Psi(t,\tau)$$

$$< \Psi(t,\tau) >= 0$$

$$< X(t) >= X(t)$$

$$< x(t,\tau) >= X(t)$$

FAST AND SLOW EQUATIONS

- After simplification and neglecting higher order terms, slow equation becomes:

$$(m+m_{01})\ddot{X}-\gamma\cdot m_{01}W\frac{X}{H_0}(<\ddot{\Psi}\sin\omega t>+\omega<\dot{\Psi}\cos\omega t>)=-<F(\dot{X}+\dot{\Psi})>+(m-\rho V_{b0})g+\gamma\cdot\rho V_{b0}W\frac{X}{H_0}\frac{A\omega^2}{2}$$

- Fast equation becomes:

$$(m+m_{01})\ddot{\Psi}=-4\rho R_0^2\psi_\infty(\dot{\Psi}^2\operatorname{sgn}\dot{\Psi}-<\dot{\Psi}^2\operatorname{sgn}\dot{\Psi}>)+(m-\rho V_{b0})A\omega^2\sin\omega t$$

Solution of Slow Equation

- Solving the fast equation transforms the slow equation to:

$$m_{01}\ddot{X}+\frac{16}{\pi}\rho\psi_{\infty}R_0^2\dot{X}B\omega=\gamma\cdot W^2\frac{X}{H_0}\frac{\rho V_{b0}g}{2}\cdot\left(1-\frac{2}{3}\frac{\theta\frac{A^2}{R_0^2}}{2\left(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}\right)+\theta\frac{A^2}{R_0^2}}\right)-\rho V_{b0}g$$

$$\theta=\frac{16^2}{\pi^4}\frac{\psi_{\infty}^2}{\chi^4}$$

- Since the bubble moves slowly, $\ddot{X}<<1$
- Equation becomes:

$$\dot{X}\approx v\left(\frac{X}{X_0}-1\right)\qquad v=\frac{\pi^2}{12\psi_{\infty}}\frac{R_0}{B}\frac{g}{\omega}$$

FIXED POINT ANALYSIS

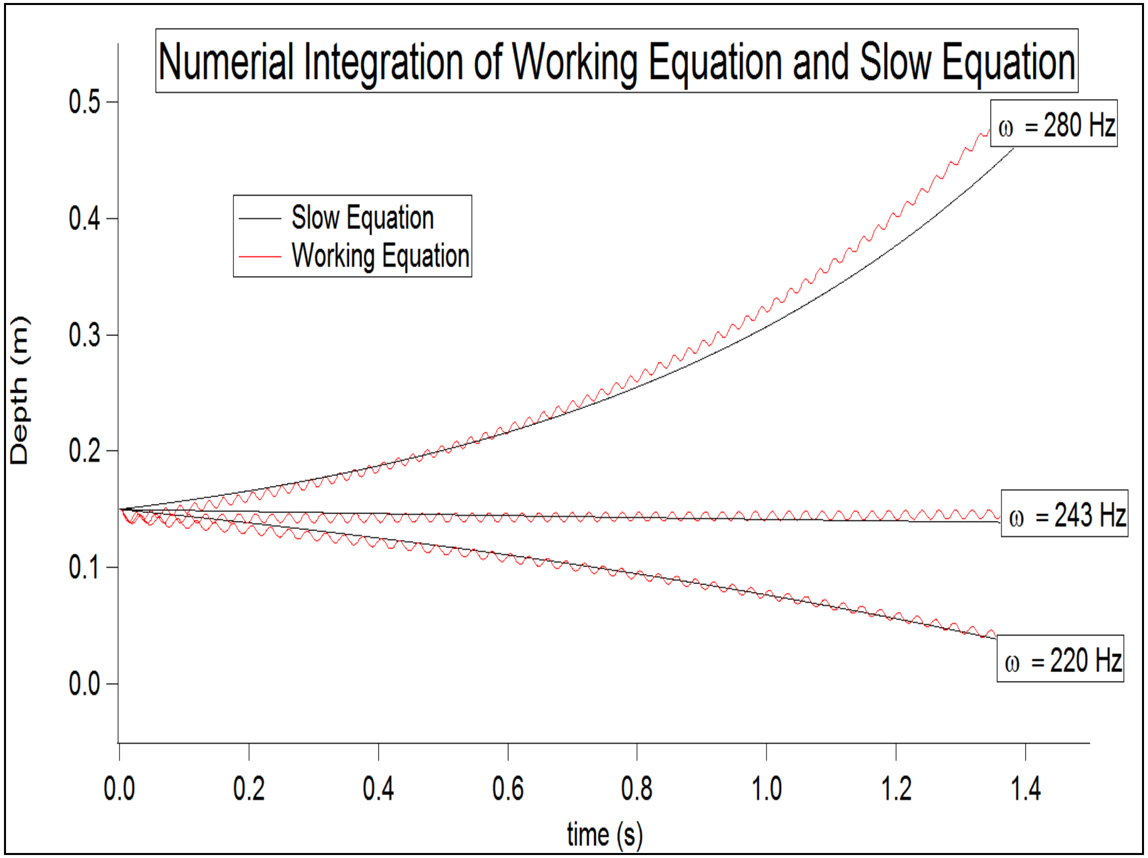
- First order equation, $\dot{X} \approx v\left(\frac{X}{X_0} - 1\right)$ has one fixed point: $X = X_0$

$$X_0 = \frac{2H_0}{\gamma \cdot W^2} \frac{2\left(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}\right)+\theta\frac{A^2}{R_0^2}}{2\left(1+\sqrt{1+\theta\frac{A^2}{R_0^2}}\right)+\frac{\theta}{3}\frac{A^2}{R_0^2}}$$

- This fixed point is unstable
- Imposing the boundary conditions gives two more fixed points: $X = 0$ and $X = H_0$
- Both boundary fixed points are stable

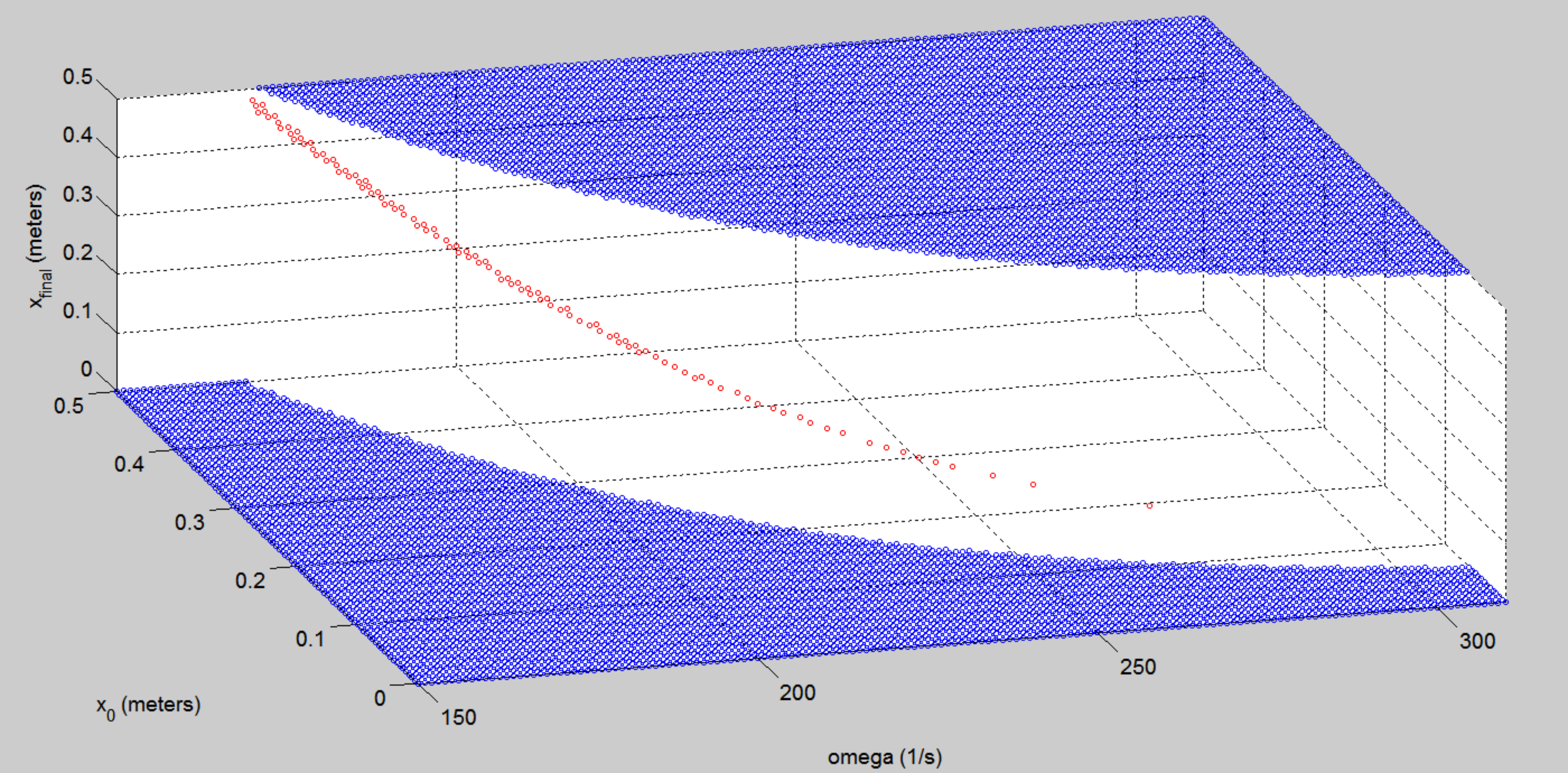
RESULTS BY DIRECT INTEGRATION

- Comparison between integration of slow equation and working equation



- Parameters
  - x_0 – 0.15 m
  - H_0 – 0.5 m
  - A – 0.002 m
  - R_0 – 0.0029 m
  - ρ_{fluid} – 1000 kg/m
  - ρ_{bubble} – 1.2 kg/m
  - P_e – 10^5 Pa

BIFURCATION DIAGRAM

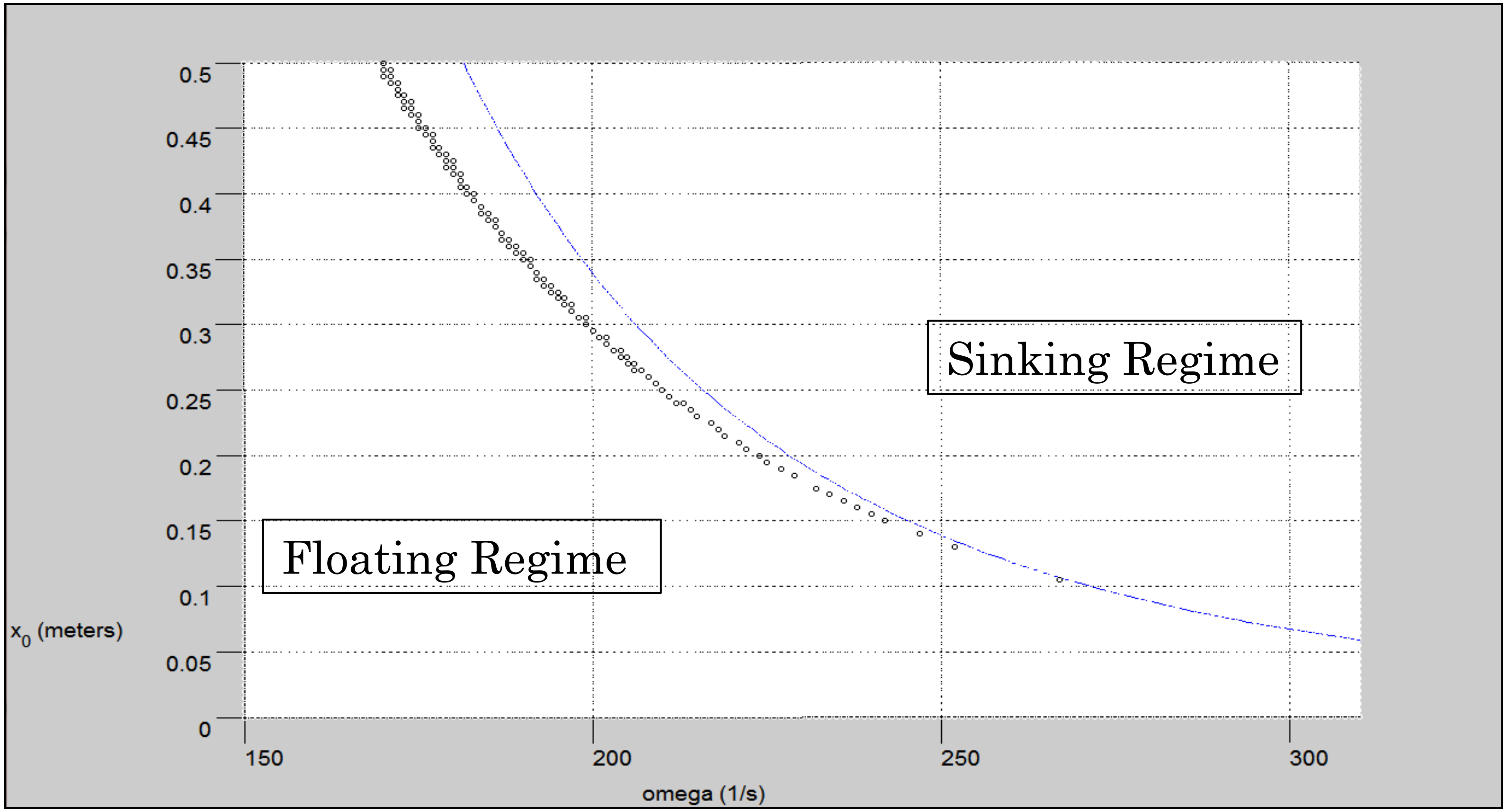


● -- Stable

● -- Unstable

PREDICTION OF INSTABILITY

- Bifurcation Diagram Prediction (circles) compared to Fixed Point Analysis Prediction (curve)



CONCLUSION

- Bubbles in oscillating fluids sometimes sink
- Extremely dependent on:
  - Height of the bubble
  - Frequency of oscillations

- Why?
- Added mass term beats out buoyancy term
- Causes average position of bubble to decrease
- Friction slows processes in either direction but not required for sinking to occur

SOURCES

[1] Sorokin, V. S., Blekhman, I. I., Vasilkov, V. B.: *Motion of a gas bubble in fluid under vibration*. Nonlinear Dyn (2012) 67:147–158.

[2] Blekhmann, I.I.: *Vibrational Mechanics*. World Scientific, Singapore (2000)

[3] 2013, Avanced Cooling Technologies, Inc., *Oscillating Liquid Cooling*, http://www.1-act.com/advanced-technologies/pumped-liquid-cooling/oscillating-liquid-cooling/ (April 24, 2013)

QUESTIONS?

- Special thanks to Dr. Gabitov and Matt Pennybacker